THE MATHEMATICAL PRINCIPLES OF MECHANICAL PHILOSOPHY,

AND THEIR APPLICATION TO

ELEMENTARY MECHANICS AND ARCHITECTURE,

BUT CHIEFLY TO

THE THEORY OF UNIVERSAL GRAVITATION.

SECOND EDITION, REVISED AND IMPROVED.

BY

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PREFACE TO THE FIRST EDITION.

A LEADING object that I have had in view in preparing the present Treatise has been to gather into one uniform system the principles of mechanical science, beginning with the most elementary and ascending to the most general. In attempting to accomplish this I have collected the fundamental principles into separate Chapters, and placed after them Chapters of application of these principles to the demonstration of others of a second class, and have then added collections of problems, and, in some instances, hints to guide to their solution.

An attachment, and that in most respects a laudable attachment, to the geometry of the *Principia* had, till of late years, led to the practice of retaining in our course of University reading some parts of that immortal work, rather for the beauty and elegance of its demonstrations, than for the importance of the theorems demonstrated. But this practice has been gradually sinking into disuse, a result which we owe to Professor Woodhouse's *Physical Astronomy*, to M. Poisson's *Traité de Mécanique*, which has been extensively used amongst us, and very largely to Mr. Whewell's Treatises on *Statics* and *Dynamics* and

Mr. Airy's *Mathematical Tracts*. But notwithstanding the great and happy changes thus brought about we still cling to the old methods, not as a whole, but just so far as to derange our system and give to it the ambiguous character of being neither strictly geometrical, nor strictly analytical. But I wish not to be misunderstood; I mean not to imply that geometry should be discarded and banished from our academical course of study; far from it; for the analyst will find his analysis of little benefit if he have not the power of gathering from his formulæ geometrical conceptions. Neither would I have it for a moment conceived, that I would in the least degree repudiate the profound veneration, which is so justly due even to the letter of the *Principia:* my own admiration of the clearness and conciseness of its demonstrations rather induces me to invite others to participate of the pleasures they may enjoy from its attentive and diligent study. But this I desire, that we should pay more regard to system than we hitherto have done; if our course is to be geometrical let us adhere to geometry, if analytical to analysis; if we are to admit both (the preferable course) let us keep our systems well apart, and not have our course of reading confused, here analysis and there geometry.

My own experience has impressed me also with the conviction, that many of our candidates for University honours are debarred the high enjoyment of penetrating into the sublimer investigations of Physical Astronomy from the want of some treatise that would lead them by

a clear and distinct path, and with an undivided attention, through the train of reasoning which leads from elementary mechanical principles to the demonstration of celestial phenomena. Some, it is true, of our first-rate students do attain this eminence; but might not this few be considerably augmented, if their path were well pointed out and disencumbered of many of the obstacles which lie in their way and impede their course?

Let it not be imagined, however, that I send forth the present volume with the presumptuous confidence, that the want of a complete analytical system of mechanics is supplied by its appearance; though I will so far commit myself as to confess, that to supply this want has been my earnest desire;—no, I would rather use the experience of a distinguished Author, whose name I have already used, who is a far better judge, in such a case, than myself, and say in his words, "a few years experience has a great tendency to diminish the confidence of producing what shall satisfy himself and others, with which a young author sets out: and he learns that the vivid impression of fancied deficiencies and imperfections of preceding works which at first induced him to write, is a very insufficient warrant of his own skill and judgment." But yet my object has been unique; and it has not been till after much time and thought spent upon the subject, that I have ventured to lay my work before the public: how I have satisfied my own desire I leave to the candour of my readers to determine.

In the first, second, and third Chapters of Statics will be found the principles of the composition and equilibrium of statical forces acting, first on a particle, then on a rigid body, and lastly on a system of bodies connected in any manner. From the conditions of equilibrium the principle of Virtual Velocities has been deduced. Afterwards this principle has been demonstrated independently after the method of Lagrange, and from it are deduced the conditions of equilibrium. Then follow Chapters of application. The fourth Chapter is a collection of examples of finding the centre of gravity of bodies, being an application of the formulæ for the co-ordinates of the centre of parallel forces. In this Chapter I have aimed at explaining and illustrating integration between limits: see particularly Examples 8, 13, 14, 19, 25, 26. The fifth Chapter contains the application of the principles to the six mechanical powers, and concludes with some notices of the laws of friction. The sixth Chapter is upon roofs, arches, and bridges, which form interesting applications of the principles of equilibrium. In this Chapter will be found some remarks upon the roofs of Trinity College Hall and Westminster Hall, the action of buttresses, and the stone vaulting of King's College Chapel, as well as notices of other noted edifices and structures. An example is given of the method of calculating the lengths and weights of the supporting rods, chains, and road-way of suspension bridges, that the strain may in every part be proportional to the strength of the chain. Then follows a Chapter of statical problems, beginning with some general re-

marks on their solution. And the treatise on Statics closes with a Chapter on Attractions. After calculating the attraction of spherical and spheroidal bodies of homogeneous mass, I have proceeded to the more general investigation of the attraction of a body differing but little from a sphere in form, with a view to the calculation of the Figure of the Earth in a future part of the work. This has led me to introduce Laplace's Coefficients, a subject unknown in our University course, till introduced a few years since by Mr. Murphy in his Treatise on Electricity. I have followed Laplace's course, and not the inverse method of Mr. Murphy. The frequent occurrence of the equation

$$\frac{d^2 V}{dx^2} + \frac{d^2 V}{dy^2} + \frac{d^2 V}{dz^2} = 0$$

in physical investigations makes it highly desirable, that a knowledge of the profound analysis of Laplace should be made as familiar as possible to the higher class of students in the University. For this reason I have introduced, in as concise and at the same time as clear a manner as I was able, the principal properties of the Coefficients of that great analyst, breaking up and arranging the subject in the form of propositions. In Art. 181 will be found a new proof of the proposition, that *a function can be expanded in* only one *series of Laplace's Coefficients.* This fact is of the utmost importance, as will appear in Art. 182 and other places. Mr. Airy has shewn that Laplace's proof is not satisfactory: and adds a very clear one in

its place; but it is restricted to the case of the number of coefficients being finite. See *Cambridge Phil. Soc. Trans.* Vol. II.

The treatise on Dynamics opens with a Chapter upon the fundamental principles of the motion of bodies, which I gather wholly from experiment and observation. After explaining the conventional method of measuring motion, I proceed to an enquiry into the laws that regulate the motion of bodies when uninfluenced by external causes: a variety of experiments and facts of ordinary occurrence point to the principle called the First Law of Motion. This leads to an explanation of the conventional method of measuring force dynamically. In explaining the means of estimating force, I have aimed at giving a distinct idea of the nature of forces that require a finite time to develop their effects, and those which generate velocity in an indefinitely short time. An investigation is then made into the laws which regulate the motion of bodies when acted on simultaneously by different causes, and this leads to the principle called the Second Law of Motion. This law enables us to introduce a method of referring the curvilinear motion of a body to three rectangular axes. The necessity is then shewn of obtaining a relation between the two arbitrarily assumed measures of force, viz. pressure, and velocity generated: this leads to the principle called the Third Law of Motion. The introductory Chapter of Dynamics concludes with the enunciation of a self-evident principle, first, for finite forces, and then for impulsive, analogous

to that first introduced by D'Alembert, whereby we deduce the equations for calculating the motion of a system from the equations of equilibrium. This principle is, in fact, the interpretation of the Three Laws of Motion into analytical language.

The second Chapter is upon the motion of a single particle. In this I have entered fully into the properties of central forces, and calculated the motion in various cases: and at the close of the Chapter Kepler's Laws are made use of to guide us to the discovery of the nature of the force acting on the planets; and we thus catch a first glimpse of the theory of gravitation: after shewing that there is sufficient ground to justify us in undertaking the task of calculating the consequences of this law, a large portion of the remainder of the work is devoted to that enquiry. In the third Chapter the motion of two particles attracting each other according to the law of gravitation is calculated. In the fourth the perturbations in this motion by the introduction of a third attracting body are explained upon the supposition that the disturbing body is very distant. This Chapter contains the Sixty-sixth Proposition of the First Book of the *Principia* and its corollaries and some Propositions of the Third Book put into an algebraical form: this is a digression from the chain of exact reasoning which is the professed object of the work; but being in a separate Chapter the student, if he choose, may pass over this and proceed to the fifth Chapter, in which the distance, longitude, and latitude

of the Moon are calculated to a second approximation upon the theory of gravity. The sixth Chapter contains the calculation of the perturbations of the planets. In the Lunar Theory, a part peculiar to this work, is the way in which I have introduced the constants *c* and *g*, which give rise to the motion of the line of apsides and the line of nodes. In the Planetary Theory I have used M. Pontecoulant's method of integrating the equations of motion of an undisturbed planet, and then applied Lagrange's principle of the variation of parameters to calculate the variations of the elliptic elements. This Chapter closes with a demonstration of the Stability of the Planetary System, retaining the squares of the eccentricities and inclinations. The next is upon the motion of a particle on curves and surfaces, and also on the oscillations and perturbations of pendulums. A collection of problems on the motion of bodies considered as particles follows.

The ninth Chapter is occupied with the demonstration of certain geometrical properties which are necessary for the following parts of the work. The tenth and eleventh Chapters are upon the motion of one or more rigid bodies acted on by finite forces. This subject introduces several interesting and important propositions: among others, the calculations of the Precession of the Equinoxes and the Nutation of the Earth's Axis; the formula for Precession is prepared for numerical calculation: the reduction to numbers will be found in the Chapter on the Figure of the Earth. Then follows

a Chapter on the motion of a flexible body. I have not entered at large upon this subject; but have taken only the case of a vibratory cord: but the principles developed in the course of the work admit of application to the solution of the most general questions of the motion of flexible bodies. The thirteenth Chapter is upon the motion of one or more rigid bodies acted on by impulsive forces. In this and the tenth and eleventh Chapters will be found some general dynamical principles which lead to interesting general remarks regarding the effect of the impact of the Ocean, the radiation of the Earth, heat, the concussion of descending avalanches, the eruption of volcanoes, the attraction of comets, and other causes upon the motion of the Earth. The last Chapter of Dynamics contains various problems on the motion of rigid bodies, and remarks upon the methods of solution.

In the treatises on Hydrostatics and Hydrodynamics the general principles of those sciences are developed, and applied to the determination of the Figure of the Earth upon the hypothesis of its mass having been at some former epoch in a semi-fluid state, the form of the Atmospheres of the Planets, the Tides, and the effect of a Resisting Medium upon the elements of the planetary orbits.

Lastly, a summary of the arguments in favour of the Theory of Universal Gravitation closes the Work. A reference to the Table of Contents will give a far better view of the character of the Work, than I have deemed it necessary to give in this place.

The prevailing argument with me for using the old differential and integral notation is the excellence of Fourier's notation for definite integrals: I much prefer that to any other that I have seen, and this naturally led me back to the old form of differentials and integrals. In case any of my readers are not acquainted with Fourier's notation I now give it, $\int_a^b udx$ represents the integral of the differential coefficient u, or of the differential udx, with respect to x taken between the limiting values a and b of x. In successive integration the order of arrangement of the integrals is the same as that of the differentials: thus $\int_{-1}^{1}\int_0^{2\pi} Pd\mu d\omega$ represents the double integral of P with respect to μ and ω, the limits of μ being -1 and 1, and the limits of ω being 0 and 2π.

I repeat, that it is not with the expectation that I have fully succeeded in satisfying even my own desires, that I present this volume to the students of the University; but with the earnest wish, that it may be found useful, and that my labours may not have been altogether spent in vain. Should any of my readers favour me with any suggestions of improvement I shall receive them with the greatest thankfulness.

J. H. P.

CAIUS COLLEGE,
Nov. 26, 1836.

Postscript. This Preface has been very slightly altered, that the outline given of the Work may not differ from the arrangement in the New Edition.

PREFACE TO THE SECOND EDITION.

In presenting a Second Edition of this Work to the public, I cannot refrain from expressing the pleasure I have derived from the very favorable manner in which the First has been received. That Edition consisted of 1000 copies, and has been disposed of in less than five years. My highest desire was, that it should meet the approval of those best able to judge of its merits; but so rapid a sale was very far from my warmest expectations.

I do not presume, however, to attribute this success to any excellence in the manner of execution, for I am aware that the Work has many imperfections: I would rather attribute it to the peculiarity and unity of the plan. While various other publications, admirable in their respective departments, develop the several parts of Mechanical Philosophy, none of them profess to conduct the student through a complete course from elementary principles to the highest branches of the Mechanism of the Heavens. A Work of this description, comprised within a moderate compass, appears to have been a desideratum: and to the existence of this desideratum I attribute the success of the present undertaking.

No pains have been spared in preparing the present Edition for the press. As soon as the First was published in 1836, a thorough revise was commenced; this was completed during a voyage to India at the close of 1838. Many improvements were introduced in making explanations clearer and more perspicuous, in re-arranging some parts, in changing methods of demonstration, in removing matter where it appeared to be inappropriate or superfluous, and in adding new. The copy thus revised was sent to England, and has from time to time received further alterations and additions as they were suggested to my mind.

When a new Edition was called for, a gentleman kindly undertook to correct the sheets as they passed through the press. As they were struck off, a copy of each sheet was sent to me at Calcutta by the monthly overland mail, that I might have the opportunity of seeing the print, and of making any remarks before the Edition was finally laid before the public. The first sheets reached me in May, and the last in September; and by this month's mail I send back the Appendix, the Preface, Table of Contents, and Errata. I should be obliged to my readers if they would make the necessary corrections in their own copies with the pen: some are improvements rather than corrections of errors. I must say, in justice to the compositors and corrector of the press, that the list of errata is large in a great measure, in consequence of the peculiar circumstances under which this Edition has been prepared. I laid down the strict injunction that my

copy should be exactly followed; and there were several little errors, which, in consequence of this restriction, the corrector of the press was not at liberty to remove; but which would have come to my knowledge in time for correction, had I been able to correct the sheets myself. As a substitute for this, I have read every page with great care, and have drawn up the list of errata and corrigenda, which will be found at the end.

I will now give an outline of the principal points in which this Edition differs from the First. The demonstration of the conditions of equilibrium of a rigid body derived from those of a single particle is made complete by shewing that the six equations derived by elimination are the *only* equations: see p. 34, Note. A general proposition regarding the stability and instability of any system is added: Art. 78. Several new propositions are given upon the subject of Arches: Arts. 126-128. In the Chapter on Attractions a very material improvement is introduced in the demonstration, that *every function of* μ *and* ω *which does not become infinite between the limiting values* $-1, 1$ *of* μ *and* $0, 2\pi$ *of* ω, *can be expanded in a series of Laplace's Coefficients.* An Appendix is added to that Chapter to explain the method of actually calculating the values of Laplace's Coefficients.

The Introductory Chapter of Dynamics has undergone a most complete revise; and everything has been done to make it clear and perspicuous. I have endeavoured to lead the student on from step to step in

such a manner, that he may clearly perceive the *necessity* and also the *sufficiency* of the Three Laws of Motion for enabling us to obtain equations, for calculating the relations between the forces and the motion of the bodies that move under their influence. In the Chapter which contains explanations of the Lunar Perturbations after the method of Newton, I have omitted some parts which appeared irrelevant to this Work, and which will be found better explained in Mr. Airy's *Gravitation*, and Sir John Herschel's *Treatise on Astronomy*.

The part of the Work which treats of the motion of one or more rigid bodies has been re-arranged: additional explanations have been given: and the demonstrations regarding the action of impulsive forces are thrown entirely into one Chapter.

In the calculations of the Figure of the Earth a very considerable improvement has been introduced in shewing that $Y_i = 0$ except when $i = 2$. This occupied four pages in the former Edition, but does not occupy one in this: see Art. 532. Mr. O'Brien in his *Mathematical Tracts, Part* I., has given an excellent demonstration, and shorter than Laplace's, but the one now given is shorter and simpler even than his. In Art. 556, we have added a proposition, in which the Earth's figure is determined from the motion of the Moon in latitude.

These are the principal changes in the body of the Work. But an Appendix has also been added con-

taining Problems of rather a difficult kind. In adding this I have had a double object in view; first, to exercise the skill of the higher class of students in filling up the skeleton demonstrations which follow the enunciations; and secondly, to introduce new subjects into the work without adding much to its bulk; this is done by adding hints only, and not full demonstrations. A reference to the following Table of Contents will give an idea of the nature of these Problems.

I have thus done what I could to make the volume still acceptable to my readers: but the distance at which I now reside from England, and the constant press of duties more immediately connected with my profession prevent my taking that personal interest in Mathematical Studies that I used. Nevertheless, though far removed by God's providence from the venerated walls of the University to which I owe so much, I constantly maintain a lively interest in all that concerns my Alma Mater; and shall ever aim at imparting to others, especially in this dark land, whatever lessons of sound learning and religious education I may have imbibed under her fostering care.

J. H. P.

Bishop's Palace, Calcutta,
Oct. 5, 1841.

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INTRODUCTION AND DEFINITIONS.

1. The uniformity which characterises the operations of nature leads us to conjecture that the phenomena of the material world are regulated by certain fixed laws. The never-ceasing alternation of light and darkness, the unvarying succession of the Seasons, the periodical flux and reflux of the Ocean, the constant tendency of bodies downwards, and numberless such like appearances, mutually strengthen the suspicion that they are the necessary consequences of some universal principles with which matter has been endued by the Creator of the World. It is the object of Mechanical Philosophy to search out these Principles.

But at the very outset we are overwhelmed with such a variety of causes, all in simultaneous action, that it becomes no easy task to disentangle the simple laws from the maze in which they are involved.

The whole universe is in perpetual fluctuation, changes are incessantly taking place, and while we are occupying ourselves with the investigation of present appearances, the appearances themselves are in the act of transition from one state to another, and new phenomena press themselves upon our notice.

It is only by a careful and attentive examination of the phenomena perpetually presenting themselves to our view, arranging them in groups, selecting, re-examining and re-arranging, that we are able to rise, by a process of induction and generalization from the mass of facts accumulated by observation, to the laws from which they flow. Having once reached this summit, we descend, making these laws our guides, and follow out, by a deductive process, the phenomena which must naturally result from their operation. A comparison of

the calculated results with the phenomena observed determines, by their agreement or disagreement, whether the laws, to which our investigations have conducted us, are laws of nature or not. It is by a process of this nature that we are convinced of the truth of the Law of Universal Gravitation: and the chief object which we have in view in the present undertaking is to lead the student step by step up to this great Principle, and then shew him the real foundation on which we rest our belief of its truth by displaying its power of explaining accurately every astronomical phenomenon with which we are acquainted.

2. We give the name *Matter* to everything that affects our senses in any manner whatever. *Bodies* are portions of matter limited in every direction, and are consequently of a determinate *form* and *volume*. The *Mass* of a body is the quantity of matter of which it is composed. A *material particle* is a body infinitely small in every dimension.

3. We may consider a body of finite dimensions to be an assemblage of an infinite number of material particles, and its mass to be the sum of all their infinitely small masses.

The mass of a body is said to be *homogeneous* when the same quantity of matter is contained in equal volumes of the body. When this is not the case, the mass is said to be *heterogeneous*.

Bodies of different material have different quantities of matter comprised in the same volume. The term *density* is used to indicate the quantity of matter contained in a given volume of a mass, and serves to measure the quantities of matter in different bodies. The density of a homogeneous mass is measured by the quantity of matter in a unit of a volume: when the mass is heterogeneous, the density at any point is measured by the quantity of matter of the same nature as that at the given point that would occupy a unit of volume.

4. A body is *in motion* when the body or its parts occupy successively different positions in space. But since space is infinite in extent and in every part identical, we cannot judge of the state of rest or motion of a body without comparing it with other bodies: and, for this reason, all mo-

tions which come under our observation are necessarily *relative* motions.

All bodies are capable of motion; but experience shews us that matter will not move spontaneously. Also it is a matter of experiment, as it is indeed of ordinary experience, that when a body is passing from a state of rest to a state of motion, we can always attribute the change to the action of a foreign cause.

5. Any cause which produces or tends to produce motion in a body is termed *Force*.

6. MECHANICS is the Science which treats of the Laws of Rest and Motion of Bodies, whether Solid or Fluid.

We divide this science into four branches.

STATICS, which treats of the laws of the equilibrium of solid bodies.

DYNAMICS, of the laws of motion of solid bodies.

HYDROSTATICS, of the laws of the equilibrium of fluid bodies, and

HYDRODYNAMICS, which treats of the laws of motion of fluid bodies.

7. In Statics force is estimated by the *pressure* it causes a body when at rest to exert against another with which it is in contact and is said to be estimated *statically*. In Dynamics, however, the estimate used is *the space through* which the force causes a body to move in a given time, and the measure is said to be *dynamical*. We shall endeavour to make this more intelligible.

8. Let us begin with the consideration of the ordinary phenomenon of a falling body. Experience teaches us that if a body be let free from the hand, it will fall downwards in a certain determinate direction: however frequently the experiment be made, the result is the same, the body strikes the same spot on the ground in each trial, provided the place from which it is dropped remain the same. Now this undeviating effect must be the result of some cause equally undeviating. The cause is assumed to be an affinity which

all bodies have for the earth, and is termed the force of *Attraction*. It is found to prevail in all parts of the globe.

The direction in which the body falls is called the *vertical line* of the place where the experiment is made: and a plane perpendicular to this is called the *horizontal plane* of the place. If the motion be prevented by interposing the hand, the body exerts a *pressure*, and it requires a muscular effort to keep the body from falling.

In one case then the attraction of the Earth produces a pressure, in the other motion: now of these, *viz.*, the pressure exerted by the body when at rest, and the space through which the body falls when in motion, either may be taken as a means of estimating the intensity of the force of attraction at different places on the Earth, at different elevations above or depressions below its surface.

9. And the same may be said of any force: as another instance let us consider the force exerted by a constrained spring. If the force of the spring be estimated by the pressure it produces on a body holding it in its constrained position, the estimate is said to be statical. But if the force be estimated by the magnitude of the motion generated in a body which it causes to move, the estimate is dynamical.

10. *Weight* is the name given to the pressure which the attraction of the Earth causes a body to exert on another with which it is in contact. Since the gravitation of bodies downwards is unceasing, weight becomes a very useful means of estimating all statical forces. Thus the force of a constrained spring, may be measured by the weight which will just hold the spring in its constrained position. The force of attraction of a magnet may be measured by the weight it will sustain; and so of other forces.

STATICS.

CHAPTER I.

THE COMPOSITION AND EQUILIBRIUM OF FORCES ACTING UPON A MATERIAL PARTICLE.

11. When a single force acts upon a particle, it is clear, from the meaning we attach to the term force, that the particle cannot be at rest.

Experience shews us, however, that two forces may counteract each other's effects in producing motion. In such a case the forces, even though they originate from different causes, are said to be *equal*; since they are measured by their effects. Experience likewise shews us that three or more forces may be in equilibrium with each other, if their directions and magnitudes are properly adjusted. The object of the Science of Statics is to determine the relations which must exist among the forces, in magnitude, direction, and points of application, that they may produce equilibrium when acting on a body.

12. Now any given number of forces acting upon a particle must either be in equilibrium, or else produce an effect on the particle which some single but unknown force would produce. For if the forces be not in equilibrium, the particle will begin to move in some determinate curve line immediately the particle is abandoned to the action of the forces. It is clear, then, that a single force may be found of such a magnitude, that if it act along the tangent at the commencement of the curve, and in a direction opposite to that in which the mo-

tion would take place, this force would prevent the motion, and would consequently be in equilibrium with the other forces which act upon the particle. If, then, we were to remove the original forces, and replace them by a single force, equal in magnitude to that described above, but acting in an opposite direction, the particle would still remain at rest. This force, which is equivalent in its effect to the combined effect of the original forces, is called their *resultant*, and the original forces are called the *components* of the resultant.

13. It will be necessary, then, to begin by deducing rules for the *composition of forces*; that is, for finding their resultant force. After we have determined these, it will be an easy matter to deduce the analytical relations which forces in equilibrium must satisfy, by equating the expression which gives the magnitude of their resultant to zero.

PROP. *To find the resultant of a given number of forces acting upon a particle in the same straight line: and to find the condition that they must satisfy, that they may be in equilibrium.*

14. When two or more forces act on a particle in the same direction, it is evident that the resulting force is equal to their sum, and acts in the same direction.

When two forces act in opposite directions on a particle, it is equally clear that their resultant force is equal to their difference, and acts in the direction of the greater component.

When several forces act in different directions, but in the same straight line, on a particle, the resultant of the forces acting in one direction equals the sum of these forces, and acts in the same direction: and so of the forces acting in the opposite direction. The resultant, therefore, of all the forces equals the difference of these sums, and acts in the direction of the greater.

If the forces acting in one direction are reckoned positive, and those in the opposite direction negative, then their resultant equals their algebraical sum; its sign determining the direction in which it acts.

15. In order that the forces may be in equilibrium, their resultant, and therefore their algebraical sum must equal zero.

Prop. *To find the resultant of two forces acting upon a particle not in the same straight line**.

16. Let P and Q represent the magnitudes of the two forces: A the particle (fig. 1.), AP, AQ the directions in which the forces act: α the angle between these directions. Let R

* The following proof of the Parallelogram of Forces given by M. Duchayla, is well worthy of attention for the simplicity of its demonstration.

1. To find the *direction* of the resultant of two forces acting upon a point.

When the forces are equal, it is clear that the direction of the resultant will *bisect* the angle between the directions of the forces: or, if we represent the forces in magnitude and direction by two lines drawn from the point where they act, the diagonal of the parallelogram described on these lines will be the direction of the resultant.

Let us assume that this is true for two ~~unequal~~ forces, p and m: and also for p and n. We can prove that it must then necessarily be true for two forces, p and $m+n$.

Let A (fig. 2.) be the point on which the forces p and m act, AB, AC their directions and proportional to them in magnitude: complete the parallelogram BC, and draw the diagonal AD: then by hypothesis, the resultant of p and m acts along AD.

Again, take CE in the same ratio to AC that n bears to m. Since it is an experimental fact that the point of application of a force may be transferred to any point of its direction, without disturbing the equilibrium, so long as the two points of application are invariably connected, we may suppose the force n to act at A or C: and therefore the forces p, m and n, in the lines AB, AC, and CE are the same as p and $m+n$ in the lines AB and AE.

Now, replace p and m by their resultant, and transfer its point of application from A to D: then resolve this force at D into two, parallel to AB and AC; these resolved parts must evidently be p and m, p acting in the direction DF, and m in the direction DG. Transfer these two forces, p to C and m to G.

But by the hypothesis, p and n acting at C have a resultant in the direction CG; let them p and n be replaced by their resultant, and transfer its point of application to G. But m acts at G.

Hence by this process we have, without disturbing the equilibrium, removed the forces p and $m+n$ which acted at A to the point G.

Therefore the resultant of p and $m+n$ acts in the direction of the diagonal AG, provided our hypothesis is correct.

But the hypothesis is correct for equal forces, as p, p, and therefore it is true for forces p, $2p$; consequently for p, $3p$, and so it is true for p, $r.p$.

Hence it is true for p, $r.p$ and p, $r.p$, and consequently for $2p$, $r.p$, and so forth; and it is finally true for $s.p$ and $r.p$, r and s being positive integers.

We have still to shew that the Proposition is true for *incommensurable* forces.

Let AB, AC (fig. 3.) represent two such forces. Complete the parallelogram BC. Then if their resultant do not act along AD, suppose it to act along AE; draw EF parallel to BD. Divide AC into a number of equal portions, each less than DE; mark

represent the magnitude of the resultant, and suppose AR is the direction in which it acts, this line being in the same plane as AP, AQ, and lying between them: let θ be the angle between AP and AR. Draw a line P_2AQ_2 in the plane of the forces through the point A, and perpendicular to AR.

Now let us imagine that P is the resultant of two forces P_1 and P_2 acting in the directions AR, AP_2; and that Q is the resultant of two forces Q_1 and Q_2, acting in the directions AR and AQ_2. Then (Art. 14.)

$$\left.\begin{aligned} R &= P_1 + Q_1 \\ \text{and } 0 &= P_2 - Q_2 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots \ (1),$$

P_1 and P_2 are functions of P and θ; and Q_1 and Q_2 are similar functions of Q and $\alpha - \theta$. Since P, P_1, P_2 are merely the numerical ratios which the corresponding forces bear to the unit of force, and since the relation they bear to one another must manifestly be independent of the unit we choose to adopt, the relation between P and P_1 must be of the form

$$\frac{P_1}{P} = \text{function of } \theta = f(\theta) \text{ suppose};$$

$$\text{and } \therefore \frac{P_2}{P} = f(\tfrac{1}{2}\pi - \theta).$$

We have, then, to determine the form of $f(\theta)$.

mark off from CD portions equal to these, and let K be the last division, this evidently falls between D and E; draw GK parallel to AC. Then two forces represented by AC, AG have a resultant in the direction AK, because they are commensurable: and this is nearer to AG than the resultant of the forces represented by AC, AB, which is absurd, since AB is greater than AG.

In the same manner we may shew that every direction besides AD leads to an absurdity, and therefore the resultant must act along AD, whether the forces be commensurable or incommensurable.

2. To find the *magnitude* of the resultant.

Let AB, AC be the directions of the given forces, AD that of their resultant: (fig. 4.) take AE opposite to AD, and of such a length as to represent the *magnitude* of the resultant. Then the forces represented by AB, AC, AE balance each other. Complete the parallelogram BE.

Hence AC is in the same straight line with AF: hence FD is a parallelogram: and therefore $AE = FB = AD$.

Or the resultant is represented in *magnitude* as well as in direction by the diagonal of the parallelogram.

We assume that a force can produce no effect in a direction perpendicular to its own direction.

This principle points out to us two general conditions which P_1 and P_2 must fulfil; for since P can produce no effect in a direction at right angles to its own, it follows that the sum of the resolved parts of P_1 and P_2 in a direction at right angles to that of P must equal zero; and the sum of their resolved parts in the direction of P must equal P.

These conditions lead to the equations

$$P_1 f(\tfrac{1}{2}\pi - \theta) - P_2 f(\theta) = 0,$$
$$P_1 f(\theta) + P_2 f(\tfrac{1}{2}\pi - \theta) = P.$$

Then, by putting for P_1 and P_2 their values $Pf(\theta)$ and $Pf(\tfrac{1}{2}\pi - \theta)$, and dividing by P, we have the first equation identical, and the second gives

$$\{f(\theta)\}^2 + \{f(\tfrac{1}{2}\pi - \theta)\}^2 = 1 \ldots\ldots\ldots (2).$$

This is the equation which $f(\theta)$ is to satisfy; but it admits of an infinite variety of solutions, and we assume (as the result of experiments allows us) that P_1 bears a determinate ratio to P, or $f(\theta)$ has a determinate value, for every value of θ. There must consequently be some other conditions, arising from the nature of the question, which $f(\theta)$ must satisfy; and which are to be our guides in selecting the proper solution of the equation just deduced.

The direct process would be, first to obtain the general solution of the above equation, and then to determine the values of the arbitrary quantities involved in the general solution by the particular values of $f(\theta)$ for particular values of θ given by the nature of our problem. We may, however, reverse the process, and first search for the particular values of $f(\theta)$, and use these as our guides in detecting the proper solution.

Now the principle which has hitherto guided us—viz. that a force produces no effect in a direction at right angles to its own—furnishes us with new conditions, which point out which of the solutions of equation (2) is to be chosen.

For whenever the direction of P_1 is at right angles to the direction of P, and in no other case, $P_1 = 0$ and $P_2 = P$ or $-P$; and whenever the direction of P_2 is at right angles to that of

P, and in no other case, $P_2 = 0$ and $P_1 = P$ or $-P$ as exhibited below;

$$\begin{array}{lllll} \text{when } \theta = 0, & P_2 = 0, & P_1 = P; & \therefore & f(0) = 1, \\ \theta = \frac{1}{2}\pi, & P_1 = 0, & P_2 = P; & \therefore & f(\frac{1}{2}\pi) = 0, \\ \theta = \pi, & P_2 = 0, & P_1 = -P; & \therefore & f(\pi) = -1, \\ \theta = \frac{3}{2}\pi, & P_1 = 0, & P_2 = -P; & \therefore & f(\frac{3}{2}\pi) = 0, \\ \theta = 2\pi, & P_2 = 0, & P_1 = P; & \therefore & f(2\pi) = 1, \\ \ldots\ldots & & & & \end{array}$$

and all these cases are comprised in the formula

$$f(\tfrac{1}{2}n\pi) = \cos(\tfrac{1}{2}n\pi)\ldots\ldots\ldots\ldots(3),$$

n being an integer: and, moreover, this formula embraces no cases, which are not contained in the column of values written above.

These equations (2) and (3) are the only conditions which $f(\theta)$ is to satisfy: and since, as we have observed, $f(\theta)$ must, from the nature of the question, have a determinate form, it follows that there is only one form of $f(\theta)$ which satisfies both equations (2) and (3); consequently if we can find one, this is the solution we are seeking.

Now equation (3) suggests $f(\theta) = \cos\theta$; this fully satisfies both (2) and (3), and is consequently the required solution.

Hence equations (1) become

$$R = P\cos\theta + Q\cos(a - \theta)$$
$$0 = P\sin\theta - Q\sin(a - \theta)\ldots\ldots(4);$$

adding the squares of these,

$$R^2 = P^2 + Q^2 + 2PQ\cos a\ldots\ldots\ldots(5).$$

Equation (4) determines the *direction* of the resultant, and (5) its *magnitude*.

17. These equations point out the following geometrical construction, (fig. 1).

Take AB, AC in the ratio of P to Q, through B draw BD parallel to AC and cutting AR in D: join CD. Then by Trigonometry,

$$BD = AB \frac{\sin\theta}{\sin(\alpha - \theta)} = AB \frac{Q}{P} \text{ by (4)} = AC$$

by the construction.

Hence BC is a parallelogram; and its diagonal is the *direction* in which the resultant of P and Q acts.

Again, by Trigonometry,

$$AD^2 = AB^2 + AC^2 + 2AB\,AC\cos\alpha.$$

Comparing this with equation (5) we see, that the diagonal represents the *magnitude* of the resultant on the same scale that the sides of the parallelogram represent the forces P and Q.

This Proposition is, in consequence of the property just proved, called The Proposition of the *Parallelogram of Forces*.

18. Cor. 1. Any force acting on a particle may be replaced by two others, if the sides of a triangle, drawn parallel to the directions of the forces, have the same relative proportion that the forces have.

This is called the *resolution* of a force.

19. Cor. 2. When three forces acting on a particle are in equilibrium, they are respectively in the same proportion as the sines of the angles included by the directions of the other two.

For if we refer to fig. 4, we have

$$\begin{aligned} P : Q : R &:: AB : AC \text{ (or } BD) : AD \\ &:: \sin ADB : \sin BAD : \sin ABD \\ &:: \sin CAE : \sin BAE : \sin BAC. \end{aligned}$$

Prop. *Three forces act upon a particle in directions making right angles with each other: required to find the magnitude and direction of their resultant.*

20. Let AB, AC, AD represent the three forces X, Y, Z in magnitude and direction: fig. 5.

Complete the parallelogram BC, and draw AE: then AE represents the resultant of X and Y in magnitude and direction, by Art. 17. Now the resultant of this force and Z, which are represented by AE, AD, is represented in magnitude and direction by AF, the diagonal of the parallelogram DE. Hence the resultant of XYZ is represented in magnitude and direction by AF. Let R be the magnitude of the resultant, and abc the angles the direction of R makes with those of XYZ.

Then, since $AF^2 = AE^2 + AD^2 = AB^2 + AC^2 + AD^2$;

$$\therefore R^2 = X^2 + Y^2 + Z^2.$$

$$\text{Also, } \cos a = \frac{AB}{AF} = \frac{X}{R}, \; \cos b = \frac{AC}{AF} = \frac{Y}{R}, \; \cos c = \frac{AD}{AF} = \frac{Z}{R}.$$

Whence the magnitude and direction of the resultant are determined.

21. Cor. Any force R, the direction of which makes the angles abc with three rectangular axes fixed in space, may be replaced by the three forces $R \cos a$, $R \cos b$, $R \cos c$, acting simultaneously on the particle on which R acts, and having their directions parallel to the axes of co-ordinates respectively.

Prop. *Any number of forces act upon a particle in any directions: required to find the magnitude and direction of their resultant.*

22. Let $PP_{,}\ldots\ldots$ be the forces, and $\alpha\beta\gamma$, $\alpha_{,}\beta_{,}\gamma_{,}$, $\ldots\ldots$ the angles their directions make with three rectangular axes drawn through the proposed point.

Then the component parts of P in the directions of the axes are, by Art. 21,

$$P \cos \alpha, \quad P \cos \beta, \quad P \cos \gamma, \quad (\text{or } X, Y, Z, \text{ suppose}).$$

Resolving each of the other forces in the same way, we reduce the system to three forces, by adding those which act in the same lines (Art. 14.), we thus have

$$P\cos a + P_{,}\cos a_{,} + \ldots\ldots\ldots \text{ or } \Sigma . P\cos a \text{ or } \Sigma . X,$$

$$P\cos\beta + P_{,}\cos\beta_{,} + \ldots\ldots\ldots \text{ or } \Sigma . P\cos\beta \text{ or } \Sigma . Y,$$

$$\text{and } P\cos\gamma + P_{,}\cos\gamma_{,} + \ldots\ldots\ldots \text{ or } \Sigma . P\cos\gamma \text{ or } \Sigma . Z,$$

acting in the directions of the axes of x, y and z.

The symbol Σ indicates that we are to take the sum of all the quantities in the system which are symmetrical with that before which it is placed.

If we call the resultant R, and the angles which the direction of R makes with the axes a, b, c, we have, by Art. 20,

$$R^2 = (\Sigma . X)^2 + (\Sigma . Y)^2 + (\Sigma . Z)^2,$$

$$\text{and } \cos a = \frac{\Sigma . X}{R}, \quad \cos b = \frac{\Sigma . Y}{R}, \quad \cos c = \frac{\Sigma . Z}{R},$$

to determine the resultant.

Cor. If the forces all act in the same plane, and xy be taken for that plane, then $\gamma = 90^0$, $\beta = 90^0 - a$: and we have the two equations $R\cos a = \Sigma . X$, and $R\sin a = \Sigma . Y$.

Prop. *To find the conditions of equilibrium when any number of forces act upon a material particle.*

23. When the forces are in equilibrium, we must have $R = 0$;

$$\therefore\ (\Sigma . X)^2 + (\Sigma . Y)^2 + (\Sigma . Z)^2 = 0;$$

$$\therefore\ \Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

and these are the conditions among the forces, that they may be in equilibrium.

These conditions may be expressed under another form: and lead to a principle denominated the Principle of Virtual Velocities.

VIRTUAL VELOCITIES.

Prop. *To prove the Principle of Virtual Velocities when forces acting on a particle are in equilibrium.*

24. Let xyz be the co-ordinates to the point of application of the forces, and $x + \delta x$, $y + \delta y$, $z + \delta z$ the co-ordinates

to a point near the former. Draw perpendiculars from this latter point upon the directions of the forces P, $P_{,}$... and let δp, $\delta p_{,}$... be the distances of these perpendiculars from the point xyz: hence

$$\delta p = \delta x . \cos \alpha + \delta y . \cos \beta + \delta z . \cos \gamma$$

$$\delta p_{,} = \delta x . \cos \alpha_{,} + \delta y . \cos \beta_{,} + \delta z . \cos \gamma_{,}$$

. . . .

If then we multiply the equations

$$\Sigma . P \cos \alpha = 0, \quad \Sigma . P \cos \beta = 0, \quad \Sigma . P \cos \gamma = 0$$

by δx, δy, δz, and add the equations, bearing in mind that δx, δy, δz are independent of the forces, and may therefore be written inside the symbol Σ, we have

$$\Sigma . P (\delta x \cos \alpha + \delta y \cos \beta + \delta z \cos \gamma) = 0,$$

$$\text{or } \Sigma . P \delta p = 0,$$

which proves the following principle:—That if any number of forces acting upon a particle be in equilibrium, and the point of application be moved geometrically through any small space, then the sum of the products of the forces and the spaces described by the point of application relatively to the directions of the forces will vanish; these spaces being reckoned positive when drawn in the direction in which the force acts, and vice versâ.

This is termed the *Principle of Virtual Velocities*, since the spaces above mentioned measure the relative velocity of the geometric motion in the direction of the forces. We shall see in the next Chapter that this Principle is true for any system of forces.

CHAPTER II.

THE COMPOSITION AND EQUILIBRIUM OF FORCES ACTING ON A RIGID BODY.

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25. A SOLID or fluid body is conceived to be an aggregation of indefinitely small material particles or molecules, which are held together by their mutual affinities. This appears to be a safe hypothesis, since experiments shew that any body is divisible into successively smaller and smaller portions without limit, if sufficient force be exerted to overcome the mutual action of the parts of the body.

26. By the term *rigid* we mean to express that the molecules of the body are held together in an invariable form; so that the intensity of the molecular forces is infinitely greater than that of the other forces which act upon the body. Were this not the case, the figure of the body would depend upon the forces which act upon it.

NOW, in matter of fact, no body is perfectly rigid; every body yields more or less to the forces by which it is acted on. If, then, in any case this compressibility is of a sensible magnitude, we shall suppose that the body has assumed its figure of equilibrium, and then consider the points of application of the forces as a system of invariable form.

27. We are quite unacquainted at present with the laws according to which the molecules of a mass of matter act upon each other. In consequence of this, we must look for some principle which will enable us to calculate the effect of forces acting upon a rigid body, without bringing the molecular forces into the calculation.

And now we fall upon a case of the action of force totally different from anything we have yet met with. In considering its action on a single particle, the force was supposed to act

on the whole of the particle: but now we have to consider the effect of forces acting on individual particles of an assemblage held rigidly together by their mutual affinities. The force which acts upon any particle of the body must in some way have its effect propagated through the whole system of particles, in consequence of their invariable connexion. Sundry experiments have led philosophers to the following principle; which, as will be seen, exactly answers our purpose.

28. *When a force, acting in combination with others, holds a solid body in equilibrium, the equilibrium of the body will not be disturbed if we transfer the point of application of the force to any other point whatever in the line in which the force is acting.*

We shall now commence with the simplest case of a rigid body acted on by forces, and so ascend to the most general.

PARALLEL FORCES.

PROP. *To find the magnitude and direction of the resultant of two parallel forces acting in the same plane on a rigid body.*

29. Let P and Q be the forces; A, B (fig. 7.) their points of application: let P and Q act in the same direction, making angles α with AB. The state of equilibrium of the body will not be altered if we apply two equal and opposite forces, each equal to S, at the points A, B, acting in the line AB.

Then P and S acting at A, are equivalent to some force P' acting in some direction AP' (Art. 16.); and Q and S acting at B, are equivalent to some force Q' acting in some direction BQ' inclined to AP.

Produce $P'A$, $Q'B$ to cut each other in C, and draw CD parallel to AP and BQ, and cutting AB in D.

Transfer P' and Q' to C, (Art. 28.) C being rigidly connected with AB, and resolve them along CD and parallel to AB; the latter parts will be S and S acting in opposite directions, and the sum of the former is $Q + P$.

Hence R, the resultant of P and Q, $= Q + P$, and acts parallel to P and Q in the line CD. We shall now determine the point where this line cuts AB.

Since the sides of the triangle ACD are parallel to the directions of the forces P, S, P'; (see Art. 18.)

$$\therefore \frac{P}{S} = \frac{CD}{DA}, \quad \text{and similarly, } \frac{S}{Q} = \frac{DB}{CD};$$

$$\therefore \frac{P}{Q} = \frac{DB}{DA} = \frac{a - x}{x}; \quad \text{if } AB = a \text{ and } AD = x:$$

$$\therefore \frac{x}{a} = \frac{Q}{Q + P},$$

this determines the point D, through which the direction of the resultant passes.

30. If the force P act in a direction opposite to that of Q, (fig. 8.) a similar process will lead us to

$$R = Q - P, \text{ and } \frac{x}{a} = \frac{Q}{Q - P};$$

but these are included in the formulæ of last article by putting $-P$ for P.

31. The point D possesses this remarkable property; that however P and Q are turned about their points of application A and B, their directions remaining parallel, D, determined as above, remains the same. This point is, in consequence, called the *centre* of the parallel forces P and Q.

Prop. *To find the magnitude and direction of the resultant of any number of parallel forces acting upon a rigid body, and to determine the* centre *of parallel forces.*

32. Let the points of application of the forces be referred to a system of rectangular co-ordinate axes (fig. 9.) $m_1 m_2 \ldots\ldots$ the points of application: $x_1 y_1 z_1$, $x_2 y_2 z_2$, ...their co-ordinates, $P_1 P_2 \ldots\ldots$ the forces acting at these points, those being reckoned positive which act in the direction of P_1 and those negative

which act in the opposite direction. Join $m_1 m_2$: and take the point n_1 on $m_1 m_2$ such that

$$m_1 n_1 = \frac{P_2}{P_1 + P_2} \cdot m_1 m_2,$$

then the resultant of P_1 and P_2 is $P_1 + P_2$, and it acts through n_1 parallel to P_1 and P_2: we shall now find the co-ordinates to n_1: Arts. ~~30, 31~~. 29 30

Draw $m_1\,a'$, $n_1\,b'$, $m_2\,c'$ perpendicular to the plane of xy and meeting that plane in $a'\,b'\,c'$: also draw $a'a$, $b'b$, $c'c$ perpendicular to the axis of x; and $m_1 de$ parallel to $a'b'c'$ cutting $n_1 b'$, $m_2 c'$ in d and e. Then, by similar triangles,

$$\frac{m_1 n_1}{m_1 m_2} = \frac{m_1 d}{m_1 e} = \frac{a'b'}{a'c'} = \frac{ab}{ac} = \frac{Ab - x_1}{x_2 - x_1};$$

$$\therefore\ Ab - x_1 = \frac{P_2}{P_1 + P_2}\,(x_2 - x_1);$$

$$\therefore\ Ab, \text{ the abscissa to } n_1, = \frac{P_1 x_1 + P_2 x_2}{P_1 + P_2}.$$

Then, supposing P_1 and P_2 to be replaced by $(P_1 + P_2)$ acting at n_1, the abscissa to the point of application of the resultant of $(P_1 + P_2)$, and P_3

$$= \frac{(P_1 + P_2)\,.\,Ab + P_3\,.\,x_3}{(P_1 + P_2) + P_3} = \frac{P_1 x_1 + P_2\,.\,x_2 + P_3\,.\,x_3}{P_1 + P_2 + P_3}:$$

and the same will evidently be true for any number of parallel forces: and for each of the axes. Let R be the resultant of all the forces, and $\bar{x}$, $\bar{y}$, $\bar{z}$ the co-ordinates to its point of application, determined as above;

$$\therefore\ R = \Sigma\,.\,P_1; \quad \text{and } \bar{x} = \frac{\Sigma\,.\,P_1 x}{\Sigma\,.\,P_1},\ \bar{y} = \frac{\Sigma\,.\,P_1 y}{\Sigma\,.\,P_1},\ \bar{z} = \frac{\Sigma\,.\,P_1 z}{\Sigma\,.\,P_1}.$$

These determine the magnitude of the resultant; and a point in the direction of its action: this point being known the line in which the resultant acts is known also, because its direction is parallel to that of the forces P_1, P_2......

33. These co-ordinates are independent of the angle, which the directions of the forces make with the axes. Hence if these directions be turned about the points of application of the forces, and their parallelism be preserved, the point of application of the resultant will not move. For this reason that point is called the *centre of the parallel forces.*

34. A heavy body consists of an aggregation of material particles, each of which, in consequence of the Earth's attraction, tends towards the Earth's centre.

The weight, then, of a body may be considered as the resultant of the weights of the different elementary portions of the body acting in parallel and vertical lines.

In this case the centre of parallel forces is termed *the centre of gravity* of the body. The obvious property of this point is, that if it be fixed, the body will rest in any position; no forces but the body's weight being supposed to act.

35. The expression $P.x$ is denominated *the moment of the force P with respect to the plane yz.*

In consequence of the above definition, the equations for determining the position of the centre of parallel forces shew that *the sum of the moments of any number of parallel forces with respect to any plane equals the moment of their resultant.*

36. If $P = Q$ in Art. 30., then $R = 0$ and $x = \infty$, a result perfectly nugatory. It shews us, that two equal and opposite parallel forces do not admit of a resultant. In fact the addition of the forces S, S still gives, in this case, two equal forces parallel and opposite in their directions.

Such a system of forces is called a *Couple.*

We shall investigate the laws of the composition and resolution of couples; since to these we shall reduce the composition and resolution of forces of every description acting upon a rigid body.

37. Definitions. The *arm* of a couple is the perpendicular distance between the directions of its forces.

The product of a force and the distance of its direction from a given point is called *the moment of the force with*

respect to the point. If through the point an *axis* be drawn at right angles to the plane, passing through the point and the direction of the force, this product is called *the moment of the force with respect to the axis.*

And hence the *moment* of a couple is the product of the force at either extremity and the arm.

The *axis* of a couple is a straight line perpendicular to the plane of the couple and proportional in length to the moment.

PROP. *The effect of a couple upon the equilibrium of a body is not altered, if its arm be turned through any angle about one extremity in the plane of the couple.*

38. Let the plane of the paper be the plane of the couple (fig. 10.) and AB the arm: AB' its new position: the forces P_1 P_2 are equal, and act on the arm AB.

At A and B' let the two pair of equal and opposite forces P_3P_5, P_4P_6, each $= P_1$ or P_2 be applied, acting perpendicular to AB': this will not affect the equilibrium.

Let BP_2, $B'P_3$ cut in C: join AC: AC manifestly bisects the angle BAB'.

Now P_2 and P_3 are equivalent to some force in direction CA,

P_1 and P_4 .... same force .... AC;

$\therefore$ $P_1P_2P_3P_4$ are in equilibrium with each other;

therefore the remaining forces P_5, P_6 acting at B' A produce the same effect as P_1 and P_2 acting on AB. Hence the proposition is true.

PROP. *The effect of a couple on the equilibrium of a body is not altered if we transfer the couple to any plane parallel to its own, the arm remaining parallel to itself.*

39. See fig. 11. AB the arm: $A'B'$ the new position parallel to AB. Join AB', $A'B$ bisecting each other in G.

At A' B' apply two equal and opposite forces each $= P_1$ or P_2: and let these forces be $P_3P_4P_5P_6$: this will not alter the effect of the couple.

But P_1 and P_4 are equivalent to $2P_1$ acting at G in direction Ga,

and P_2 and P_3 .... Gb.

Hence $P_1 P_2 P_3 P_4$ are in equilibrium with each other, and may be removed; therefore the remaining forces P_5 P_6, acting at A' and B' produce the same effect as P_1 and P_2 acting on AB. Hence the proposition is true.

40. Cor. Combining these two propositions, we see that a couple may be any how transferred so long as its plane remains parallel to itself.

Prop. *The effect of a couple on a body at rest will not be altered if we replace it by another of which the moment is the same: the plane remaining the same, and the arms being in the same line, and having a common extremity.*

41. Let AB be the arm, (fig. 12.): P, P the forces: and suppose $P = Q + R$: let $AB = a$: and make AC, a new arm, $= b$: at C apply two equal and opposite forces Q_1 Q_2 each $= Q$: this will not alter the effect of the couple.

Now R at A and Q_1 at C will balance $Q + R$ or P at B,

$$\text{if } AB : BC :: Q_2 : R \text{ (Art. 29.)},$$

$$\text{or if } AB : AC :: Q_2 : Q_2 + R = P,$$

$$\text{or if } Q \cdot b = P \cdot a,$$

we then have remaining the couple Q_1 Q_2 acting on the arm AC.

Hence the couple P, P acting on AB, may be replaced by the couple Q_1 Q acting on AC, if $Q \cdot b = P \cdot a$; that is, if their moments are the same.

Prop. *To find the resultant of any number of couples acting upon a body, the planes of the couples being parallel to each other.*

42. First suppose the couples all transferred to the same plane (Art. 39.): next let them all be transferred so as to have their arms in the same straight line, and one extremity common (Art. 38.): and lastly let them all be replaced by others having the same arm (Art. 41.).

Thus if P, Q, R, S, ... be the forces, and

a, b, c, d, ... be their arms,

we shall have replaced them by the following forces, (supposing a the length of the common arm)

$$P \cdot \frac{a}{a}, \quad Q \cdot \frac{b}{a}, \quad R \cdot \frac{c}{a}, \ldots\ldots\ldots\ldots \text{ acting on the arm } a.$$

Hence their resultant will be a couple of which the force

$$= P \cdot \frac{a}{a} + Q \cdot \frac{b}{a} + R \cdot \frac{c}{a} + \ldots\ldots\ldots\ldots \text{and arm} = a,$$

or of which the moment $= P \cdot a + Q \cdot b + R \cdot c + \ldots\ldots$

Hence the moment of the resultant couple is equal to the sum of the moments of the original couples.

If one of the couples, as (S, S), act in a direction opposite to the couple (P, P), then the force at each extremity of the arm of the resultant couple will be

$$P \cdot \frac{a}{a} + Q \cdot \frac{b}{a} + R \cdot \frac{c}{a} - S \cdot \frac{d}{a} + \ldots\ldots\ldots$$

and the moment of the resultant couple will be

$$P \cdot a + Q \cdot b + R \cdot c - S \cdot d + \ldots\ldots\ldots\ldots$$

or the algebraical sum of the moments of the original couples; the moments of those couples which tend in the direction opposite to the couple (P, P) being reckoned negative.

PROP. *To find the resultant of two couples not acting in the same plane.*

43. Let the planes of the couples intersect in the line AB, which is perpendicular to the plane of the paper (fig. 13.), and let the couples be referred to the common arm AB, and let their forces, thus altered, be P and Q.

In the plane of the paper draw Aa, Ab perpendicular to the planes of the couples (P, P) and (Q, Q): and equal in length to their *axes*, (Art. 37).

Let R be the resultant of the forces P, Q at A, acting in the direction AR; and of P, Q at B, in the direction BR.

Since AP, AQ are parallel to BP, BQ respectively, therefore AR is parallel to BR.

Hence the two couples are equivalent to the single couple (R, R) acting on the arm AB.

Draw Ac perpendicular to the plane of (R, R), and in the same proportion to Aa, Ab that the moment of the couple (R, R) has to those of (P, P), (Q, Q).

Then Ac is the *axis* of (R, R).

Now the three lines Aa, Ac, Ab make the same angles with each other that AP, AR, AQ make with each other; also they are in the same proportion in which

$AB \cdot P$, $AB \cdot R$, $AB \cdot Q$ are,

or in which P, R, Q are.

But R is the resultant of P and Q;

therefore Ac is the diagonal of the parallelogram on Aa, Ab (see Art. 17).

Hence if two straight lines, having a common extremity, represent the axes of two couples, that diagonal of the parallelogram described on these lines, which passes through their common extremity is equal in magnitude and direction to the axis of the resultant couple.

PROP. *To find the magnitude and position of the couple which is the resultant of three couples which act in planes at right angles to each other.*

44. Let AB, AC, AD be the axes of the given couples, (fig. 5). Complete the parallelogram CB: and draw AE the diagonal. Then AE is the axis of the couple which is the resultant of the two couples of which the axes are AB, AC.

Complete the parallelogram DE, and draw AF the diagonal. Then AF is the axis of the couple which is the resultant of the couples of which the axes are AE, AD, or of those of which the axes are AB, AC, AD.

$$\text{Now } AF^2 = AE^2 + AD^2 = AB^2 + AC^2 + AD^2.$$

Let G be the moment of the resultant couple, L, M, N those of the given couples;

$$\therefore\ G^2 = L^2 + M^2 + N^2;$$

and if λ, μ, ν be the angles the axis of the resultant makes with those of the components

$$\cos\lambda = \frac{AB}{AF} = \frac{L}{G}; \quad \cos\mu = \frac{M}{G}; \quad \cos\nu = \frac{N}{G}.$$

45. Cor. Hence conversely any couple may be replaced by three couples acting in planes at right angles to each other, their moments being

$$G\cos\lambda, \quad G\cos\mu, \quad G\cos\nu,$$

where G is the moment of the given couple, and λ, μ, ν the angles its axis makes with the axes of the three couples.

Prop. *To find the resultant of any number of forces acting on a rigid body in the same plane.*

46. Let the system be referred to any pair of rectangular co-ordinate axes Ax, Ay in the given plane; (fig. 14).

Let P, P_1, P_2, ... be the forces,

a, a_1, a_2, ... the angles which their directions make with the axis of x.

xy, x_1y_1, x_2y_2, ... the co-ordinates to their points of application.

Let B be the point of application of P: join BA: the points B and A are rigidly connected. At A apply two equal and opposite forces, each equal and parallel to P. This will not affect the equilibrium. Draw Ap perpendicular to PB produced if necessary.

Hence P acting at B is replaced by P acting at A, together with a couple (P, P) acting on the arm Ap, or a couple of which the moment $= P\,.\,Ap$, and tending to turn the body from the axis of x to the axis of y.

$$\text{Now } Ap = x\sin a - y\cos a.$$

Hence the moment of the couple (P, P)

$$= P\,.\,(x\sin a - y\cos a).$$

The moments of those couples are reckoned positive that tend to turn the body from the axis of x to the axis of y: and those negative that tend the other way.

The other forces may be similarly replaced.

Hence our system is reduced to the forces

$$P,\ P_1,\ P_2 \ldots\ldots\ldots \text{ acting at } A$$

in directions parallel to those of the original forces; and the couples of which the moments are

$$P\ \{x\ \sin\alpha\ - y\ \cos\alpha\},$$
$$P_1\ \{x_1 \sin\alpha_1 - y_1 \cos\alpha_1\},$$
$$P_2\ \{x_2 \sin\alpha_2 - y_2 \cos\alpha_2\},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

acting in the plane of the paper.

Let R be the resultant of the forces acting at A, angle which R makes with the axis of x; G the moment resultant couple: then, by Art. 22,

$$R\cos a = \Sigma \,.\, P\cos\alpha,\ \ R\sin a = \Sigma \,.\, P\sin\alpha,$$

and, by Art. 42, $G = \Sigma \,.\, P\,(x\sin\alpha - y\cos\alpha)$,

and if $P\cos\alpha = X$, and $P\sin\alpha = Y$, these may be writt

$$R^2 = (\Sigma \,.\, X)^2 + (\Sigma \,.\, Y)^2,\ \ \tan a = \frac{\Sigma \,.\, Y}{\Sigma \,.\, X},$$

$$\text{and } G = \Sigma \,.\, \{Y \,.\, x - X \,.\, y\}.$$

47. Let the arm of the resultant couple be turned plane of the forces and about its extremity A, till it is p dicular to the direction of R. Art. 38: (fig 15).

Let AR be the direction of R: $AB = a$, the arm resultant couple; and, consequently, $G \div a$ the force at extremity; let this $= R'$.

Hence the forces are all reduced to a force $R + R'$ at A in the direction AR, and R' acting at B in the dir

BR', parallel to AR. The resultant of these is R, acting at a point C in the direction CR parallel to AR, the distance AC being $= \frac{R'}{R} AB$ (by Art. 29.) $= \frac{G}{R}$.

Wherefore the resultant of all the forces P, P_1,...... is a force R acting in the straight line of which the equation is

$$y + AC \cos a = \tan a \, (x - AC \sin a),$$

which simplified becomes

$$x \tan a - y = AC \sec. a,$$

$$\text{or} \quad x \sin a - y \cos a = AC,$$

$$\text{or} \quad x . \Sigma . Y - y . \Sigma . X = G,$$

the *direction* in which R acts will be determined by the sign of $\tan a$.

48. Cor. If it should happen that the forces are such that $R = 0$, then we are left with the couple of which the moment is G, and there is not a single resultant force.

Prop. *To find the conditions of equilibrium of any number of forces acting on a rigid body in the same plane.*

49. We have shewn in the last Prop. that the resultant of any number of forces acting in the same plane on a rigid body equals a force R acting about the origin of co-ordinates, at a distance $= G \div R$, where

$$R^2 = (\Sigma . X)^2 + (\Sigma . Y)^2, \quad \text{and } G = \Sigma . \{Yx - Xy\}.$$

Now when the forces are in equilibrium their resultant must vanish, therefore $R = 0$; also $G = 0$, since the distance $G \div R$ must be indeterminate and not infinite; for if it were infinite, then the resultant of the forces would be a couple: see Art. 36.

Hence the conditions of equilibrium are

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \text{and} \quad \Sigma . (Yx - Xy) = 0.$$

These may be written

$$\Sigma . P\cos\alpha = 0, \quad \Sigma . P\sin\alpha = 0,$$

$$\text{and } \Sigma . P(x\sin\alpha - y\cos\alpha) = 0.$$

50. Since $Ap = x\sin\alpha - y\cos\alpha$ (see Art. 46. and fig. 15) these equations shew, that, when the forces are in equilibrium, the sums of their resolved parts parallel to any two straight lines at right angles to each other, and lying in the plane of the forces, must equal zero: and also the sum of the moments of the forces with respect to any point in the plane (Art. 37.) must equal zero.

Prop. *To find the two resultants of any number of forces acting upon a rigid body in any directions.*

51. Let the forces be referred to three rectangular axes Ax, Ay, Az; and suppose PP_1P_2... are the forces, xyz, $x_1y_1z_1$, $x_2y_2z_2$...the co-ordinates to their points of application, and $\alpha\beta\gamma$, $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$...the angles their directions make with the axes: fig. 16.

Let m be the point of application of P, mP its direction $Ar = x$, $rn = y$, $nm = z$: An in the plane xy, also nz parallel to Ax.

Now P may be replaced by its three components

$$P\cos\alpha, \; P\cos\beta, \; P\cos\gamma, \text{ (or } X, Y, Z \text{ suppose)}$$

parallel to the axes, (Art. 21.)

Z produces the same effect if it be transferred to n. Now the equilibrium of the body will not be disturbed if we apply at A and also at r two opposite forces, each equal and parallel to Z. Then Z at m is equivalent to Z at A, and the two couples of which the moments are $Z . rn$ and $Z . Ar$ and the axes coincide respectively with the co-ordinate axes of x and y.

Hence Z at m is replaced by Z at A, and the two couples $Z . y$ and $-Z . x$ acting in the planes perpendicular to x and y respectively: the moments of those couples which tend to turn the body from the axis of x to that of y about the axis of z,

from y to z about x, and from z to x about y, are reckoned positive, and those in the opposite direction negative.

In the same manner we may substitute for Y and X.

Wherefore the force P acting at m may be replaced by X, Y, Z acting at A along the axes, together with the couples

$Z \,.\, y$ and $-\ Y \,.\, z$ in plane perpendicular to axis of x

$X \,.\, z$ and $-\ Z \,.\, x$ .... y

$Y \,.\, x$ and $-\ X \,.\, y$ .... z

or, by adding the moments of the couples acting in the same or parallel planes (Art. 42.)

P is replaced by X, Y, Z acting at A and the couples of which the moments are

$Z \,.\, y - Y \,.\, z$ in plane perpendicular to axis of x

$X \,.\, z - Z \,.\, x$ .... y

$Y \,.\, x - X \,.\, y$ .... z.

By a similar resolution of all the forces, we shall have them replaced by the forces

$$\Sigma \,.\, X, \quad \Sigma \,.\, Y, \quad \Sigma \,.\, Z,$$

acting at A along the axes: and the couples

$\Sigma \,.\, \{Z \,.\, y - Y \,.\, z\} = L$ acting in the plane perpendicular to axis of x

$\Sigma \,.\, \{X \,.\, z - Z \,.\, x\} = M$ .... y

$\Sigma \,.\, \{Y \,.\, x - X \,.\, y\} = N$ .... z.

Let R be the resultant of the forces acting at A; a, b, c the angles its direction makes with the axis of co-ordinates: then (Art. 22.)

$$R^2 = \{\Sigma \,.\, X\}^2 + \{\Sigma \,.\, Y\}^2 + \{\Sigma \,.\, Z\}^2$$

$$\cos a = \frac{\Sigma \,.\, X}{R}, \quad \cos b = \frac{\Sigma \,.\, Y}{R}, \quad \cos c = \frac{\Sigma \,.\, Z}{R}.$$

Let G be the moment of the couple which is the resultant of the three couples above mentioned; λ, μ, ν the angles its

axis makes with the axes of co-ordinates; then (Art. 44.)

$$G^2 = L^2 + M^2 + N^2,$$

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

52. We may still further reduce the forces in the following manner.

Let the plane of the couple be turned round its axis till the projection of the direction of R on this plane is perpendicular to the arm. Let a be the length of the arm chosen arbitrarily. Then $G \div a$ is the force at each extremity. Also let θ be the angle between the direction of R and the axis of G. Hence the whole force at A is $=$

$$\sqrt{R^2 + \frac{G^2}{a^2} + \frac{2RG}{a}\sin\theta}, \text{ (Art. 17.)}$$

The couple has been thus turned round, because in this particular position the angle θ is given by the following simple formula,

$$\cos\theta = \cos a \cos\lambda + \cos b \cos\mu + \cos c \cos\nu.$$

The second force is $G \div a$, acting at the other extremity of the arm.

These two forces cannot in general be reduced to a single force, since their directions do not meet. When their directions do meet, then the forces can be reduced to one.

Prop. *Required to find the condition among the forces that they may have a single resultant.*

53. In order that this may be the case, it is clear that the force R must be in the plane of the couple G. For then the force resulting from the composition of R with one of the forces of the couple will, when produced, meet the other force of the couple, and, being compounded, will thus produce a single resultant. Now this condition is satisfied when the angle between R and the axis of G equals 90^0: or when the cosine of this angle equals zero: that is, when

$$\cos a \cos\lambda + \cos b \cos\mu + \cos c \cos\nu = 0;$$

therefore the condition is that

$$\frac{(\Sigma . X) L + (\Sigma . Y) M + (\Sigma . Z) N}{R . G} = 0,$$

$$\text{or } (\Sigma . X) L + (\Sigma . Y) M + (\Sigma . Z) N = 0,$$

unless R or G vanishes.

This is no condition when $R = 0$: that is, when $\Sigma . X = 0$, $\Sigma . Y = 0$, $\Sigma . Z = 0$, for the above equation is then identical.

In fact we then have only the couple G: which does not admit of a single resultant.

Also this is no condition when $G = 0$, for then $L = 0$, $M = 0$, $N = 0$, and the equation is again identical.

But in this case it is evident we have a single resultant R.

Prop. *When the forces are reducible to a single resultant, required the magnitude of this force and the equations to the line in which it acts.*

54. In this case the force R is in the plane of the couple of which the moment is G.

Let the arm of the couple be turned about its extremity A (see fig. 15), and in the plane of the couple, till it is perpendicular to the force R: and let $AB = a$ be the arm of the couple: then the force of the couple $(R') = G \div a$; and the single resultant equals R acting at C in the direction CR parallel to AR, C being in BA produced and determined by the equation

$$AC = \frac{R'}{R} AB = \frac{G}{R}.$$

55. We must now find the equations to the line in which this resultant acts.

Let $x_1 y_1 z_1$ be the co-ordinates to some point in this line; then, transferring the origin to this point, it is clear that the body must have no tendency to revolve about the origin.

Therefore the new values of $L\,M\,N$ when we put $x_1 + x$, $y_1 + y$, $z_1 + z$ for $x\,y\,z$ must $= 0$;

$$\therefore\ 0 = \Sigma . P\left\{(y_1 + y)\cos\gamma - (z_1 + z)\cos\beta\right\},$$

$$\text{or } 0 = L + y_1\Sigma . Z - z_1\Sigma . y \ \ldots\ldots\ldots\ldots\ldots\ (1).$$

Similarly,

$$0 = M + z_1\Sigma . X - x_1\Sigma . Z \ \ldots\ldots\ldots\ldots\ldots\ (2),$$

$$0 = N + x_1\Sigma . Y - y_1\Sigma . X \ \ldots\ldots\ldots\ldots\ldots\ (3).$$

These three equations are equivalent to only two: for if we eliminate z_1 from (1) and (2), we have

$$0 = L\Sigma . X + M\Sigma . Y - x_1\Sigma . Z . \Sigma . Y + y_1\Sigma . Z . \Sigma . X.$$

But $L\Sigma . X + M\Sigma . Y + N\Sigma . Z = 0$, by Art. 53;

$$\therefore\ 0 = N + x_1\Sigma . Y - y_1\Sigma . X;$$

and therefore equation (3) is a necessary consequence of (1) and (2): wherefore any two of equations (1), (2), (3) are the equations to the line in which the single resultant acts.

PROP. *To find the conditions of equilibrium of any number of forces acting upon a rigid body in any directions.*

56. We have shewn that the forces are in the general case reducible to two acting in different planes. These forces, then, must each vanish when there is equilibrium.

Hence (Art. 52.)

$$R^2 + \left(\frac{G}{a}\right)^2 + \frac{2RG}{a}\sin\theta = 0, \text{ and } \frac{G}{a} = 0,$$

$\therefore\ R^2 = 0$, and $G^2 = 0$, a being arbitrary; or

$$(\Sigma . X)^2 + (\Sigma . Y)^2 + (\Sigma . Z)^2 = 0 \text{ and } L^2 + M^2 + N^2 = 0,$$

and these lead to the six conditions

$$\Sigma . X = 0,\quad \Sigma . Y = 0,\quad \Sigma . Z = 0,$$

$$\Sigma . (Zy - Yz) = 0,\ \Sigma . (Xz - Zx) = 0,\ \Sigma . (Yx - Xy) = 0.$$

These may be thus written:

$$\Sigma . P\cos\alpha = 0,\ \Sigma . P\cos\beta = 0,\ \Sigma . P\cos\gamma = 0,$$

$$\Sigma . P(y\cos\gamma - z\cos\beta) = 0,\ \Sigma . P(z\cos\alpha - x\cos\gamma) = 0,$$

$$\Sigma . P(x\cos\beta - y\cos\alpha) = 0.$$

57. The resolved parts of P, which act in the three planes at right angles to the axes of x, y, z, are $P\sin\alpha$, $P\sin\beta$, $P\sin\gamma$ respectively: and the distances of their directions from those axes are

$$\frac{y\cos\gamma - z\cos\beta}{\sin\alpha}, \quad \frac{z\cos\alpha - x\cos\gamma}{\sin\beta}, \quad \frac{x\cos\beta - y\cos\alpha}{\sin\gamma}.$$

The products of these forces and their distances, each by each, are the moments of P about the three axes of co-ordinates respectively (Art. 37). Hence the six conditions of equilibrium deduced in the last Art. may be stated as follows.

The sums of the resolved parts of the forces parallel to any three lines at right angles to each other must respectively equal zero; and the sums of the moments of the forces with respect to these lines (Art. 37.) must also equal zero.

58. If we derive the conditions of equilibrium from the case where the forces admit of a single resultant, we shall arrive at the same conclusion. For we must have the force $R = 0$, and also the distance $G \div R$ at which it acts must be arbitrary and not necessarily infinite: hence also $G = 0$, and the conclusions are the same as before*.

* We have remarked in Art. 25, that the property of the divisibility of matter, leads us to the supposition that every body consists of an assemblage of material particles, or molecules, which are held together by their mutual attraction. Now we are totally unacquainted with the nature of these molecular forces: if, however, we assume the two hypotheses, that the action of any two molecules on each other is the same, and also that it acts in the line joining their centres, two suppositions which appear to be perfectly legitimate, then we shall be able to deduce the conditions of equilibrium of a rigid body from those of a single particle.

PROP. *To find the conditions of equilibrium of a rigid body from those of a single molecule.*

Let the body be referred to three rectangular co-ordinate axes: and let xyz be the co-ordinates to one of its constituent particles: XYZ the sums of the resolved parts parallel to the axes of the forces which act upon this particle, neglecting the molecular forces: $P, P' \dots$ the molecular forces acting on this particle; $\alpha\beta\gamma$, $\alpha'\beta'\gamma' \dots$ the angles their respective directions make with the three axes of co-ordinates.

Then we may suppose the rest of the body to be removed, and this particle held in equilibrium by the above forces. Hence, by Art. 23,

$$\left.\begin{aligned} X + P\cos\alpha + P'\cos\alpha' + \dots\dots &= 0 \\ Y + P\cos\beta + P'\cos\beta' + \dots\dots &= 0 \\ Z + P\cos\gamma + P'\cos\gamma' + \dots\dots &= 0 \end{aligned}\right\} \dots\dots (a).$$

We shall have a similar system of equations for each particle in the body: if there be n particles, we shall have $3n$ equations. These $3n$ equations will be connected one with another, since any molecular force which enters into one system of equations must enter into a second system; this is in consequence of the *mutual* action of the molecules.

There are two considerations which will enable us to deduce from these $3n$ equations, *six* equations of condition, independent of the molecular forces. These will be the equations which the other forces must satisfy, in order that the equilibrium may be established.

Prop. *To find the conditions of equilibrium of forces acting upon a rigid body when one point is fixed.*

59. Let the fixed point be taken as the origin of co-ordinates.

The first consideration is this, that the molecular actions are *mutual;* and that, consequently, if $P\cos\alpha$ represent the resolved part parallel to the axis of x of any one of the molecular forces involved in the $3n$ equations, we shall likewise meet with the term $-P\cos\alpha$ in another of those equations which have reference to the axis of x. Consequently, if we add all those equations together which have reference to the same axis, we have the three following equations of condition independent of the molecular forces.

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0.$$

The second consideration is this:—that the straight lines joining the different particles are the directions in which the molecular forces act.

Thus let P be the molecular action between the particles whose co-ordinates are (xyz) and $(x_1y_1z_1)$:

$$\begin{array}{ccc} P\cos\alpha, & P\cos\beta, & P\cos\gamma, \\ -P\cos\alpha, & -P\cos\beta, & -P\cos\gamma, \end{array}$$

the corresponding resolved parts of P for the two particles.

Then $$\cos\alpha = \frac{x_1 - x}{\sqrt{(x_1-x)^2+(y_1-y)^2+(z_1-z)^2}},$$

$$\cos\beta = \frac{y_1 - y}{\sqrt{(x_1-x)^2+(y_1-y)^2+(z_1-z)^2}},$$

$$\cos\gamma = \frac{z_1 - z}{\sqrt{(x_1-x)^2+(y_1-y)^2+(z_1-z)^2}}.$$

These enable us to obtain three more equations free from molecular forces: for if we multiply the first and second of equations (a) by y and x respectively, and then subtract them, we have

$$Yx - Xy + \ldots\ldots + P\{x\cos\beta - y\cos\alpha\} + \ldots\ldots = 0,$$

and by the same process, we obtain from the system of equations which refer to the particle $(x_1y_1z_1)$,

$$Y_1x_1 - X_1y_1 + \ldots\ldots - P\{x_1\cos\beta - y_1\cos\alpha\} + \ldots\ldots = 0.$$

But the values of $\cos\alpha$ and $\cos\beta$, given above, lead to the condition

$$(x_1 - x)\cos\beta - (y_1 - y)\cos\alpha = 0.$$

Wherefore the equation

$$\left.\begin{array}{l} Y.x - X.y + \ldots\ldots \\ +Y_1.x_1 - X_1y_1 + \ldots\ldots \end{array}\right\} = 0,$$

will not involve P, the molecular action between the particles whose co-ordinates are xyz and $x_1y_1z_1$, respectively.

It follows readily from what we have shewn, that if we form all the equations

$$Y_2.x_2 - X_2.y_2 + \ldots\ldots\ldots\ldots = 0,$$
$$Y_3.x_3 - X_3.y_3 + \ldots\ldots\ldots\ldots = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and add them to those above, we shall have a final equation

$$\Sigma.(Y.x - X.y) = 0,$$

independent of the molecular forces. In

Now the action of the forces on the body will produce a pressure on the fixed point, and this will act in some definite direction.

Let $X' Y' Z'$ be the resolved parts of this pressure parallel to the axes. Then the fixed point will exert forces $-X'$, $-Y'$, $-Z'$ against the body, and if we take these forces in connexion with the given forces, we may suppose the body to be free, and the equations of equilibrium give

$$\Sigma . X - X' = 0, \quad \Sigma . Y - Y' = 0, \quad \Sigma . Z - Z' = 0,$$

$$L = 0, \quad M = 0, \quad N = 0.$$

The first three equations give the resolved parts of the pressure on the fixed point: and the last three are the only conditions to be satisfied by the given forces.

PROP. *To find the conditions of equilibrium of a body which has two points in it fixed.*

60. Let the axis of x pass through the two fixed points: and let the distances of the points from the origin be x' and x''. Also let $X' Y' Z'$, $X'' Y'' Z''$ be the resolved parts of the pressures on these points.

In like manner we should obtain

$$\Sigma . (X . z - Z . x) = 0,$$
$$\Sigma . (Z . y - Y . z) = 0.$$

By introducing the condition, that the body is *rigid*, we can shew that these six equations are the *only* equations free from the molecular forces, and are therefore the sole conditions of equilibrium.

The condition of the invariability of the system of molecules requires, that each molecule should be acted on by three molecular forces at least: and a little consideration will shew, that, to effect this, there must be at least $3n-6$ independent molecular forces. For if the body consist of three molecules, there must evidently be three independent molecular forces to keep them invariable: if to these a fourth be added, we must introduce three new forces to hold it to the others; if we add a fifth, we must introduce three forces to hold this invariably to any three of those which are already rigidly connected: and so on: from which we see that there must be at least $3+3(n-3)$ or $3n-6$ forces. Hence the $3n$ equations resembling the equations (a) contain at least $3n-6$ independent quantities to be eliminated: and therefore there cannot be more than *six* equations of condition connecting the external forces and the co-ordinates of their points of application. The six equations already obtained are therefore the only conditions of equilibrium.

Then, as in the last Prop. the equations of equilibrium will be

$$\Sigma . X - X' - X'' = 0,\ \Sigma . Y - Y' - Y'' = 0,\ \Sigma . Z - Z' - Z'' = 0,$$

$$L + Y' . z' + Y'' . z'' = 0,\ M - X' . z' - X'' z'' = 0,$$

$$N = 0.$$

The first, second, fourth and fifth of these equations will determine $X' X'' Y' Y''$: and the third equation gives $Z' + Z''$, shewing that the pressures on the fixed points in the direction of the line joining them are indeterminate, being connected by one equation only.

The last is the only condition of equilibrium, viz. $N = 0$.

PROP. *To find the conditions of equilibrium of a rigid body resting on a plane.*

61. Let this plane be the plane of xy: and let $x'\ y'$ be the co-ordinates to one of the points of contact, R' the pressure which the body exerts against the plane at that point. Then the force $-R'$, and similar forces for the other points of contact, taken in connexion with the given forces ought to satisfy the equations of equilibrium.

$$\text{Hence } \Sigma . X = 0,\ \ \Sigma . Y = 0,\ \ \Sigma . Z - \Sigma . R' = 0,$$

$$L - \Sigma . R'y' = 0,\ \ M + \Sigma . R'x' = 0,\ \ N = 0.$$

If only one point be in contact with the plane, then the third equation gives the pressure, and we have five equations of condition,

$$\Sigma . X = 0,\ \ \Sigma . Y = 0,\ \ L - y' \Sigma . Z = 0,$$

$$M + x' \Sigma . Z = 0,\ \ N = 0.$$

If two points be in contact, then

$$R'y' + R''y'' = L,\ \ R'x' + R''x'' = -M,$$

$$\text{give } R' = \frac{Lx'' + My''}{y'x'' - x'y''},\ \ R'' = \frac{-Lx' - My'}{y'x'' - x'y''},$$

and the equations of condition are

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z - \frac{L(x'' - x') + M(y'' - y')}{y'x'' - x'y''} = 0,$$

$$\text{and } N = 0.$$

If three points be in contact, then the pressures are determined from the equations

$$\left.\begin{aligned} R' + R'' + R''' &= \Sigma . Z \\ R'y' + R''y'' + R'''y''' &= L \\ R'x' + R''x'' + R'''x''' &= -M \end{aligned}\right\}$$

and the conditions of equilibrium are

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad N = 0.$$

If more than three points are in contact, then the pressures are indeterminate, since they are connected by only three equations: but the conditions of equilibrium are still

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad N = 0.$$

62. In Art. 51. we shewed, that if the forces acting upon a rigid body are compounded, their resultant moment with respect to a given centre (taken as the origin of co-ordinates) $= G = \sqrt{L^2 + M^2 + N^2}$. Here L, M, N are the resultant moments about the axes of co-ordinates chosen arbitrarily. Now it is clear that L, M, and N cannot be greater than G. Hence, for a given centre, the *resultant* moment, G, is greater than the moment of the forces about any other axis. For this reason G is called the *Principal Moment* of the forces.

63. The values of L, M, N shew, that the moment about an axis through the given centre, and making an angle ϕ with the axis of principal moment, equals $G \cos \phi$.

PROP. *To find the locus of the centres, which give the least principal moments; the magnitude of these moments and the position of their axes.*

64. Let $x_1 y_1 z_1$ be the co-ordinates to a centre, which gives a minimum principal moment: let $L_1 M_1 N_1 G_1$ be the values of

$LMNG$ at that point: then these are found by putting $x - x_1$, $y - y_1$, $z - z_1$ for xyz in $LMNG$;

$$\therefore\ L_1 = L - y_1 \Sigma . Z + z_1 \Sigma . Y,$$

$$M_1 = M - z_1 \Sigma . X + x_1 \Sigma . Z,$$

$$N_1 = N - x_1 \Sigma . Y + y_1 \Sigma . X,$$

$$G_1^2 = L_1^2 + M_1^2 + N_1^2$$

$$= (L - y_1 \Sigma . Z + z_1 \Sigma . Y)^2 + (M - z_1 \Sigma . X + x_1 \Sigma . Z)^2$$

$$+ (N - x_1 \Sigma . Y + y_1 \Sigma . X)^2.$$

When this is a minimum, its three partial differential coefficients with respect to $x_1 y_1 z_1$ must vanish: hence three equations which may easily be written in the form

$$R^2 . x_1 = (x_1 \Sigma . X + y_1 \Sigma . Y + z_1 \Sigma . Z) \Sigma . X + N \Sigma . Y - M \Sigma . Z,$$

$$R^2 . y_1 = (x_1 \Sigma . X + y_1 \Sigma . Y + z_1 \Sigma . Z) \Sigma . Y + L \Sigma . Z - N \Sigma . X,$$

$$R^2 . z_1 = (x_1 \Sigma . X + y_1 \Sigma . Y + z_1 \Sigma . Z) \Sigma . Z + M \Sigma . X - L \Sigma . Y.$$

If we multiply these respectively by $\Sigma . X$, $\Sigma . Y$, and $\Sigma . Z$ and add, we find an identical equation: which shews that these three are equivalent to only two equations: and since they are simple equations in $x_1 y_1 z_1$, we learn that the centres of minimum principal moments lie in a straight line, any two of the above equations being the equations to this line.

65. If we eliminate the second terms of the above equations, they become of the ordinary form of equations to a line:

$$L - y_1 \Sigma . Z + z_1 \Sigma . Y = \frac{(L \Sigma . X + M \Sigma . Y + N \Sigma . Z) \Sigma . X}{R^2},$$

$$M - z_1 \Sigma . X + x_1 \Sigma . Z = \frac{(L \Sigma . X + M \Sigma . Y + N \Sigma . Z) \Sigma . Y}{R^2},$$

$$N - x_1 \Sigma . Y + y_1 \Sigma . X = \frac{(L \Sigma . X + M \Sigma . Y + N \Sigma . Z) \Sigma . X}{R^2}.$$

66. The minimum principal moment is

$$G_1 = \frac{L\Sigma.X + M\Sigma.Y + N\Sigma.Z}{R}.$$

It is evident, that the general value of G_1, given above, is capable of becoming infinitely great, and therefore does not admit of a maximum value.

67. Let $\alpha_1\beta_1\gamma_1$ be the angles which the axis of G_1 makes with lines parallel to the co-ordinate axes: then

$$\cos\alpha_1 = \frac{L_1}{G_1}, \quad \cos\beta_1 = \frac{M_1}{G_1}, \quad \cos\gamma_1 = \frac{N_1}{G_1}:$$

and these become, by the above equations,

$$\cos\alpha_1 = \frac{\Sigma.X}{R}, \quad \cos\beta_1 = \frac{\Sigma.Y}{R}, \quad \cos\gamma_1 = \frac{\Sigma.Z}{R},$$

which shew, that the axes of all the minima principal moments are parallel to each other, and to the direction of the resultant.

CHAPTER III.

THE EQUILIBRIUM OF A SYSTEM OF RIGID BODIES.

68. In order to obtain the conditions of equilibrium of two or more rigid bodies connected together in any way whatever, we must substitute unknown forces in the place of the mutual actions at the points of connexion, and then write down the equations of equilibrium of each body. These systems of equations will be connected together, since a force depending on the mutual actions of any two of the bodies must enter *both* the systems of equations, which correspond to those bodies.

Prop. *To find the conditions of equilibrium of a system of bodies acted on by given forces.*

69. Let any one of the bodies A be acted on by the given forces $X_1Y_1Z_1$, ... at the points $(x_1y_1z_1)$, ... also suppose that in consequence of the connexion of the system a mutual force P acts between the bodies A and B, making angles $\alpha\beta\gamma$ with the axes: and let $x_1'y_1'z_1'$ be its point of application in A, and $x_2'y_2'z_2'$ the point of application in B.

Now if we suppose the force P to act on A and analogous forces for all the other mutual actions arising from the connexion of the system, the body A may be supposed to be in equilibrium under the action of these forces and the forces $X_1Y_1Z_1$ ...... Hence by Art. 56,

$$\Sigma . X_1 + P\cos\alpha + \ldots\ldots = 0, \quad \Sigma . Y_1 + P\cos\beta + \ldots\ldots = 0,$$

$$\Sigma . Z_1 + P\cos\gamma + \ldots\ldots = 0,$$

$$\Sigma . (Z_1y_1 - Y_1z_1) + P(y_1'\cos\gamma - z_1'\cos\beta) + \ldots\ldots = 0,$$

$$\Sigma . (X_1z_1 - Z_1x_1) + P(z_1'\cos\alpha - x_1'\cos\gamma) + \ldots\ldots = 0,$$

$$\Sigma . (Y_1x_1 - X_1y_1) + P(x_1'\cos\beta - y_1'\cos\alpha) + \ldots\ldots = 0.$$

In the same manner for the body B,

$$\Sigma . X_2 - P \cos \alpha + \ldots\ldots = 0, \quad \Sigma . Y_2 - P \cos \beta + \ldots\ldots = 0,$$

$$\Sigma . Z_2 - P \cos \gamma + \ldots\ldots = 0,$$

$$\Sigma . (Z_2 y_2 - Y_2 z_2) - P (y_2' \cos \gamma - z_2' \cos \beta) + \ldots\ldots = 0,$$

$$\Sigma . (X_2 z_2 - Z_2 x_2) - P (z_2' \cos \alpha - x_2' \cos \gamma) + \ldots\ldots = 0,$$

$$\Sigma . (Y_2 x_2 - X_2 y_2) - P (x_2' \cos \beta - y_2' \cos \alpha) + \ldots\ldots = 0,$$

If we add the second set of equations to the first set, each to each, we shall have six equations free from P: for P evidently vanishes from the first three: and it enters the fourth in the form $P \{ (y_1' - y_2') \cos \gamma - (z_1' - z_2') \cos \beta \}$: and this vanishes when the bodies are in contact, because then $y_1' = y_2'$ and $z_1' = z_2'$: also it vanishes when the bodies are not in contact, because then P must act in the line passing through the points $(x_1' y_1' z_1')$, $(x_2' y_2' z_2')$; and then, r being the distance between these points,

$$r \cos \gamma = z_1' - z_2', \quad r \cos \beta = y_1' - y_2';$$

$$\therefore \ (y_1' - y_2') \cos \gamma - (z_1' - z_2') \cos \beta = 0;$$

and in the same way P disappears from the fifth and sixth equations.

Hence the six final equations are free from P: and, by adding together the equations referring to all the bodies, each to each, we shall have finally

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

$$\Sigma . (Zy - Yz) = 0, \quad \Sigma . (Xz - Zx) = 0, \quad \Sigma . (Yx - Xy) = 0,$$

free from all the mutual actions of the bodies of the system.

Also these are the *only* equations, which are independent of the peculiar manner of connexion of the bodies one with another. For if n be the number of bodies in the system, there must be at least $n - 1$ unknown mutual forces, and therefore $6(n - 1)$ forces, angles, and co-ordinates, depending upon the mutual connexion of the bodies, to be eliminated from the $6n$

equations of equilibrium of the n bodies. More equations may be obtained from these $6n$ equations by introducing the peculiar condition of connexion; such as, that the mutual action of two bodies in contact will always be in the common normal at the point of contact, or that two given points in two bodies shall be kept at a constant distance by an inflexible rod; and so on. But the six equations of equilibrium are the only conditions, which are general for all cases, and independent of the peculiar circumstances of any proposed problem.

Cor. We learn from this Proposition (what indeed is of itself obvious), that the equilibrium of the system would not be disturbed by uniting the bodies rigidly at the points of mutual action, and so considering the system as one rigid body.

Prop. *To prove that the Principle of Virtual Velocities is true of any system of forces, which keep any material system in equilibrium.*

70. We shall first enunciate this Principle.

Suppose a material system is held in equilibrium by the action of a system of forces: suppose the points of application of the forces are geometrically moved through very small spaces in a manner consistent with the connexion of the parts of the system one with another. Suppose perpendiculars drawn from the new positions of the points upon the directions of the forces acting at the points in their positions of equilibrium. The distance of any perpendicular from the original point of application of the corresponding force is called the *virtual velocity of the point with respect to that force*, and is estimated positive or negative, according as the perpendicular falls on the side of the point towards which the force acts or the opposite side: then the Principle is this,

The algebraical sum of the products of each force of the system and the corresponding virtual velocity vanishes.

71. I. Suppose the system consists of only one rigid body.

We must cause the different particles to describe small spaces consistent with their connexion; this will, in the case of

a rigid body, be as well accomplished by supposing the co-ordinate axes to receive a slight alteration of position.

Suppose the axes to revolve round z through a small angle θ: then

$$x,\ y,\ z \text{ become } x + y\theta,\quad y - x\theta,\quad z,$$

neglecting small quantities of the second and higher orders.

Next, suppose these new axes to revolve through a small angle ϕ about the new axis of y: by these means the original values x, y, z become

$$(x + y\theta) - z\phi,\quad y - x\theta,\quad z + (x + y\theta)\phi,$$

$$\text{or } x + y\theta - z\phi,\quad y - x\theta,\quad z + x\phi.$$

Next, suppose the axes to revolve about the new axis of x, through a small angle ψ, and the co-ordinates become

$$x + y\theta - z\phi,\quad y - x\theta + z\psi,\quad z + x\phi - y\psi,$$

omitting small quantities of the second and higher orders.

Lastly, let the origin be shifted to a point whose co-ordinates are a, b, c: hence, if δx, δy, δz be the total changes in x, y, z produced by these changes of axes,

$$\delta x = a + y\theta - z\phi \ \dots\dots\dots\dots \ (1),$$

$$\delta y = b + z\psi - x\theta \ \dots\dots\dots\dots \ (2),$$

$$\delta z = c + x\phi - y\psi \ \dots\dots\dots\dots \ (3).$$

Now multiply the equations of equilibrium

$$\Sigma . X = 0,\quad \Sigma . Y = 0,\quad \Sigma . Z = 0,$$

$$\Sigma . (Xy - Yx) = 0,\quad \Sigma . (Zx - Xz) = 0,\quad \Sigma . (Yz - Zy) = 0,$$

by $a, b, c, \theta, \phi, \psi$ respectively, and add;

$$\therefore \Sigma . \{X(a + y\theta - z\phi) + Y(b + z\psi - x\theta) + Z(c + x\phi - y\psi)\} = 0;$$

and, consequently,

$$\Sigma . \{X\delta x + Y\delta y + Z\delta z\} = 0.$$

72. Let R be the force, of which XYZ are the components: a, b, c the angles which the direction of R makes with the axes;

$$\therefore X = R\cos a, \quad Y = R\cos b, \quad Z = R\cos c.$$

Also let δs be the small geometric displacement of the point of application of R, of which δx, δy, δz are the resolved parts: a', b', c' the angles δs makes with the axes: then

$$\delta x = \delta s \,.\, \cos a', \quad \delta y = \delta s \,.\, \cos b', \quad \delta z = \delta s \,.\, \cos c';$$

$$\therefore X\delta x + Y\delta y + Z\delta z = R\delta s\,(\cos a \cos a' + \cos b \cos b' + \cos c \cos c') = R\,\delta r,$$

where δr is the resolved part of δs in the direction of R's action; that is, the virtual velocity of the point (xyz) with respect to R.

$$\text{Hence } \Sigma \,.\, R\delta r = 0,$$

and the Principle of Virtual Velocities is true of a system of forces holding a rigid body in equilibrium.

73. II. Suppose the system consists of any number of rigid bodies.

Let Q be the mutual action of any two of the rigid bodies, whether by contact or by any means of connexion whatever: let $\alpha\beta\gamma$ be the angles which its direction makes with the axes, and let xyz, $x'y'z'$ be the co-ordinates to the points where the force Q acts.

Now each of these bodies is in equilibrium under the action of its own forces, together with the force Q, and the mutual actions it has with the other bodies of the system. Hence, by the first case,

$$\Sigma \,.\, R\delta r + Q\,(\delta x \cos\alpha + \delta y \cos\beta + \delta z \cos\gamma) + \ldots = 0 \ldots (1);$$

also for the other body on which Q acts,

$$\Sigma \,.\, R'\delta r' - Q\,(\delta x' \cos\alpha + \delta y' \cos\beta + \delta z' \cos\gamma) + \ldots = 0 \ldots (2);$$

and Q will not occur in any of the equations that have reference to the other bodies.

Adding equations (1) and (2), will give

$$\Sigma \cdot R\delta r + \Sigma \cdot R'\delta r' + Q \{(\delta x - \delta x')\cos\alpha$$

$$+ (\delta y - \delta y')\cos\beta + (\delta z - \delta z')\cos\gamma\} + \ldots\ldots = 0.$$

Now in consequence of the geometric displacement of the system, suppose the points (xyz) and $(x'y'z')$ describe the small spaces δt and $\delta t'$, making respectively with the axes the angles m, m', m'' and n, n', n'': hence

$$\delta x = \delta t \cos m, \quad \delta y = \delta t \cos m', \quad \delta z = \delta t \cos m'',$$

$$\delta x' = \delta t' \cos n, \quad \delta y' = \delta t' \cos n', \quad \delta z' = \delta t' \cos n''.$$

Hence $(\delta x - \delta x')\cos\alpha + (\delta y - \delta y')\cos\beta + (\delta z - \delta z')\cos\gamma$

$= \delta t\,(\cos m \cos\alpha + \cos m' \cos\beta + \cos m'' \cos\gamma)$

$\qquad - \delta t'\,(\cos n \cos\alpha + \cos n' \cos\beta + \cos n'' \cos\gamma)$

= resolved part of δt − resolved part of $\delta t'$ in direction of Q

= sum of virtual veloc. of the pnts. (xyz), $(x'y'z')$ with resp. to Q

$= \delta q + \delta q'$ suppose.

Hence $\Sigma \cdot R\delta r + \Sigma \cdot R'\delta r' + Q(\delta q + \delta q') + \ldots\ldots = 0.$

Wherefore if we form the equations analogous to equations (1) and (2) for all the bodies, and add them together, we shall have, supposing Σ now to extend through the whole system,

$$\Sigma \cdot R\delta r + \Sigma \cdot Q\,(\delta q + \delta q') = 0:$$

or, if P represent any one of the forces R, R', ... Q, ...

$$\Sigma \cdot P\delta p = 0,$$

and the Principle of Virtual Velocities is still true.

74. Cor. 1. If the force Q be the mutual normal pressure of two surfaces in contact, then by giving the system such a geometric motion, that these surfaces shall remain in contact, we shall cause Q to disappear from this equation, because then

$$\delta q + \delta q' = 0.$$

75. Cor. 2. If the points (xyz) and $(x'y'z')$ are connected invariably, as for instance by an inextensible string, then Q disappears from the equation of Virtual Velocities, since

$$\delta q + \delta q' = 0.$$

76. Cor. 3. If $\Sigma . P\delta p$ be a complete differential of a function Π; then, when Π is a maximum or minimum, the system is in equilibrium: the converse of this is not always true.

77. Def. When a system of bodies is in equilibrium and an indefinitely small motion is given to the parts of the system so as to disturb the state of rest, the equilibrium is said to be *stable* or *unstable* according as the parts of the system do or do not return towards their original positions of rest.

Prop. *The equilibrium of the system is* stable *or* unstable *according as* Π *is a maximum or minimum: and, conversely,* Π *is a maximum or minimum according as the equilibrium is stable or unstable.*

78. Suppose that the forces are reduced as much as possible by composition: and suppose that the final resultants (after this reduction) are R, R',...... ; and δr, $\delta r'$, ... the virtual velocities of their points of application: then δr, $\delta r'$, ... are independent of each other, since there is no relation connecting the forces R, R', ...

$$\text{Now } \delta\Pi = \Sigma . P\delta p = R\delta r + R'\delta r' + \ldots\ldots$$

$$\therefore\ \delta^2\Pi = \delta R . \delta r + \delta R' . \delta r' + \ldots + R . \delta^2 r + R' . \delta^2 r' + \ldots$$

And, when there is equilibrium, since δr, $\delta r'$, ... are independent, we have $R = 0$, $R' = 0$, ...

$$\therefore\ \delta^2\Pi = \delta R . \delta r + \delta R' . \delta r' + \ldots\ldots$$

and, that Π may be a maximum or minimum, the terms $\delta R . \delta r$, $\delta R' . \delta r'$, ... must be all negative or all positive. If, then, Π is a maximum (for instance), δR and δr have different signs; and therefore when δr is positive δR is negative, and

vice versâ: hence the small force δR, brought into play by the slight displacement of the system, always acts in a direction opposite to the direction of the displacement δr; and therefore tends to restore the point of application to its position of equilibrium. The same may be said of the other forces. Hence, when Π is a maximum, the equilibrium is stable. In the same manner it may be shewn, that when Π is a minimum, the equilibrium is unstable. When Π is neither a maximum nor a minimum, then the equilibrium is said to be ambiguous; because for some displacements the points of application of the forces will return towards their positions of rest, and for others they will not: this is, virtually, unstable equilibrium.

The converse proposition is obviously true. For stable equilibrium δR and δr must have different signs; hence $\delta R . \delta r$ is negative: and so for the other forces: and therefore Π is a maximum: and so on.

Cor. If $R\delta r + R'\delta r' + \ldots\ldots$ be not a perfect differential $\delta\Pi$, still this Prop. is true if we say instead of Π being a maximum or minimum, that $\delta\Pi$ is negative or positive.

The following is an example of this Proposition.

Prop. *A system of rigid bodies under the action of no forces but their weights, mutual forces, and pressures upon smooth immoveable surfaces, will be in equilibrium, if placed so that the centre of gravity is in the lowest or highest position it can possibly attain by moving the system consistently with the connexion of its parts one with another.*

79. For let the axis of z be taken vertically downwards: and let $P_1, P_2, \ldots$ be the vertical forces with which the different particles tend downwards by reason of the attraction of the Earth: $z_1\, z_2 \ldots$ the vertical ordinates to their points of application, $\bar{z}$ the vertical ordinate to the centre of gravity (see Art. 32.)

$$\therefore \bar{z} = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots}.$$

Now suppose the system to receive a slight displacement of its parts consistent with their connexion, and let $\delta z_1\, \delta z_2\, \delta z_3 \ldots$

be the vertical displacements of the points of application of $P_1 P_2 P_3 \ldots$ (these are the virtual velocities of the points); and let $\bar{z}$ become $\bar{z} + \delta\bar{z}$;

$$\therefore \bar{z} + \delta\bar{z} = \frac{P_1 z_1 + P_2 z_2 + P_3 z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots}$$

$$+ \frac{P_1 \delta z_1 + P_2 \delta z_2 + P_3 \delta z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots};$$

$$\therefore \delta\bar{z} = \frac{P_1 \delta z_1 + P_2 \delta z_2 + P_3 \delta z_3 + \ldots\ldots}{P_1 + P_2 + P_3 + \ldots\ldots}.$$

But by the principle of Virtual Velocities, the numerator of this fraction ultimately vanishes when there is equilibrium;

$$\therefore \delta\bar{z} = 0, \text{ ultimately},$$

and when $\bar{z}$ is a maximum, or minimum, or the centre of gravity is in its highest or lowest position, this is satisfied.

It appears from Art. 78, that the equilibrium is stable or unstable according as the centre of gravity is lowest or highest respectively: it may also be shewn as follows.

PROP. *To prove that when the centre of gravity has its lowest position the equilibrium is* stable, *and when it has its highest position the equilibrium is* unstable.

80. Suppose the pressures (mentioned in the last Prop.) and the weight of the parts of the system are not in equilibrium. We shall prove that the centre of gravity cannot ascend, but must descend.

The resultant of the weights of the different parts of the system passes through the centre of gravity of the system. Let W be the weight of the whole system: and suppose the centre of gravity would move in the direction GG' (fig. 17.) making an angle θ with the vertical drawn downwards from G, if not prevented by a force P acting in the opposite direction, and combining with the pressures to preserve equilibrium: $GG' = a$: then by Virtual Velocities,

$$W \cdot a \cos\theta - P \cdot a = 0;$$

$$\therefore \cos\theta = \frac{P}{W};$$

therefore $\cos\theta$ cannot be negative, or θ cannot lie between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$; that is, G cannot move upwards but must move downwards when the system is not in equilibrium.

Now if the system be in equilibrium with its centre of gravity as high as possible, any slight disturbance must bring it lower; and since, by what we have just proved, it can never rise again, it follows that the equilibrium will be *unstable*. But if the equilibrium be such, that the centre of gravity is in its lowest position, any disturbance must raise it higher; and since when left to itself it must fall, it follows, that the centre of gravity will return to its former position, or the equilibrium is *stable*.

81. We have in the foregoing part of this work deduced the conditions of equilibrium of a material system from the simplest principles, commencing with the equilibrium of a single material particle: and we have from these conditions proved the Principle of Virtual Velocities. But we might have pursued an inverse course and commenced with proving the Principle of Virtual Velocities, and thence deducing the conditions of equilibrium of a material system.

PROP. *To prove the Principle of Virtual Velocities independently of the Parallelogram of Forces.*

82. The following is Lagrange's Proof of this Principle. Let us suppose that the forces are P_1, P_2, P_3 ... and that they are commensurable and in the proportion of the numbers n_1, n_2, n_3,.... let A_1, A_2, A_3,...(fig. 18.) be their points of application: A_1a_1, A_2a_2, A_3a_3,... their directions.

Now imagine a_1 and b_1 to be two blocks consisting each of n_1 wheels of equal size, the wheels in the same block turning freely about the same axis: and let the centres of these blocks be in the straight line A_1a_1 produced. Let a_1 be connected with A_1 by an inextensible string: and suppose b_1 is firmly fixed to an immoveable beam B_1; and a_1, b_1 connected by an

inextensible string passing round their wheels alternately, one end of the string being attached to a fixed point M any where in the plane of the first wheel of b_1 over which it passes; and the other end being carried (as represented in the figure) to another system of blocks corresponding to the force P_2, each block having n_2 wheels; and so on: and lastly, let the string be passed over a simple wheel at C and be stretched by a weight W hanging by it.

The string is imagined to be perfectly flexible, and the wheels perfectly smooth: consequently the string will be stretched uniformly throughout, with a tension equal to the weight W. It is very evident, then, that since the wheels of a_1 and b_1 are all equal, the portions of string connecting them are parallel, and (they being $2n_1$ in number) the tension of A_1a_1 equals the weight $2n_1W$; in the same manner the tension of A_2a_2 is $2n_2W$; and so on.

Consequently by this imaginary contrivance the weight W produces forces at the points $A_1 A_2$ .... in the directions A_1a_1, A_2a_2 ...... and in the proportion of n_1 n_2 ...... ; that is, in the proportion of $P_1 P_2$ ......

But P_1 P_2 ...... are in equilibrium: and since the unit of force may be any force, a system of forces *in the same proportion as* P_1 P_2 ...... acting at the same points and in the same directions as P_1 P_2 ...... will be in equilibrium.

Hence if we remove the forces P_1 P_2 ...... and replace them in the manner described above, W will be at rest: and this will be the case of whatever magnitude W be, since by increasing or diminishing W, the forces P_1 P_2 ...... are altered so as to retain their proportion unchanged.

Wherefore, however much we alter W, we cannot thereby cause the moveable block (a_1) of any of the systems (as a_1 b_1) to move.

This shews that the relation of the magnitudes of the forces P_1 P_2 ..., their directions, and points of application is such, that if we forcibly make the block a_1, or any other block, to approach or recede from the other block b_1 of the system by an indefinitely small space, then the other moveable blocks will

so shift, that on the whole the length of string given off from the blocks which approach will exactly equal the length of string taken in by the blocks which separate. If this were not the case, this indefinitely small displacement of the system would give W an indefinitely small motion, and this would shew conversely, that it is possible to move W, which (as we have proved) cannot be done, however much we alter W in magnitude.

Hence, if $\delta p_1\ \delta p_2 \ldots\ldots$ be the spaces through which $a_1\ a_2 \ldots\ldots$ move in consequence of the indefinitely small displacement, those being reckoned positive when the blocks approach, or string is given off, and the others negative. Then $n_1 \delta p_1$, $n_2 \delta p_2$, will be the lengths of string given off or taken on the wheels, according as they are positive or negative;

$$\therefore\ n_1 \delta p_1 + n_2 \delta p_2 + \ldots\ldots\ldots\ldots = 0,$$

$$\text{or } P_1 \delta p_1 + P_2 \delta p_2 + \ldots\ldots\ldots\ldots = 0,$$

which is the Principle of Virtual Velocities.

The displacements δp_1, $\delta p_2 \ldots\ldots$ must be taken indefinitely small, otherwise the equilibrium will be sensibly disturbed, and W will not remain at rest. In fact the best way of representing the principle is this; that when any part of the system is moved through a space less than any assignable quantity, then W will move through a small space which varies as the square or some higher power of the disturbance, so that it vanishes in the limit.

PROP. *To obtain the equations of equilibrium of a rigid body from the Principle of Virtual Velocities.*

83. By this principle we have $\Sigma . P\delta p = 0$. Let XYZ be the resolved parts of P; and $\delta x, \delta y, \delta z$ the virtual velocities of the point (xyz) with respect to P;

$$\therefore \Sigma . (X\delta x + Y\delta y + Z\delta z) = 0.$$

Now, by Art. 71, we must put

$$\delta x = a + y\theta - z\phi, \quad \delta y = b + z\psi - x\theta, \quad \delta z = c + x\phi - y\psi,$$

in which a, b, c, θ, ϕ, ψ are arbitrary small quantities: hence

$$a\Sigma . X + b\Sigma . Y + c\Sigma . Z$$
$$+ \psi\Sigma . (Yz - Zy) + \phi\Sigma . (Zx - Xz) + \theta\Sigma . (Xy - Yx) = 0,$$

and because a, b, c, θ, ϕ, ψ are arbitrary,

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$
$$\Sigma . (Yz - Zy) = 0, \quad \Sigma . (Zx - Xz) = 0, \quad \Sigma . (Xy - Yx) = 0,$$

which are the six equations of equilibrium deduced in Art. 56.

CHAPTER IV.

CENTRE OF GRAVITY.

84. It was shewn in Art. 33. that there is a point in every body such that, if the particles of the body be acted on by parallel forces and this point be fixed, the body will rest in whatever position it be placed.

85. Now the weight of a body may be considered as the resultant of the weights of the different elementary portions of the body acting in parallel and vertical lines. In this case the point above described, the centre of parallel forces, is called *the centre of gravity of the body*. We intend to devote the present Chapter to the determination of this point in bodies of various forms.

86. We shall first give a few geometrical calculations of the position of the centre of gravity.

Ex. 1. *To find the centre of gravity of a triangular figure of uniform thickness and density.*

Let ABC be one surface of the triangular figure: fig. 19. Bisect AC in D; join BD: draw adc parallel to ADC cutting BD in d. Then by similar triangles

$$ad : AD :: Bd : BD$$

$$\text{and } dc : DC :: Bd : BD$$

$$\therefore ad : AD :: dc : DC$$

$$\text{but } AD = DC; \quad \therefore ad = dc.$$

Hence BD bisects every line parallel to the side AC: and therefore each of these lines will balance on BD, and consequently the whole triangle will balance on BD: and therefore the centre of gravity must be in the line BD.

Bisect AB in E and join CE; let this cut BD in F. Then, as before, the centre of gravity must be in CE: but it must be in BD: and therefore F is the centre of gravity.

Join DE. Then $\because AD = DC$ and $AE = EB$;

$\therefore DE$ is parallel to BC and $BC = 2 . DE$,

and by similar triangles

$$\frac{DF}{DE} = \frac{BF}{BC}; \quad \therefore DF = \tfrac{1}{2} BF;$$

$$\therefore DF = \tfrac{1}{3} DB.$$

Hence to find the centre of gravity of a triangle, bisect any side, join the point of bisection with the opposite angle, and the centre of gravity lies a third of the way up this line.

Ex. 2. *To find the centre of gravity of a pyramid on a triangular base.*

Let ABC be the base; V the vertex: fig. 20, bisect AC in D; join BD, DV: take $DF = \tfrac{1}{3} . DB$, then F is the centre of gravity of ABC. Join FV: and draw abc parallel to ABC cutting VF in f: join bf; and produce it to meet DV in d.

Then by similar triangles, we easily see that $ad = dc$: also

$$\frac{bf}{BF} = \frac{Vf}{VF} = \frac{df}{DF}: \text{ but } DF = \tfrac{1}{2} BF;$$

$$\therefore df = \tfrac{1}{2} bf;$$

therefore f is the centre of gravity of the triangle abc: and if we suppose the pyramid to be made up of an infinitely great number of infinitely thin triangular figures parallel to the base, each of these has its centre of gravity in VF. Hence the centre of gravity of the pyramid is in VF.

Again, take $DH = \tfrac{1}{3} DV$: join HB cutting VF in G. Then, as before, the centre of gravity of the pyramid must be in BH: but it is in VF: hence G, the point of intersection of these lines, is the centre of gravity.

Join FH: then FH is parallel to VB:

also $\because DF = \tfrac{1}{3} DB$; $\therefore FH = \tfrac{1}{3} VB$:

$$\text{and } \frac{FG}{FH} = \frac{VG}{VB} \text{ but } FH = \tfrac{1}{3}\, VB;$$

$$\therefore\ FG = \tfrac{1}{3}\, GV = \tfrac{1}{4}\, FV.$$

Hence the centre of gravity is found to be one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

Ex. 3. *To find the centre of gravity of any pyramid having a plane base.*

Divide the base into triangles: if any part of the base is curvilinear, then suppose the curve to be divided into an indefinitely great number of indefinitely short straight lines. Join the vertex of the pyramid with the centres of gravity of all the triangles, and also with all their angles. Draw a plane parallel to the base at a distance from the base equal to one-fourth of the distance of the vertex from the base: then this plane cuts every line drawn from the vertex to the base in parts, having the same ratio 3 : 1; and therefore the triangular pyramids have their centres of gravity in this plane, and therefore the whole pyramid has its centre of gravity in this plane.

Again, join the vertex with the centre of gravity of the base; then every section of the pyramid parallel to the base will be similar to the base, and will have its centre of gravity in this line. Hence the whole pyramid has its centre of gravity in this line.

Wherefore the centre of gravity is one-fourth of the way up the line joining the centre of gravity of the base with the vertex.

Ex. 4. *To find the centre of gravity of the frustum of a pyramid, formed by parallel planes.*

Let $ABC\,abc$ be the frustum, fig. 20: G, g the centres of gravity of the pyramids $VABC$, $Vabc$: it is clear that the centre of gravity of the frustum must be in gG produced; at G' suppose.

Let $G'F = x$; $Ff = c$; $AB = a$, $ab = b$.

Now the smaller pyramid and the frustum supposed to act at their centres of gravity are in equilibrium about G: hence, by Art. 29,

$$\frac{GG'}{Gg} = \frac{\text{weight of smaller pyd.}}{\text{weight of frustum}}$$

$$= \frac{\text{vol. of small pyd.}}{\text{vol. of large} - \text{vol. of small pyd.}} = \frac{b^3}{a^3 - b^3},$$

$$Gg = VG - Vg = \tfrac{3}{4}(VF - Vf) = \tfrac{3}{4}c;$$

$$\therefore GG' = \frac{3c}{4}\frac{b^3}{a^3 - b^3}.$$

Also $GF = \frac{1}{4}VF = \frac{1}{4}(VF - Vf)\dfrac{a}{a-b}$ by similar figures,

$$= \frac{c}{4}\frac{a}{a-b};$$

$$\therefore FG' = FG - G'G = \frac{c}{4}\left\{\frac{a}{a-b} - \frac{3b^3}{a^3 - b^3}\right\}$$

$$= \frac{c}{4}\frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

This is true of a frustum of a pyramid on any base, a and b being homologous sides in the two ends.

We proceed now to the analytical calculations.

PROP. *To obtain formulæ for the calculation of the co-ordinates of the centre of gravity of a body.*

87. Let xyz be the rectangular co-ordinates to an elementary parallelopiped of the body, the mass of the element being dm: then if g be the constant ratio of the mass of a body to its weight, $g.dm$ is the weight of this element: or the force with which it presses downwards in a vertical line: $gdm.x$ is the moment of this force with respect to the plane of yz (see Art. 35.), and $\int gxdm$ is the sum of the moments of the forces which the parts of the body exert downwards in vertical lines:

also $\int g\,dm$ is the sum of the forces. Hence if $\bar{x}$ be that co-ordinate of the centre of gravity of the body which is parallel to the axis of x, (Art. 32.)

$$\bar{x} = \frac{\int gx\,dm}{\int g\,dm} = \frac{\int x\,dm}{\int dm}.$$

$$\text{Similarly, } \bar{y} = \frac{\int y\,dm}{\int dm}, \quad \bar{z} = \frac{\int z\,dm}{\int dm},$$

the limits of integration being determined by the form of the body.

When the body is not bounded by continuous surfaces, these formulæ cannot be used, except in some particular cases, as we shall see when we come to apply them to examples. When these formulæ will not apply we must divide the body into distinct portions, of which the respective centres of gravity can be calculated by the above formulæ; and must finally find the centre of gravity of the whole body by considering these constituent portions as condensed, each into its centre of gravity, and so forming an assemblage of particles to which the formulæ of Art. 32. can be applied.

Ex. 1. *A straight rod of uniform thickness and density:* (fig. 21.)

AB the rod: P, Q two transverse sections, $AP = x$, $PQ = dx$, M the mass of the whole rod and l its length: then the mass of $PQ = M\dfrac{dx}{l} = dm$;

$$\therefore \bar{x} = \frac{\int_0^l . x\,dx}{\int_0^l . dx}, \text{ since } \frac{M}{l} \text{ divides out}$$

$$= \frac{\frac{1}{2}l^2}{l} = \tfrac{1}{2}l = AG.$$

Ex. 2. *A curved line of uniform density and thickness, the curvature lying in one plane:* (fig. 22.)

Let $AP = s$, $PQ = ds$; x, y co-ordinates to P;

$$\text{the mass of } PQ = M\frac{ds}{l} \text{ and } ds = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx,$$

$$\therefore \; \bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{\int ds}$$

between the proper limits. The two following examples are applications of these formulæ.

Ex. 3. *A quadrant of a circle:* (fig. 23.)

Here $y^2 = a^2 - x^2$, the centre B being origin: BA axis of x,

$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}; \quad \frac{ds}{dx} = \frac{a}{\sqrt{a^2 - x^2}}$$

the limiting values of x are 0 and $AB = a$;

$$\therefore \; \bar{x} = \frac{\int_0^a \frac{ax\,dx}{\sqrt{a^2 - x^2}}}{\int_0^a \frac{a\,dx}{\sqrt{a^2 - x^2}}} = \frac{a^2}{\frac{1}{2}\pi a} = \frac{2a}{\pi} = BH,$$

$$\bar{y} = \frac{\int_0^a \frac{ay\,dx}{\sqrt{a^2 - x^2}}}{\int_0^a \frac{a\,dx}{\sqrt{a^2 - x^2}}} = \frac{\int_0^a dx}{\int_0^a \frac{dx}{\sqrt{a^2 - x^2}}} = \frac{2a}{\pi} = HG.$$

Ex. 4. *The arc of a semi-cycloid:* (fig. 23.)

The axis AB being the axis of x and the vertex A the origin,

$$y = a \text{ vers.}^{-1} \frac{x}{a} + \sqrt{2ax - x^2}.$$

$$\frac{dy}{dx} = \frac{2a - x}{\sqrt{2ax - x^2}} = \sqrt{\frac{2a - x}{x}}, \quad \frac{ds}{dx} = \sqrt{\frac{2a}{x}},$$

the limits of x are 0 and AB or $2a$,

$$\therefore \; \bar{x} = \tfrac{2}{3} a = AH, \; \bar{y} = \pi a - \tfrac{4}{3} a = HG.$$

Ex. 5. *A curve line of double curvature.*

In this case $ds = \sqrt{1 + \dfrac{dy^2}{dx^2} + \dfrac{dz^2}{dx^2}}\, dx$, xyz being co-ordinates to the variable extremity of s: this value of ds put in $\bar{x}\,\bar{y}\,\bar{z}$ will give the required co-ordinates.

Ex. 6. *Any portion of a helix, or the curve of the thread of a screw:* (fig. 24.)

The equations are $y = \sqrt{a^2 - x^2}$, $\quad z = na \cos^{-1}\dfrac{x}{a}$;

$$\therefore \frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}, \quad \frac{dz}{dx} = \frac{-na}{\sqrt{a^2 - x^2}};$$

$$\therefore 1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2} = \frac{a^2(1 + n^2)}{a^2 - x^2},$$

and the limits of x are a and x;

$$\bar{x} = \frac{\int_a^x \dfrac{x\,dx}{\sqrt{a^2 - x^2}}}{\int_a^x \dfrac{dx}{\sqrt{a^2 - x^2}}} = \frac{\sqrt{a^2 - x^2}}{\cos^{-1}\dfrac{x}{a}}, \quad \bar{y} = \frac{\int_a^x dx}{\int_a^x \dfrac{dx}{\sqrt{a^2 - x^2}}} = \frac{a - x}{\cos^{-1}\dfrac{x}{a}},$$

$$\bar{z} = \frac{\int_a^x na \cos^{-1}\dfrac{x}{a}\, \dfrac{dx}{\sqrt{a^2 - x^2}}}{\int_a^x \dfrac{dx}{\sqrt{a^2 - x^2}}} = \frac{na}{2}\cos^{-1}\frac{x}{a}.$$

Ex. 7. *A body of uniform thickness and density bounded by a plane curve and its ordinate.*

Let the plane parallel to the plane faces of the body and bisecting it be the plane of xy: the centre of gravity is evidently in this plane: M the mass of the body, and A its area: then the mass of an elementary portion of the area at the point P, of which the co-ordinates are xy, is $M\dfrac{dx\,dy}{A}$; and since $\dfrac{M}{A}$

divides both numerator and denominator, the co-ordinates of the centre of gravity become

$$\bar{x} = \frac{\iint x\,dx\,dy}{\iint dx\,dy}, \quad \bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy}$$

between proper limits.

We shall sometimes find it convenient to use polar co-ordinates: (fig. 25.)

Let $AP = r$, $xAP = \theta$: dr, $rd\theta$ the sides of the elementary portion of the area at P; then $M\dfrac{rdrd\theta}{A}$ is the mass of the element at P: and $x = r\cos\theta$, $y = r\sin\theta$;

$$\therefore\ \bar{x} = \frac{\iint r^2 \cos\theta\,dr\,d\theta}{\iint r\,dr\,d\theta}, \quad \bar{y} = \frac{\iint r^2 \sin\theta\,dr\,d\theta}{\iint r\,dr\,d\theta},$$

between the proper limits: the following examples are applications of these; we shall sometimes use rectangular and sometimes polar co-ordinates.

Ex. 8. *Let the curve be the semi-parabola* AC, (fig 23.)

$AM = x$, $MP = y$: $QM^2 = 4mx$, the equation to AC. Now x and y are independent variables, we may consequently integrate our expressions, first considering y variable and x constant, and then with regard to x. This admits of an easy explanation. Integrating our expressions on the supposition that x is constant and y variable is the same as calculating the expressions only for the elementary masses which lie in a *strip* of the area, like QM in the figure, in which PM or y is different for each element, but AM or x is the same: the limiting values of y in this integration will be $y = 0$ and $y = MQ = 2\sqrt{mx}$, and the result will therefore be a function of x only: then integrating this result with respect to x is the same as adding together the expressions for all the strips like QM, of which the area consists: the limits of x are 0 and AB or a;

$\therefore\ \iint x\,dx\,dy$ between proper limits $= \int_0^a x\,(y + X)\,dx$

(X an arbitrary function of x) $= \int_0^a 2\sqrt{m}\,x^{\frac{3}{2}}dx = \frac{4}{5}\sqrt{m}\,a^{\frac{5}{2}}$,

$\iint dx\,dy$ between limits $= \int_0^a (y + X')\,dx = \frac{4}{3}\sqrt{m}\,a^{\frac{3}{2}}$;

$$\therefore \bar{x} = \tfrac{3}{5} a = AH,$$

in like manner $\bar{y} = \tfrac{3}{4}\sqrt{ma} = \tfrac{3}{8} BC = GH.$

If we had taken the double area CAC', the limits of y would have been $y = -MQ' = -2\sqrt{mx}$, and $y = MQ = 2\sqrt{mx}$, and we should have found $\bar{x} = \tfrac{3}{5} a = AH$, $\bar{y} = 0$, and therefore H is the centre of gravity of the whole.

Ex. 9. *Let* CAC′ *be a circular area:* (fig. 23.)

$y^2 = 2ax - x^2$: and if we take the quadrant ACB, the limits of y are 0, and $\sqrt{2ax - x^2}$, and those of x are 0 and AB or a

$$\bar{x} = \frac{\int_0^a \int_0^y x\,dx\,dy}{\int_0^a \int_0^y dx\,dy} = \frac{\int_0^a x\sqrt{2ax - x^2}\,dx}{\int_0^a \sqrt{2ax - x^2}\,dx}$$

$$= \frac{\int_0^a \dfrac{2ax^2 - x^3}{\sqrt{2ax - x^2}}\,dx}{\int_0^a \dfrac{2ax - x^2}{\sqrt{2ax - x^2}}\,dx} = a - \frac{4a}{3\pi} = AH^*.$$

$$\text{Also } \bar{y} = \frac{\int_0^a \int_0^y y\,dx\,dy}{\int_0^a \int_0^y dx\,dy} = \frac{\tfrac{1}{2}\int_0^a y^2\,dx}{\int_0^a y\,dx} = \frac{\tfrac{1}{2}\int_0^a (2ax - x^2)\,dx}{\int_0^a \sqrt{2ax - x^2}\,dx}$$

$$= \frac{\tfrac{1}{2}(a^3 - \tfrac{1}{3}a^3)}{\tfrac{1}{4}\pi a^2} = \frac{4a}{3\pi} = GH.$$

Ex. 10. *Let* CAC′ *be an ellipse.*

Then if we take the quadrant ACB, $AB = a$, $BC = b$,

$$\bar{x} = a - \frac{4a}{3\pi}, \quad \bar{y} = \frac{4b}{3\pi}.$$

* The general form is

$$\int_0^x \frac{x^n\,dx}{\sqrt{2ax - x^2}} = \frac{2n-1}{n}\, a \int_0^x \frac{x^{n-1}\,dx}{\sqrt{2ax - x^2}} - \frac{x^{n-1}\sqrt{2ax - x^2}}{n}.$$

Ex. 11. *Let* CAC′ *be a cycloid*: $AB = 2a$,

$$\bar{x} = AH = \frac{7a}{6}, \quad \bar{y} = HG = a\left(\frac{\pi}{2} - \frac{8}{9\pi}\right)$$

Ex. 12. *A triangle*: (fig. 26.)

Draw AD perpendicular to BC; A the origin, AD the axis of x: $DAB = \alpha$, $DAC = \beta$ ($= A - \alpha$), $AD = e$; $x \tan\alpha$, $-x\tan\beta$, the limits of y;

$$\therefore \bar{x} = \frac{\int_0^e x^2 (\tan\alpha + \tan\beta)\, dx}{\int_0^e x (\tan\alpha + \tan\beta)\, dx} = \frac{2e}{3} = AH.$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^e x^2 (\tan^2\alpha - \tan^2\beta)\, dx}{\int_0^e x (\tan\alpha + \tan\beta)\, dx} = \frac{e}{3}(\tan\alpha - \tan\beta) = HG;$$

$$\therefore AH = \tfrac{2}{3} AD, \quad GH = \tfrac{1}{3}(BD - CD),$$

$$\text{and } \therefore \tfrac{2}{3} DE = GH;$$

$$\therefore BD - CD = 2DE;$$

$$\therefore BD - DE = CD + DE;$$

$$\text{and } BE = CE,$$

and $AG = \frac{2}{3} AE$, as in Ex. 1. Art. 86.

As an instance of the application of polar co-ordinates, we will take the following.

Ex. 13. *A semi-ellipse* CBC′: (fig. 25.)

Let H be its centre of gravity: $AP = r$, $BAP = \theta$.

In this case we must integrate first with respect to r and then with respect to θ; but not first with respect to θ and then with respect to r. For if we first integrate with respect to r, we take the sum of the elements in AQ, and the whole area can be divided into strips like AQ: but if we begin by integrating with respect to θ, we take the elements in an annular strip through P: and the area cannot be divided into strips described after the same law, hence we should be unable to integrate again with respect to r.

The limits of r are 0 and AQ or $\dfrac{b}{\sqrt{1-e^2\cos^2\theta}}$, the limits of θ are $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

$$\therefore \bar{x} = \frac{\iint r^2 \cos\theta\, d\theta\, dr}{\iint r\, d\theta\, dr} \text{ between these limits}$$

$$= \frac{2b}{3} \frac{\int_{-a}^{a} \dfrac{\cos\theta\, d\theta}{(1-e^2\cos^2\theta)^{\frac{3}{2}}}}{\int_{-a}^{a} \dfrac{d\theta}{1-e^2\cos^2\theta}}, \quad a = \frac{\pi}{2}$$

$$= \frac{2b}{3} \frac{\int_{-a}^{a} \dfrac{d.\tan\theta}{(1-e^2+\tan^2\theta)^{\frac{3}{2}}}}{\int_{-a}^{a} \dfrac{d.\tan\theta}{1-e^2+\tan^2\theta}}, \quad a = \frac{\pi}{2}$$

$$= \frac{2b}{3} \frac{\dfrac{1}{1-e^2} \dfrac{\tan a}{\sqrt{1-e^2+\tan^2 a}}}{\dfrac{1}{\sqrt{1-e^2}} \tan^{-1} \dfrac{\tan a}{\sqrt{1-e^2}}} = \frac{2b}{3} \frac{1}{\dfrac{\pi}{2}\sqrt{1-e^2}} = \frac{4a}{3\pi}.$$

Ex. 14. *The sector of a circle:* (fig. 25.)

Let $BP'A$ be the sector: $\angle BAP' = a$.

It matters not in this example whether we integrate with respect to r first or θ first; since the area may be made up of either strips like AQ, or of annular strips like that passing through P.

$$\bar{x} = \frac{\int_0^a \int_0^a r^2 \cos\theta\, d\theta\, dr}{\int_0^a \int_0^a r\, d\theta\, dr} = \frac{2a \int_0^a \cos\theta\, d\theta}{3 \int_0^a d\theta} = \frac{2a\,(\sin a)}{3a},$$

$$\bar{y} = \frac{\int_0^a \int_0^a r^2 \sin\theta\, d\theta\, dr}{\int_0^a \int_0^a r\, d\theta\, dr} = \frac{2a \int_0^a \sin\theta\, d\theta}{3 \int_0^a d\theta} = \frac{2a\,(1-\cos a)}{3a};$$

$$\therefore \quad \angle GAB = \tan^{-1}\left(\frac{\bar{y}}{\bar{x}}\right) = \tan^{-1}\frac{1-\cos\alpha}{\sin\alpha} = \frac{\alpha}{2},$$

$$AG = \sqrt{\bar{x}^2 + \bar{y}^2} = \frac{2a}{3\alpha}\sqrt{2-2\cos\alpha} = \frac{4a\sin\frac{1}{2}\alpha}{3\alpha}$$

Ex. 15. *A surface of revolution:* (fig. 27.)

Let $AM = x$, $MP = y$; $MM' = dx$: through M and M' draw two planes at right angles to the axis of the figure; that is, the axis of x. Now every portion of the surface between these planes is equally distant from the axis, and therefore the centre of gravity of the surface $PQQ'P'$ is at M ultimately: let M be the mass of the whole surface (the thickness and density being uniform) and S the whole surface;

$$\therefore \text{ mass of the surface } PQQ'P' = M\frac{2\pi y ds}{S};$$

$$\therefore \quad \bar{x} = \frac{\int_0^a xy\frac{ds}{dx}dx}{\int_0^a y\frac{ds}{dx}dx} = AG,\ AB = a: \text{ and } \bar{y} = 0.$$

Ex. 16. *The surface of a portion of a sphere.*

$$y^2 = 2ax - x^2,\quad \frac{ds}{dx} = \frac{a}{\sqrt{2ax - x^2}},\quad \bar{x} = \tfrac{1}{2}x = AG.$$

Ex. 17. *The surface of a cone.*

$$y = ax,\quad \frac{ds}{dx} = \sqrt{1 + a^2};\quad \bar{x} = \tfrac{2}{3}x = AG.$$

Ex. 18. *Let the body be any surface of uniform thickness and density:* (fig. 28.)

Let xyz be co-ordinates to any point of the surface: the area of a small portion of the surface at that point is

$$\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}\,dx\,dy;$$

therefore mass of the corresponding element

$$= \frac{M}{S} \sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}\, dx\, dy;$$

$$\therefore \bar{x} = \frac{\iint x \sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}\, dx\, dy}{\iint \sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}\, dx\, dy}$$

between proper limits, and similar expressions for $\bar{y}$, $\bar{z}$.

Ex. 19. *The surface of an eighth part of a sphere.*

The origin being at the centre, $x^2 + y^2 + z^2 = a^2$ (fig. 28.)

$$\sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}} = \frac{a}{z} = \frac{a}{\sqrt{a^2 - x^2 - y^2}}.$$

We shall consider y to vary, x remaining constant: that is, we shall take all the elements in the strip QQ': hence the limits of y are 0 and $Q'M$ or $\sqrt{a^2 - x^2}$ $(= y')$, which is obtained from the equation to the surface by putting $z = 0$: then the limits of x are 0 and AB or a;

$$\therefore \bar{x} = \frac{\int_0^a \int_0^{y'} \cdot \frac{x\, dx\, dy}{\sqrt{a^2 - x^2 - y^2}}}{\int_0^a \int_0^{y'} \frac{dx\, dy}{\sqrt{a^2 - x^2 - y^2}}} = \frac{\int_0^a \cdot \frac{\pi}{2}\, x\, dx}{\int_0^a \cdot \frac{\pi}{2}\, dx} = \frac{1}{2}\, a,$$

in the same manner $\bar{y} = \frac{1}{2} a$, $\bar{z} = \frac{1}{2} a$.

Ex. 20. *Let the body be a solid formed by the revolution of a plane curve about the axis of* x: (fig. 27.)

The centre of gravity of the slice PQ' is evidently at M when the thickness of the slice is diminished indefinitely: and the mass of this slice $= \frac{M}{V} \pi y^2 dx$, $V =$ whole volume;

$$\therefore \bar{x} = \frac{\int_0^x xy^2\,dx}{\int^x y^2\,dx}.$$

Ex. 21. *Let the body be a hemisphere.*

$$y^2 = 2ax - x^2: \ \bar{x} = \tfrac{5}{8}a.$$

Ex. 22. *Let the body be a paraboloid.*

$$y^2 = 4mx: \ \bar{x} = \tfrac{2}{3}x.$$

Ex. 23. *The frustum of a paraboloid.*

Let a and b be the radii of the larger and smaller ends: α and β the values of x measured from the vertex to the ends;

$$\text{then } \alpha = \frac{a^2}{4m}, \quad \beta = \frac{b^2}{4m},$$

$$\bar{x} = \frac{\int_\beta^\alpha x^2\,dx}{\int_\beta^\alpha x\,dx} = \frac{2}{3}\,\frac{\alpha^3 - \beta^3}{\alpha^2 - \beta^2} = \frac{2}{3}\,\frac{\alpha^2 + \alpha\beta + \beta^2}{\alpha + \beta};$$

therefore the distance from smaller end $= \bar{x} - \beta$

$$= \frac{2\alpha^2 - \alpha\beta - \beta^2}{3(\alpha + \beta)} = (\alpha - \beta)\,\frac{2\alpha + \beta}{3(\alpha + \beta)} = \frac{c}{3}\,\frac{2a^2 + b^2}{a^2 + b^2},$$

c = length of the frustum.

Ex. 24. *Frustum of a cone.*

$$\text{Distance from smaller end} = \frac{c}{4}\,\frac{b^2 + 2ab + 3a^2}{b^2 + ab + a^2}.$$

Ex. 25. *Let the body be any solid.*

We shall first suppose the body referred to rectangular coordinates, as in fig. 29.

Let the body be divided into *slices*, like $Q'N''M$, by planes parallel to the plane yz: let these slices be divided into *prisms*, like QN, by planes parallel to the plane zx: and let these prisms be divided into *parallelopipeds*, like PP', by planes parallel to the plane xy. In this manner the body is divided

into a number of elementary parallelopipeds: those at the extremities of the prisms will not be perfect; but when the distance of the cutting planes is diminished indefinitely, the sum of these imperfect portions vanishes.

Let xyz be co-ordinates to P; dx, dy, dz the sides of the parallelopiped at P: then $dxdydz$ is the volume of this figure, and V being the volume and M the mass of the whole body, supposed homogeneous, the mass of the element at P $= \frac{M}{V}\, dxdydz$;

$$\therefore\ \bar{x} = \frac{\iiint x\, dxdydz}{\iiint dxdydz},\quad \bar{y} = \frac{\iiint y\, dxdydz}{\iiint dxdydz},\quad \bar{z} = \frac{\iiint z\, dxdydz}{\iiint dxdydz},$$

between the proper limits.

We shall now suppose the body is referred to polar co-ordinates, as in fig. 30.

Let the body be divided into *slices*, such as $CN'NA$, by planes passing through AC: let these slices be divided into *pyramids* having their vertices in A, like AQ, by the rotation of rays like AQ about AC, preserving a constant inclination to AC during the rotation: lastly, let each of these pyramids be divided into *six-sided figures*, like PP', by planes perpendicular to its length. In this manner the body is divided into a number of six-sided figures which become parallelopipeds ultimately when the distance of the cutting planes is diminished indefinitely.

Let $CAP = \theta$, $AP = r$, $BAN = \phi$; then the sides of the figure at P are dr, $rd\theta$, $r\sin\theta d\phi$, and its volume ultimately equals the product of these $= r^2\sin\theta drd\theta d\phi$.

Also $x = r\sin\theta\cos\phi$, $\quad y = r\sin\theta\sin\phi$, $\quad z = r\cos\theta$:

therefore, supposing the body homogeneous,

$$\bar{x} = \frac{\iiint r^3\sin^2\theta\cos\phi drd\theta d\phi}{\iiint r^2\sin\theta\, drd\theta d\phi},\quad \bar{y} = \frac{\iiint r^3\sin^2\theta\sin\phi drd\theta d\phi}{\iiint r^2\sin\theta\, drd\theta d\phi},$$

$$\text{and } \bar{z} = \frac{\iiint r^3\sin\theta\cos\theta drd\theta d\phi}{\iiint r^2\sin\theta drd\theta d\phi},\ \text{between limits.}$$

Ex. 26. *The eighth part of a sphere:* (fig. 29).

Now xyz, being co-ordinates to any point P in the body, are independent variables: we may therefore integrate with respect to z, considering x and y not to vary: that is the same as taking all the elements of the mass in a given prism QN, since although z or PN is different for each element, yet x and y remain the same: the limiting values of z are 0 and $QN = \sqrt{a^2 - x^2 - y^2}$ ($= z'$ suppose) obtained from the equation to the surface. This integration with respect to z between limits will leave the result a function of x and y without z. We shall then integrate with respect to y, considering x constant; this is the same as taking all the prisms in the same slice as $Q'N''$; since, although MN or y is different for each prism, yet AM or x is the same. The limits of y are 0 and MN' or $\sqrt{a^2 - x^2}$ ($= y'$ suppose) obtained from the equation to the line BN'. We shall finally integrate with respect to x from $x = 0$ to $x = AB$ or a, which is the same as taking all the slices, and therefore the whole body;

$$\therefore \bar{x} = \frac{\int_0^a \int_0^{y'} \int_0^{z'} x\,dx\,dy\,dz}{\int_0^a \int_0^{y'} \int_0^{z'} dx\,dy\,dz} = \frac{\int_0^a \int_0^{y'} x\sqrt{a^2 - x^2 - y^2}\,dx\,dy}{\int_0^a \int_0^{y'} \sqrt{a^2 - x^2 - y^2}\,dx\,dy}$$

$$= \frac{\int_0^a \frac{1}{4}\pi x\,(a^2 - x^2)\,dx}{\int_0^a \frac{1}{4}\pi\,(a^2 - x^2)\,dx} = \frac{\frac{1}{2}a^4 - \frac{1}{4}a^4}{a^3 - \frac{1}{3}a^3} = \tfrac{3}{8}a;$$

in like manner we shall find $\bar{y} = \frac{3}{8}a$, and $\bar{z} = \frac{3}{8}a$.

Ex. 27. *The same as last example, but referred to polar co-ordinates:* (fig. 30.)

We shall integrate first with respect to r, then θ, and lastly ϕ. The limits of r are 0 and AQ or a; those of θ are 0 and $\frac{1}{2}\pi$; those of ϕ are 0 and $\frac{1}{2}\pi$;

$$\therefore \bar{x} = \frac{\int_0^\alpha \int_0^\alpha \int_0^a r^3 \sin^2\theta \cos\phi\, d\phi\, d\theta\, dr}{\int_0^\alpha \int_0^\alpha \int_0^a r^2 \sin\theta\, d\phi\, d\theta\, dr},\ \alpha = \frac{\pi}{2}$$

$$= \frac{3a}{4}\,\frac{\int_0^\alpha \int_0^\alpha \sin^2\theta \cos\phi\, d\phi\, d\theta}{\int_0^\alpha \int_0^\alpha \sin\theta\, d\phi\, d\theta}$$

$$= \frac{3a}{8}\,\frac{\int_0^\alpha \cdot \frac{\pi}{2} \cos\phi\, d\phi}{\int_0^\alpha d\phi} = \tfrac{3}{8}a.$$

So also $\bar{y} = \frac{3}{8}a$, and $\bar{z} = \frac{3}{8}a$.

Ex. 28. *A hemisphere in which the density varies as the* n^th^ *power of the distance from the centre.*

We shall use polar co-ordinates.

The volume of an element at $P = r^2 \sin\theta d\phi d\theta dr$; and if ρ be the density at a distance a, the density at a distance $r = \rho\left(\frac{r}{a}\right)^n$;

$$\therefore \text{ mass of element at } P = \frac{\rho}{a^n} r^{n+2} \sin\theta d\phi d\theta dr.$$

The limits of r are 0 and a; of θ, 0 and π; of ϕ, $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$;

$$\therefore \bar{x} = \frac{\int_{-\alpha}^{\alpha}\int_0^{\pi}\int_0^{a} r^{n+3}\sin^2\theta\cos\phi d\phi d\theta dr}{\int_{-\alpha}^{\alpha}\int_0^{\pi}\int_0^{a} r^{n+2}\sin\theta d\phi d\theta dr}\left(\alpha = \frac{\pi}{2}\right) = \frac{n+3}{n+4}\cdot\frac{a}{2}.$$

GULDINUS's PROPERTIES.

PROP. *To prove that if any plane figure revolve about an axis lying in its own plane, the content of the solid generated by this figure in revolving through any angle is equal to a prism, of which the base is the revolving figure and height the length of the path described by the centre of gravity of the area of the plane figure.*

88. Let the axis of revolution be the axis of x, and the plane of the revolving figure in its initial position the plane of xy; we shall suppose the figure to be wholly on one side of the axis of x: θ the angle through which the figure revolves.

Then the elementary area $dxdy$ of the plane figure in revolving through an angle $d\theta$, generates the elementary solid whose volume is $yd\theta dxdy$; therefore whole solid

$$= \int_0^{\theta}\iint yd\theta dxdy,$$

(the limits of x and y depend upon the equation to the curve)

$$= \theta\iint ydxdy \text{ between proper limits.}$$

But if $\bar{y}$ be the ordinate to the centre of gravity of the plane figure, then by Art. 87. Ex. 7.

$$\bar{y} = \frac{\iint y\,dx\,dy}{\iint dx\,dy},$$

the limits the same as before;

therefore whole solid $= \theta \iint y\,dx\,dy = \bar{y}\theta \,.\, \iint dx\,dy$

$=$ arc descd. by centre of gravity $\times$ area of figure.

Hence the Prop. is true.

PROP. *To prove that the surface of the solid generated is equal in area to the rectangle of which the sides are the length of the perimeter of the generating figure and the length of the path of the centre of gravity of the perimeter.*

89. The surface generated by the arc ds of the figure revolving through an angle $d\theta$ equals $y\,d\theta\,ds$;

$\therefore$ whole surface $= \int_0^\theta \int y\,d\theta\,ds = \theta \int y\,ds$ between proper limits.

But $\bar{y}$ = ordinate to centre of gravity of perimeter

$= \dfrac{\int y\,ds}{\int ds}$ between same limits as before;

therefore whole surface $= \bar{y}\theta \,.\, \int ds$

$=$ arc descd. by centre of gravity $\times$ length of perimeter.

Hence the Prop. is true.

90. It is evident that these theorems are true also when the generating figure is bounded by a line not of continuous curvature.

Ex. 1. *To find the solid content and the surface of the ring of an anchor.*

Let the radius of the axis be a, and the radius of a transverse section be b: then the length of the path of the

centre of gravity of the area of the generating figure $= 2\pi a$, and the area of the figure $= \pi b^2$;

$$\therefore \text{ content of solid} = 2\pi^2 a b^2.$$

Also path of centre of gravity of the perimeter $= 2\pi a$, and the length of the perimeter $= 2\pi b$;

$$\therefore \text{ surface} = 4\pi^2 a b.$$

Ex. 2. *To find the centre of gravity of the area and also of the arc of a semi-circle.*

A semi-circle by revolving about its diameter generates a sphere: the content of the sphere $= \frac{4}{3}\pi a^3$, a the radius: the surface $= 4\pi a^2$:

the area of the semi-circle $= \frac{1}{2}\pi a^2$; the perimeter $= \pi a$;

therefore distance of centre of gravity of area from diameter

$$= \frac{\text{content of sphere}}{2\pi \,.\, \text{area of } \frac{1}{2} \text{ circle}} = \frac{4a}{3\pi};$$

and distance of centre of gravity of arc from diameter

$$= \frac{\text{surface of sphere}}{2\pi \,.\, \text{arc of } \frac{1}{2} \text{ circle}} = \frac{2a}{\pi}.$$

CHAPTER V.

MACHINES. FRICTION.

91. A Machine is an instrument, or a system of solid bodies, for the purpose of transmitting force from one part to another of the system.

It would be endless to describe all the machines that have been invented; we shall consequently confine ourselves to those of simple construction. The most simple species of machines are denominated the Mechanical Powers. These we shall explain, and also a few combinations of them.

92. A *Lever* is an inflexible rod moveable only about a fixed axis; which is called the *fulcrum*. The portions of the lever into which the fulcrum divides it are called the *arms* of the lever: when the arms are in the same straight line, it is called a *straight lever*; in other cases a *bent lever*.

Two forces act upon the lever about the fulcrum, called the *power* and the *weight*: the power is the force applied by the hand (or other means) to sustain or overcome the other force, or the weight. There are three species of levers: the first has the fulcrum between the power and weight; in the second the weight acts between the fulcrum and the power; and in the third the power acts between the fulcrum and the weight.

Prop. *To find the conditions of equilibrium of two forces acting in the same plane on a lever.*

93. Let the plane of the paper be the plane in which the forces act, and also be perpendicular to the axis, of which C is the projection, and about which the lever can move (fig. 31.); A, B the points of application of the forces P, W; α, β the angles which the directions of the forces make with any line aCb drawn through C on the paper. Let R be the pressure

upon the fulcrum, and θ the angle which it makes with the line aCb; then if we apply a force R in the direction CR, we may suppose the fulcrum removed, and the body to be held in equilibrium by the forces P, W, R.

We shall resolve these forces in directions parallel and perpendicular to aCb; and also take their moments about C; then, by Art. 49, we have the following equations of condition:

$$P\cos\alpha - W\cos\beta - R\cos\theta = 0 \ldots\ldots\ldots (1),$$
$$P\sin\alpha + W\sin\beta - R\sin\theta = 0 \ldots\ldots\ldots (2),$$
$$\text{and } P.CD - W.CE = 0 \ldots\ldots\ldots (3).$$

CD and CE being drawn perpendicular to the directions of P and W.

These three equations determine the ratio of P to W when there is equilibrium; and the magnitude and direction of the pressure on the fulcrum.

For equation (3) gives

$$\frac{P}{W} = \frac{CE}{CD} = \frac{\text{perpendicular on direction of } W}{\text{perpendicular on direction of } P}.$$

Also by transposing the last terms of (1) and (2), we have

$$R\cos\theta = P\cos\alpha - W\cos\beta,$$
$$R\sin\theta = P\sin\alpha + W\sin\beta.$$

Add their squares;

$$\therefore R^2 = P^2 + W^2 - 2PW\cos(\alpha + \beta),$$

which gives the magnitude of R.

Take the ratio of the above equations;

$$\therefore \tan\theta = \frac{P\sin\alpha + W\sin\beta}{P\cos\alpha - W\cos\beta},$$

which gives the direction of the pressure.

If we suppose B to be the fulcrum and take the moments about B instead of C, we have instead of equation (3) the following*:

* This is not a new equation of condition; but is a consequence of the three already given, (1), (2), (3). To shew this imagine AD and BE produced to meet CR:

$$\frac{P}{R} = \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P}.$$

It follows, then, that *the condition of equilibrium in a lever of any species is that the two forces must be inversely as the perpendiculars drawn upon their directions from the fulcrum.*

94. This property of the lever renders it a useful instrument in multiplying the power of a force. For any two forces, however unequal in magnitude, may be made to balance each other simply by fixing the fulcrum so that the ratio of its distances from the directions of the forces shall be equal to the ratio of the forces; an adjustment which can always be made. If the fulcrum be moved from this position, then that force will preponderate from which the fulcrum is moved and the equilibrium will be destroyed. We are thus led to understand how mechanical advantage is gained by using a crow-bar to move heavy bodies, as large blocks of stone: a poker to raise the coals in a grate: scissors, shears, nippers, and pincers; these last consisting of two levers of the first kind. The brake of a pump is a lever of the first kind. In the Stanhope printing-press we have a remarkable illustration of the mechanical advantage that can be gained by levers. The frame-work in which the paper to be printed is fixed, is acted upon by the shorter arm of a lever, the other arm being connected to a second lever, the longer arm of which is worked by the pressman. These levers are so adjusted that at the instant the paper comes in contact with the types, the perpendiculars from the fulcra upon the directions of the forces acting at the shorter arms are exceedingly short, and consequently the levers multiply the force exerted by the pressman to an enormous extent.

CR: they will meet this line in the same point, since the distances by these two constructions are CD cosec $(\theta-\alpha)$ and CE cosec $(\theta+\beta)$; and these are made equal, by equations (1), (2), (3), if we eliminate P, W. Suppose, then, F to be the point in which these lines meet. By multiplying (1), (2), respectively by $\sin\beta$ and $\cos\beta$, and adding, we have

$$\frac{P}{R} = \frac{\sin(\theta+\beta)}{\sin(\alpha+\beta)} = \frac{FB\sin(\theta+\beta)}{FB\sin(\alpha+\beta)} = \frac{\text{perpendicular on direction of } R}{\text{perpendicular on direction of } P}:$$

therefore this equation is a consequence of the equations (1), (2), (3), as might have been anticipated.

As examples of levers of the second kind, we may mention a wheelbarrow, an oar, a chipping-knife, a pair of nutcrackers.

It must be observed, however, that as the lever moves about the fulcrum the space through which the weight is moved is, in the first and second species of lever, smaller than the space passed through by the power: and therefore what is gained in power is lost in despatch. For example in the case of the crow-bar: to raise a block of stone through a given space by applying the hand at the further extremity of the lever, we must move the hand through a greater space than that which the weight describes.

But in the third species of lever the reverse is the case. The power is nearer the fulcrum than the weight, and is consequently greater; but the motion of the weight is greater than that of the power. In this kind of lever despatch is gained at the expense of power. An excellent example is the treddle of a turning lathe. But the most striking example of levers of the third kind is found in the animal frame, in the construction of which it seems to be a prevailing principle to sacrifice power to readiness and quickness of action. The limbs of animals are generally levers of this description. The condile of the bone rests in its socket as the fulcrum; a strong muscle attached to the bone near the condile is the power, and the weight of the limb together with any resistance opposed to its motion is the weight. A slight contraction of the muscle gives a considerable motion to the limb. A drawing of the human arm is given as an illustration of these remarks: (fig. 32.)

95. The lever is applied to determine the weight of substances. Under this character it is called a Balance. The Common Balance has its two arms equal, with a scale suspended from each extremity; the fulcrum being *above* the line joining the extremities of the arms. The substance to be weighed is placed in one scale, and weights placed in the other till the beam remains in equilibrium in a perfectly horizontal position; in which case the weight of the substance is indicated by the weights by which it is balanced. If the weights differ ever so slightly, the horizontality of the beam will be disturbed, and after oscillating for some time (in consequence of the fulcrum

being placed *above* the line joining the points of support of the scales) it will, on attaining a state of rest, form an angle with the horizon, the extent of which is a measure of the sensibility of the balance.

In the construction of a balance the following requisites should be attended to. 1. When loaded with equal weights the beam should be perfectly horizontal. 2. When the weights differ, even by a slight quantity, the *sensibility* should be such as to detect this difference. 3. When the balance is disturbed it should readily return to its state of rest, or it should have *stability*. We shall now consider how these may be fulfilled.

PROP. *To find how the requisites of a good balance may be satisfied.*

96. Let P and Q be the weights in the scales (fig. 33.): $AB = 2a$: C the fulcrum : h its distance from the line joining A, B: W the weight of the beam and scales: k the distance of the centre of gravity of these (*i. e.* of the point of application of W) from C measured downwards: θ the angle the beam makes with the horizon when there is equilibrium.

Let us take the moments of P, Q, W about C: their sum equals zero since there is equilibrium (Art. 49.) Then

since the distance of P's direc. from $C = a\cos\theta - h\sin\theta$

.... Q's .... $= a\cos\theta + h\sin\theta$

.... W's .... $= k\sin\theta$,

we have

$$P(a\cos\theta - h\sin\theta) - Q(a\cos\theta + h\sin\theta) - Wk\sin\theta = 0;$$

$$\therefore \tan\theta = \frac{(P-Q)a}{(P+Q)h + Wk}.$$

This determines the position of equilibrium. The first requisite—the horizontality when P and Q are equal—is satisfied by making the arms equal.

For the second we observe that for a given difference of P and Q the sensibility is greater the greater $\tan\theta$ is ; and for a given value of $\tan\theta$, the sensibility is greater the smaller the

difference of P and Q is: hence $\dfrac{\tan\theta}{P-Q}$ is a correct measure of the sensibility: and therefore the second requisite is fulfilled by making $(P+Q)\dfrac{h}{a} + W\dfrac{k}{a}$ as small as possible.

The stability is greater the greater the moment of the forces which tend to restore the equilibrium when it is destroyed. Suppose $P = Q$, then P and Q may be placed at the mid-point between A and B: and the moment of the forces tending to restore equilibrium equals $\{(P+Q)h + Wk\}\sin\theta$. Hence to satisfy the third requisite, this must be made as large as possible. This is, in part, at variance with the second requisite. They may, however, both be satisfied by making $(P+Q)h + Wk$ large, and a large also: that is, by increasing the distances of the fulcrum from the beam and from the centre of gravity of the beam and scales, and by lengthening the arms.

It must be remarked that the sensibility of a balance is of more importance than the stability, since the eye can judge pretty accurately whether the index of the beam makes equal oscillations on each side of the vertical line; that is, whether the position of rest would be horizontal: if this be not the case, then the weights must be altered till the oscillations are nearly equal.

97. Another kind of balance is that in which the arms are unequal, and the same weight is used to weigh different substances by varying its point of support, and observing its distance from the fulcrum by means of a graduated scale. The common steelyard is of this description.

Prop. *To shew how to graduate the common steelyard.*

98. Let AB be the beam of the steelyard (fig. 34.) A the fixed point from which the substance to be weighed is suspended, Q being its weight: C the fulcrum: W the weight of the beam together with the hook or scale-pan suspended from A; G the centre of gravity of these.

Suppose that P suspended at N balances Q suspended from A, then taking the moments of P, Q, W about C, we have

$$Q \cdot AC - W \cdot CG - P \cdot CN = 0;$$

$$\therefore Q = \frac{CN + \frac{W}{P} \cdot CG}{AC} P.$$

Take the point D, so that $CD = \frac{W}{P} CG$;

$$\therefore Q = \frac{CN + CD}{AC} P = \frac{DN}{AC} P.$$

Now let the arm DB be graduated by taking Da_1, Da_2, Da_3,......equal respectively to AC, $2AC$, $3AC$,......let the figures 1, 2, 3, 4.......be placed over the points of graduation, and let subdivisions be made between these. Then by observing the graduation at N we know the ratio of Q to P; and this latter being a given weight we know the weight of Q. In this way any substance may be weighed.

99. There is a remarkable balance called, after its inventor, Roberval's Balance: a representation of it is given in fig. 35. DC' is a frame of which the opposite sides are equal, and the extremities are connected by pins at D, C, D', C' so as to allow of free motion: this frame is supported by a stand $EE'A$, being connected to it by pins at E and E' so as to allow of free motion about those points: EE' must be parallel to DC and $D'C'$, but not necessarily equidistant from them: arms are fixed at right angles to the sides DD', CC', to support weights Q and P. The peculiarity of this machine is, that if P and Q balance in any given position on the horizontal arms, the equilibrium will remain undisturbed if we shift P or Q or both of them along their arms in either direction: also if we push one arm down and consequently raise the other the whole will remain at rest in the position in which it is left. We shall prove these facts, and explain the paradoxical character of the machine in the Chapter of Problems. We may however easily prove by the Principle of Virtual Velocities the facts mentioned above, though the paradox will not be removed.

If we lower the arm on which P acts through a space x, then D' sinks through a space x; and D, and therefore the arm on which Q acts, rises through a space $\frac{a'x}{a}$, which is independent of the distances of P and Q along their arms: a and a' are the lengths DE and ED'.

$$\text{Then } P.x - Q.\frac{a'x}{a} = 0 \text{ by Virtual Velocities,}$$

$$\text{or } \frac{P}{Q} = \frac{a'}{a},$$

for all positions of the frame and of the weights.

It will be seen upon referring to the Chapter of Problems, that although the equilibrium remains undisturbed when P and Q have different positions, yet the strains at the joints D, D', C, C', E, E', and the point of application (B in figure) of the downward-pressure undergo changes.

It is on the principle of this machine that the balances used of late years in shops are constructed: the scales rest each by one point upon the extremities of a lever below them, and the only motion they are capable of is in a vertical direction.

100. The second of the Mechanical Powers is the Wheel and Axle. This machine consists of two cylinders fixed together with their axes in the same line: the larger is called the wheel, and the smaller the axle: the axis of the axle is generally much larger than that of the wheel. The cord by which the weight is suspended is fastened to the axle, and then coiled round it, while the power which supports the weight acts by a cord coiled round the circumference of the wheel; by spokes acted on by the hand, as in the *capstan*; or by the hand acting on a handle, as in the *windlass*.

Prop. *To find the ratio of the power and weight in the Wheel and Axle when in equilibrium.*

101. Let AD be the wheel and $CC'B$ the axle (fig. 36.) P the power, represented by a weight suspended from the cir-

cumference of the wheel at A: W the weight hanging from the axle at B.

Then since the axis of the machine is fixed, the condition of equilibrium is that the sum of the moments of the forces about this axis vanishes (Art. 60.);

$$\therefore P . AC - W . C'B = 0;$$

$$\therefore \frac{W}{P} = \frac{AC}{BC'} = \frac{\text{rad. of wheel}}{\text{rad. of axle}}.$$

It will be seen that this machine is only a modification of the lever. In short it is an assemblage of levers all having the same axis: and as soon as one has been in action the next comes into play; and in this way an endless leverage is obtained. In this respect, then, the wheel and axle surpasses the common lever in mechanical advantage. It is much used in docks, and in shipping.

102. The third Mechanical Power is the Toothed Wheel. It is extensively applied in all machinery; in cranes, steam-engines, and particularly in clock and watch work. If two circular hoops of metal or wood having their outer circumferences indented, or cut into equal teeth all the way round, be so placed that their edges touch, one tooth of one circumference lying between two of the other (as represented in the figure 37.); then if one of them be turned round by any means, the other will be turned round also. This is the simple construction of a pair of toothed wheels.

Prop. *To find the relation of the power and weight in Toothed Wheels.*

103. Let A and B be the fixed centres of the toothed wheels on the circumferences of which the teeth are arranged, fig. 37: C the point of contact of two teeth: QCQ a normal to the surfaces in contact at C. Suppose an axle is fixed on the wheel B, and the weight W suspended from it at E by a cord: also suppose the power P acts by an arm AD: draw Aa, Bb perpendicular to QCQ. Let the mutual pressure at C be Q. Then since the wheel A is in equilibrium about the fixed axis A, the sum of the moments about A equals zero:

$$\therefore P.AD - Q.Aa = 0.$$

Also since the wheel B is in equilibrium about B, the sum of the moments about B equals zero:

$$\therefore Q.Bb - W.BE = 0.$$

Then by eliminating Q from these two equations,

$$\frac{P}{W} = \frac{P}{Q} \cdot \frac{Q}{W} = \frac{Aa}{AD} \cdot \frac{BE}{Bb};$$

$$\text{or } \frac{\text{moment of } P}{\text{moment of } W} = \frac{Aa}{Bb};$$

when the teeth are small this ratio

$$= \frac{\text{rad. of wheel } A}{\text{rad. of wheel } B} \text{ very nearly.}$$

104. Wheels are in some cases turned by means of straps passing over their circumferences. In such cases the minute protuberances of the surfaces prevent the sliding of the straps, and a mutual action takes place such as to render the calculation exactly analogous to that in the Proposition.

For the calculation of the best forms for the teeth, the reader is referred to a Paper of Mr Airy's, in the Camb. Phil. Trans. Vol. II. p. 277.

105. The fourth Mechanical Power is the Pully. There are several species of pullies: we shall mention them in order. The simple pully is a small wheel moveable about its axis: a cord passes over part of its circumference. If the axis is fixed the effect of the pully is only to change the direction of the cord passing over it: if, however, the axis be moveable, then, as will be presently seen, a mechanical advantage may be gained.

PROP. *To find the ratio of the power and weight in the single moveable Pully.*

106. I. Suppose the parts of the cord divided by the pully are parallel (fig. 38.)

Let the cord ABP have one extremity fixed at A, and after passing under the pully at B suppose it held by the hand exerting a force P. The weight W is suspended by a cord from the centre C of the pully.

Now the tension of the cord ABP is the same throughout. Hence the pully is acted on by three parallel forces, P, P, and W: hence

$$2P - W = 0; \quad \therefore \frac{W}{P} = 2.$$

II. Suppose the portions of cord are not parallel (fig. 39).

Let a and a' be the angles which Aa and Pb make with the vertical.

Now the pully is held in equilibrium by W in CW, P in aA, P in bP. Hence by Art. 49,

$$\text{horizontal forces, } P \sin a - P \sin a' = 0 \ldots (1);$$

$$\text{and vertical forces, } P \cos a + P \cos a' - W = 0 \ldots (2),$$

the equation of moments is an identical equation.

$$\text{By (1), } \sin a = \sin a' \text{ and } a = a';$$

$\therefore$ by (2), $\dfrac{W}{P} = 2 \cos a$ which is the relation required.

PROP. *To find the ratio of the power and weight in a system of pullies, in which each pully hangs by a separate string, one end being fastened in the pully above it and the other end on a fixed beam: all the strings being parallel.*

107. Let n be the number of pullies (fig. 40).

I. Let us neglect the weights of the pullies themselves.

Then the tension of $b_1 W = W$; $\therefore$ the tension of $a_1 b_1 b_2 = \frac{1}{2} W$;

$$\therefore \text{ tension of } a_2 b_2 b_3 = \frac{1}{2^2} W, \text{ tension of } a_3 b_3 c = \frac{1}{2^3} W,$$

and so on; and the tension of the string passing under the n^{th} pully $= \dfrac{1}{2^n} W$, and this $= P$;

$$\therefore \frac{W}{P} = 2^n.$$

II. Let us suppose the weights of the pullies to be considered: and let $\omega_1 \omega_2 \omega_3 \ldots \omega_n$ be these weights.

Then if $p_1 p_2 p_3 \ldots p_n$ be the weights which they would sustain at P, and P_1 the weight W would sustain at P, we have

$$p_1 = \frac{\omega_1}{2^n},\ p_2 = \frac{\omega_2}{2^{n-1}}, \ldots\ldots p_n = \frac{\omega_n}{2},\ P_1 = \frac{W}{2^n};$$

$$\therefore\ P = p_1 + p_2 + \ldots\ldots + p_n + P_1,$$

$$\text{or } P = \frac{1}{2^n}\left\{W + \omega_1 + 2\omega_2 + 2^2\omega_3 + \ldots\ldots + 2^{n-1}\omega_n\right\}.$$

If $\omega_1 = \omega_2 = \omega_3 = \ldots\ldots = \omega_n$,

$$P = \frac{1}{2^n}\left\{W + (2^n - 1)\,\omega_1\right\}.$$

PROP. *To find the ratio of the power and weight when the system is the same as in the last Proposition; but the strings are not parallel.*

108. We shall neglect the weights of the blocks. The pullies will evidently so adjust themselves that the string at their centre will bisect the angle between the strings touching their circumference.

Let $2\alpha_1, 2\alpha_2, 2\alpha_3 \ldots\ldots 2\alpha_n$ be the angles included between the strings touching the first, second, third,......n^{th} pullies respectively: (fig. 41.)

Then, by Art. 106, tension of $a_1 b_1 b_2 = \dfrac{W}{2\cos\alpha_1}$;

$$\text{therefore tension of } a_2 b_2 b_3 = \frac{W}{2^2 \cos\alpha_1 \cos\alpha_2},$$

$$\text{tension of } a_3 b_3 c = \frac{W}{2^3 \cos\alpha_1 \cos\alpha_2 \cos\alpha_3},$$

$$\text{tension of the last string} = \frac{W}{2^n \cos\alpha_1 \cos\alpha_2 \cos\alpha_3 \ldots\ldots \cos\alpha_n},$$

and this $= P$;

$$\therefore\ \frac{W}{P} = 2^n \cos\alpha_1 \cos\alpha_2 \cos\alpha_3 \ldots\ldots \cos\alpha_n.$$

PROP. *To find the relation of the power and weight in a system of pullies where the same string passes round all the pullies.*

109. This system consists of two blocks; each containing a number of the pullies with their axes coincident. The weight is suspended from the lower block, which is moveable, and the power acts at the loose extremity of the string, which passes round the respective pullies of the upper and lower block alternately.

Since the same string passes round all the pullies, its tension will be everywhere the same, and equal to the power P. Let n be the number of portions of string at the lower block; then $n \cdot P$ will be the sum of their tensions;

$$\therefore W = n \cdot P.$$

If we take into account the weight of the lower block, and call it B, then

$$W + B = n \cdot P.$$

If the strings at the lower block are not vertical, we must take the sum of the parts resolved vertically, and equate it to W. But, in general, this deviation from the vertical is so slight, that it is neglected.

110. As the weight is rising or falling, it will be observed that in general the pullies move with different angular motions. The degree of angular motion of each pully depends upon the magnitude of its radius. Mr James White took advantage of this, to choose the radii of the pullies in such a manner, as to give those in the same block the same angular motion, and so to prevent the wear and resistance caused by the friction of the pullies against each other. This being the case, the pullies in each block might be fastened together: or, instead of this, cut out of one mass.

It will be seen without much difficulty, that if the weight W be raised through a space a, each of the portions of string between the two blocks will be shortened by the length a; and, therefore, that the portions of string, which move over the pullies in the two blocks taken alternately, will have their

lengths equal to a, $2a$, $3a$, $4a$,......If, then, the radii of the pullies of the upper block be proportional to the odd numbers 1, 3, 5,......these pullies will move with the same angular velocity, and might be made all in one piece, as mentioned above. And if the radii of the lower pullies be proportional to the even integers 2, 4, 6,......these also will move with a common angular velocity, and might therefore be cut out of one piece.

PROP. *To find the ratio of the power to the weight when all the strings are attached to the weight.*

111. If we neglect the weight of the pullies (fig. 42.) the tension of the strings $b_1 a_1 = P$; the tension of $a_2 b_2 = 2P$; and so on: if there be n pullies, then the sum of the tensions of the strings attached to the weight

$$= P + 2P + 2^2 P + \ldots\ldots + 2^{n-1} P = (2^n - 1)\ P;$$

$$\therefore \frac{W}{P} = 2^n - 1.$$

If we suppose the weights of the pullies are $\omega_1 \omega_2 \omega_3 \ldots\ldots\ldots$ reckoning from the lowest, and $\omega' \omega'' \omega''' \ldots\ldots$ the portions of W which they respectively support (since they evidently assist P), and W' the portion of W supported by P; then

$$W' = (2^n - 1)\ P,$$

$$\omega' = (2^{n-1} - 1)\ \omega_1,$$

$$\omega''' = (2^{n-2} - 1)\ \omega_2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\omega^{(n-1)} = (2 - 1)\ \omega_{n-1};$$

$$\therefore\ W = W' + \omega' + \ldots\ldots = (2^n - 1)\ P + (2^{n-1} - 1)\ \omega_1$$
$$+ (2^{n-2} - 1)\ \omega_2 + \ldots\ldots + (2 - 1)\ \omega_{n-1}.$$

If $\omega_1 = \omega_2 = \omega_3 \ldots\ldots\ldots$

$$W = (2^n - 1)\ P + \left\{2^{n-1} + 2^{n-2} + \ldots\ldots + 2 - (n-1)\right\} \omega_1$$
$$= (2^n - 1)\ P + (2^n - n - 1)\ \omega_1.$$

112. The fourth Mechanical Power is the Inclined Plane.

By an inclined plane we mean a plane inclined to the horizon. A weight W may be supported on an inclined plane by a power P less than W.

Prop. *To find the ratio of the power and weight in the inclined plane.*

113. Let AB be the inclined plane (fig. 43.): α the angle it makes with the horizon. Let the power P act on the weight in the direction CP, making an angle ϵ with the plane. Now the weight at C is held at rest by P in CP, W in the vertical CW, and a pressure R in CR, at right angles to the plane.

Hence, by Art. 23, if we resolve these forces perpendicular and parallel to the plane, we have

$$R + P\sin\epsilon - W\cos\alpha = 0 \;\ldots\ldots\; (1),$$

$$P\cos\epsilon - W\sin\alpha = 0 \;\ldots\ldots\; (2).$$

The second gives the required relation $\dfrac{P}{W} = \dfrac{\sin\alpha}{\cos\epsilon}$: and the first equation gives the magnitude of the pressure R.

Cor. 1. If P act horizontally, $\epsilon = -\alpha$, and $P = W\tan\alpha$.

Cor. 2. If P act parallel to the plane, $\epsilon = 0$, $P = W\sin\alpha$.

Cor. 3. If P act vertically, $\epsilon = \frac{1}{2}\pi - \alpha$, $P = W$.

114. The fifth Mechanical Power is the Wedge. This is a triangular prism, and is used to separate obstacles by introducing its edge between them and then thrusting the wedge forward. This is effected by the blow of a hammer or other such means, which produces a violent pressure for a short time, sufficient to overcome the greatest forces.

Prop. *An isosceles wedge being introduced between two obstacles, required to find its tendency to separate the obstacles when the wedge is prevented from being thrust back by a given force.*

115. Let $2P$ be the force acting at the back of the wedge (fig. 44). In the figure we suppose the obstacles to be the two

halves of a tree. The portions of the tree we suppose similarly situated on the two sides of the wedge: let A and A' be the points of contact between the wedge and the obstacles: AN, $A'N'$ normals to the wedge at A and A': R, R the mutual resistances of the wedge and obstacles at A and A'.

Now if the wedge were to move backwards or to be thrust forwards the points A and A' would move in some unknown curve line: the nature of this curve would depend upon the elasticity and strength of the material of the obstacles and upon other circumstances. Draw AT, $A'T'$ tangents to these curves at the points A and A'. Then it will be seen, that the parts of the pressures at A and A', which measure the tendency of the obstacles to separate, will be their resolved parts along these tangent lines; since if they separate it must be by A and A' moving along these lines. The resolved parts perpendicular to these tangents are counteracted by the resistance of the ground at E.

Let W be the resolved part of R along the tangent on either side: and suppose the angle $NAT = i$. Also let the angle of the wedge be 2α.

Then the wedge being sustained by the forces $2P$, R and R; we have by resolving them vertically,

$$2P - 2R \sin \alpha = 0 \text{ .... } (1);$$

the horizontal parts counteract each other of necessity, also the equation of moments is an identical equation.

$$\text{Again } W = R \cos i \text{ .... } (2);$$

$$\therefore \frac{P}{W} = \frac{\sin \alpha}{\cos i}.$$

If, then, we know the angle i we shall know W: but we have no means of ascertaining the value of i, and consequently the preceding calculation is of little importance.

When i is very small, then $W = \dfrac{P}{\sin \alpha}$ nearly.

116. The last Mechanical Power is the Screw.

This machine in its simple construction consists of a cylinder (fig. 45.) AB with a uniform projecting thread $abcd$...

traced round its surface, and making a constant angle a with lines parallel to the axis of the cylinder. This cylinder fits into a block D pierced with an equal cylindrical aperture, on the inner surface of which is cut a groove the exact counterpart of the projecting thread $abcd$.

It is easily seen from this description, that when the cylinder is introduced into the block, the only manner in which it can move is backwards or forwards by revolving about its axis, the thread sliding in the groove. Suppose W is the weight acting on the cylinder (including the weight of the cylinder itself) and P is the power acting at the end of an arm AC at right angles to the axis of the cylinder: the block D is supposed to be firmly fixed, and the axis of the cylinder to be vertical.

PROP. *To find the ratio of the power and weight in the Screw when they are in equilibrium.*

117. Let $AC = a$: rad. of cylinder $= b$.

Now the forces which hold the cylinder in equilibrium are W, P and the reactions of the pressures of the various portions of the thread on the corresponding portions of the lower surface of the groove in which the thread rests: these reactions are indeterminate in their number; but they all act in directions perpendicularly to the surface of the groove, and therefore their directions make a constant angle a with a horizontal plane. If, then, R be one of these reactions, $R \sin a$, $R \cos a$ are the resolved parts vertically and horizontally: the horizontal portions of the reactions act each at right angles to a radius of the cylinder. Hence resolving the forces vertically, and also taking the moments of the forces in horizontal planes, we have

$$W - \Sigma . R \sin a = 0 \ldots\ldots\ldots (1),$$

$$Pa - \Sigma . R \cos a b = 0 \ldots\ldots\ldots (2),$$

we might write down the other four equations of equilibrium; but they introduce unknown quantities with which we are unconcerned in our question.

Hence $\dfrac{W}{P} = \dfrac{a \sin a \, \Sigma . R}{b \cos a \, \Sigma . R}$, because b and a are constant:

$$= \frac{a \sin \alpha}{b \cos \alpha} = \frac{2\pi a}{2\pi b \cot \alpha}$$

$$= \frac{\text{circumference of circle of which the rad. is } AC}{\text{vertical dist. between two successive winds of the thread}}.$$

The screw is used to gain mechanical power in many ways. In excavating the Thames Tunnel the heavy iron frame-work, which supported the workmen, was gradually advanced by means of large screws.

FRICTION.

118. In the investigations of this Chapter we have supposed that the surfaces of the bodies in contact are perfectly smooth. Now in practice this is not the fact; for no surface can be so entirely freed from roughness and asperities as to be perfectly smooth, although their effect may in many cases be greatly diminished. By a *smooth* surface is meant a surface which opposes no resistance whatever to the motion of a body upon it, and therefore the resistance is wholly perpendicular to the surface. A surface which does oppose a resistance to the motion of a body upon it is said to be *rough*.

The friction of a body on a surface is measured by the least force which will put the body in motion along the surface.

In the Chapter of Problems examples will be given where friction is taken into account.

Coulomb made a series of experiments upon the friction of bodies against each other and deduced the following laws: *Mémoires des Savans Etrangers*, Tom. x.

(1) *The friction varies as the pressure, when the materials of the surfaces in contact remain the same.* When the pressures are very great indeed it is found that the friction is somewhat less than this law would give.

(2) *The friction is independent of the extent of the surfaces in contact so long as the pressure remains the same.* When the surfaces in contact are extremely small, as for in-

stance a cylinder resting on a surface, this law gives the friction much too great.

These two laws are true when the body is on the point of moving and also when it is actually in motion: but in the case of motion the magnitude of the friction is much less than when the body is in a state bordering upon motion.

(3) *The friction is independent of the velocity when the body is in motion.*

It follows from these laws that if P be the normal pressure of the body upon the surface then the friction $= \mu . P$, where μ is a constant quantity for the same materials, and is called *the coefficient of friction.*

In the state bordering on motion and when the surfaces in contact are of finite extent, we have the following results from experiment:

$\mu = \frac{1}{2}$ surfaces wood, the grain being in same direction.

$= \frac{1}{4}$ .... opposite ......

$= \frac{1}{4}$ metallic surfaces.

$= \frac{1}{5}$ one surface wood and the other metal.

Oil and grease considerably diminish friction; fresh tallow reduces it to half its value.

In the state bordering on motion and when the surfaces in contact are single lines, then $\mu = \frac{1}{12}$ for wood. When the surface in contact is a physical point the statical friction is inconsiderable.

But for full particulars on this subject we refer the reader to Coulomb's papers, and also to two Memoirs recently published in the *Mémoires de l'Institut.* by M. Morin.

PROP. *To find the greatest angle which the direction of the mutual pressure of two surfaces in contact may make with their common normal at the point where the pressure acts without sliding; the coefficient of friction being given.*

119. Let P be the mutual pressure, its direction making an angle β with the normal. Then $P \cos \beta$ is the *direct* or normal pressure of the surfaces, and $P \sin \beta$ is the force balanced by the friction acting wholly or in part.

Hence μ is the greatest value of the ratio $\dfrac{P \sin \beta}{P \cos \beta}$;

$$\therefore \ \beta = \tan^{-1} \mu \text{ is the greatest value of } \beta.$$

CHAPTER VI.

ROOFS, ARCHES AND BRIDGES.

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120. In the present Chapter we shall apply the principles of equilibrium to explain in what manner the thrusts, strains, and pressures in general act in roofs and arches. We refer the reader to Robison's *Mechanical Philosophy*, Vol. I. for two Articles on Roofs and Arches, which contain many interesting details, which would be entirely out of place in these pages.

Prop. *In a simple isosceles truss-roof required to calculate the tension of the tie-beam.*

121. Let AB, BC be two beams of the roofing connected by the tie-beam AC (fig. 46.): the truss resting on walls, as drawn in the figure. Let B be the weight of each sloping beam and the portion of the tiling or thatching supported by the beam: let G be the point at which this weight acts. Also suppose, that the weight on the vertex arising from other appendages equals W: let a be the angle the roof makes with the horizon: $AG = b$, $AB = a$.

Now the forces acting on AB are a pressure at A perpendicular to the wall, $= R$ suppose, the tension of the tie-beam acting at A in the direction AC, $= T$ suppose, the weight B acting vertically at G: and lastly, some force P acting at B in direction BP making an unknown angle θ with the beam in the plane of the paper, and arising from the weight W and the action of the beam BC.

In order to find the connexion between P and W, we remark that the point B is held at rest by the force W downwards, and the two reactions P, P acting along the dotted lines.

For the equilibrium of AB we have

$$\text{horizontal forces} = T - P\cos(a-\theta) = 0 \ldots\ldots\ldots (1),$$

$$\text{vertical forces} = R - B - P\sin(a-\theta) = 0 \ldots\ldots\ldots (2),$$

$$\text{moments about } A = B \,.\, b\cos a - P \,.\, a\sin\theta = 0 \ldots\ldots\ldots (3).$$

For the equilibrium of the point B,

$$\text{vertical forces} = 2P\sin(a-\theta) - W = 0 \ldots\ldots\ldots (4).$$

Here, then, we have four equations and four unknown quantities T, P, R, θ: and therefore we can determine the unknown quantities, and therefore T.

By (1) (4) $T = \frac{1}{2} W\cot(a-\theta)$, eliminating P:

and by (3) (4), eliminating P, we have

$$Bb\cos a = \tfrac{1}{2} Wa\frac{\sin\theta}{\sin(a-\theta)} = \tfrac{1}{2} Wa\frac{\sin\{a-(a-\theta)\}}{\sin(a-\theta)}$$

$$= \tfrac{1}{2} Wa\{\sin a\cot(a-\theta) - \cos a\};$$

$$\therefore \cot(a-\theta) = \frac{2Bb + Wa}{Wa}\cot a;$$

$$\therefore T = \frac{2Bb + Wa}{2a}\cot a.$$

This measures the horizontal thrust of the roofing against the supporting walls supposing the tie-beams to give way: and we learn that this will be less the larger a is, or the steeper the roof is, the other quantities remaining the same. Also the smaller b is in proportion to a, or the nearer G is to A, the smaller is this thrust.

Prop. *To explain the manner in which buttresses act in supporting a roof: and to calculate their angle of elevation.*

122. Let (as before) AB, BC (fig. 47.) be two beams of the roofing: AD a piece of timber firmly attached to AB running down the inside of the wall, and resting on a *corbel* E: FH a beam to strengthen the attachment of AD, AB. Let R be the pressure on the top of the wall and corbel; N the point at which the resultant of the reactions of all the hori-

zontal pressures on the wall acts; T this resultant; $AN = x$; G the point through which the weight on the slanting timbers acts; $AG = b$, $AB = a$.

Now the action of the beams on each other at B must be in a horizontal direction, since we suppose there is no extra weight acting at B, as in the last proposition; let P be this mutual pressure.

For the equilibrium of BAD,

$$\text{horizontal forces} = T - P = 0 \quad\text{.........................}\quad (1),$$

$$\text{vertical forces} = R - B = 0 \quad\text{.........................}\quad (2),$$

$$\text{moments about } A = Tx - Bb\cos a + Pa\sin a = 0 \quad\text{......}\quad (3).$$

Here we have three equations and four unknown quantities T, P, R, x: and another relation connecting these quantities cannot be found; hence the problem is indeterminate: we shall see in the solution what quantities are indeterminate.

By (2) $R = B$, and is therefore not indeterminate.

By (1) (3), eliminating P, we have

$$T = \frac{Bb\cos a}{x + a\sin a}, = P \text{ by } (1).$$

Hence P, T, and x are indeterminate: but when a value is given to one of them, then, that the equilibrium may subsist, the other two must satisfy the two conditions just deduced.

Let ϕ be the angle which the resultant of R and T makes with the vertical; then

$$\tan\phi = \frac{T}{R} = \frac{b\cos a}{x + a\sin a}.$$

Now our object is to find the least angle at which a buttress need be built to support the roof. If the roof be on the point of sinking it must be so by tending to turn about the extremities D, D of the framework: in which case the force T acts at D, and x will then have its greatest value: also we perceive that both T and also ϕ are smaller the greater x is: hence the least angle at which the buttress need be built is given by the equation

$$\tan\phi = \frac{A'G'}{AD + AA'} = \frac{A'G'}{A'D},$$

BA' being horizontal; AA', GG' vertical.

Hence the dotted line $G'D$ represents the limiting angle, which the buttress must make with the vertical in order that the roof may not fall.

This calculation shews the great use of the part of the framework which runs down the wall.

We have drawn in our figure but one connecting beam HF: but there might have been more, and the calculation would have been precisely the same, supposing that ABD is a *rigid* framework: and GG' the vertical line in which the weight of this framework and the superincumbent tiling or other covering acts. The simple rule is to draw a horizontal BG' from the vertex of the roof, cutting GG' in G' and join G' with the lowest point D of the framework: $G'D$ gives the least inclination of the buttress. Also the buttress need not extend higher up the wall than the level of D.

The roofs of Westminster Hall and of Trinity College Hall, Cambridge, are good illustrations of this kind of roof.

Before quitting this subject we will investigate the following Proposition.

PROP. *To calculate the conditions of equilibrium of any number of beams forming a framework in a vertical plane, symmetrical with respect to a vertical line through the highest point.*

123. Let the lengths of the beams be $a_1 a_2 a_3 \ldots$ reckoning from the lowest: $G_1 G_2 G_3 \ldots$ the points at which the weights of the beams and the weights with which they may be loaded act; $b_1 b_2 b_3 \ldots$ the distances of these points from the lower extremities of the beams; $\alpha_1 \alpha_2 \alpha_3 \ldots$ the angles which the beams make with the horizon; R the vertical pressure on the walls of support of each of the two lowest beams; T the horizontal thrust of these beams on the walls; or the tension of the tie-beam connecting them if there be one.

Now the actions of any two beams on each other at the points of junction must be in the same line, since there is no third force to keep them in equilibrium. Let then $P_1 P_2 \ldots$ be these mutual actions between the first and second, the second and third beams, and so on; $\theta_1 \theta_2 \ldots$ the angles which the directions of these forces make with the horizon.

For the equilibrium of the lowest beam,

$$T - P_1 \cos\theta_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$B_1 - R + P_1 \sin\theta_1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$B_1 b_1 \cos\alpha_1 - P_1 a_1 \sin(\alpha_1 - \theta_1) = 0 \ldots\ldots (3).$$

For the equilibrium of the second beam,

$$P_1 \cos\theta_1 - P_2 \cos\theta_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

$$B_2 - P_1 \sin\theta_1 + P_2 \sin\theta_2 = 0 \ldots\ldots\ldots\ldots (5),$$

$$B_2 b_2 \cos\alpha_2 - P_2 a_2 \sin(\alpha_2 - \theta_2) = 0 \ldots\ldots (6),$$

and so on, till we come to the highest beam in which the angle θ_n must $= 0$, since the two highest beams which form the vertex have no third force at their point of junction to keep P_n and P_n in equilibrium. Hence for the last (the n^{th}) beam,

$$P_{n-1} \cos\theta_{n-1} - P_n = 0 \ldots\ldots\ldots\ldots (3n - 2),$$

$$B_n - P_{n-1} \sin\theta_{n-1} = 0 \ldots\ldots\ldots\ldots (3n - 1),$$

$$B_n b_n \cos\alpha_n - P_n a_n \sin\alpha_n = 0 \ldots\ldots (3n).$$

Also we have the following analytical relation connecting $a_1 a_2 \ldots$, $\alpha_1 \alpha_2 \ldots$

$$a_1 \cos\alpha_1 + a_2 \cos\alpha_2 + \ldots + a_n \cos\alpha_n = D \ldots (3n + 1),$$

$2D$ being the distance of the opposite walls from each other.

We have then $3n + 1$ equations from which to eliminate the quantities $P_1 P_2 \ldots P_n$, $\theta_1 \theta_2 \ldots \theta_{n-1}$, R, T, which are $2n + 1$ in number, and we have n equations remaining to determine the n angles $\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n$; and these being known we know the position of equilibrium of the beams.

If we add together all the second equations we have

$$R = B_1 + B_2 + \ldots\ldots + B_n.$$

Now by (2) (3), eliminating P_1, we have

$$\frac{\sin(a_1 - \theta_1)}{\sin\theta_1} = \frac{B_1 b_1 \cos a_1}{(R - B_1)\,a_1};$$

$$\therefore\ \sin a_1 \cot\theta_1 = \left\{1 + \frac{B_1 b_1}{(R - B_1)\,a_1}\right\}\cos a_1;$$

$$\therefore\ \cot\theta_1 = \frac{(R - B_1)\,a_1 + B_1 b_1}{(R - B_1)\,a_1}\cot a_1;$$

$$\therefore\ T = (R - B_1)\cot\theta_1 \text{ by (1) (2)}$$

$$= \frac{(R - B_1)\,a_1 + B_1 b_1}{a_1}\cot a_1 = \left\{B_2 + \ldots + B_n + B_1\frac{b_1}{a_1}\right\}\cot a_1.$$

Again, by adding equations (1), (4); (2) (5) respectively, we have

$$T - P_2\cos\theta_2 = 0,$$

$$B_1 + B_2 - R + P_2\sin\theta_2 = 0,$$

$$\text{also by (6) } B_2 b_2 \cos a_2 - P_2 a_2 \sin(a_2 - \theta_2) = 0.$$

Hence, as by solving (1) (2) (3), we have

$$T = \left\{B_3 + \ldots + B_n + B_2\frac{b_2}{a_2}\right\}\cot a_2,$$

and in the same manner we should obtain

$$T = \left\{B_4 + \ldots + B_n + B_3\frac{b_3}{a_3}\right\}\cot a_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$T = \left\{B_n\frac{b_n}{a_n}\right\}\cot a_n.$$

These n values of T, being equated, give $n - 1$ relations connecting the angles $a_1 a_2 \ldots a_n$; and these combined with equation $(3n + 1)$ determine these angles.

The relation of one angle a_m to the preceding a_{m-1} is given by

$$\tan a_m = \frac{B_{m+1} + \ldots + B_n + B_m \dfrac{b_m}{a_m}}{B_m + \ldots + B_n + B_{m-1} \dfrac{b_{m-1}}{a_{m-1}}} \tan a_{m-1}$$

This shews that every one of the angles $a_1 a_2 \ldots a_n$ is greater or every one less than 90^0 and equation $(3n + 1)$ shews that they must be all less than 90^0, otherwise we should have $D =$ a sum of negative quantities.

Also the beams must be less and less inclined to the horizon as we ascend, since $\tan a_m$ is less than $\tan a_{m-1}$.

124. By an *Arch* is meant an assemblage of bodies supported, as represented in fig. 48, by their mutual pressures and the pressures of the two extreme bodies against fixed obstacles.

We shall suppose the bodies to have the usual form, that of truncated wedges; and to be placed so as to have their sides, which are in contact, perpendicular to the same vertical plane. These bodies are then called *voussoirs:* the highest voussoir is called the *key-stone* of the arch: the surfaces which separate the voussoirs are called the *joints:* the external curve of the arch is called the *extrados;* the internal curve the *intrados:* the solid mass against which the lowest voussoir at each end rests is called the *pier* or *abutment*.

125. It is found in practice, that the friction of the voussoirs against each other is so great, that they are generally incapable of sliding past each other; and in many cases all possibility of sliding is prevented by the voussoirs being *joggled;* that is, being united by a piece of stone or iron which is partly imbedded in one voussoir and partly in the voussoir in contact with it. It would therefore appear, that the conditions of equilibrium of an arch reduce themselves to the condition, that the arch shall not break at any part by the rotation of one voussoir upon another; or, which is the same thing, by the opening of any of the joints.

When, however, we consider, that the artificial connexions of the masonry cannot be so capable of resisting pressure and strain as the masonry itself, it will be evident, that the most stable form of an arch, under all circumstances, will be that, which it must have, if the voussoirs are considered smooth, and the effects of the friction, cohesion of mortar, and all artificial connexions are neglected. If therefore such a form of arch can be discovered as shall satisfy this condition, and at the same time not present counteracting disadvantages (as want of elegance, inconvenience to navigation, and so on), such a form is decidedly to be preferred. We shall shew in the following Articles, that such an arch can be constructed: and shall afterwards shew the important office that friction performs in giving stability to the equilibrium.

PROP. *To explain how an arch is supported, when the voussoirs are supposed smooth: and to calculate the conditions of equilibrium.*

126. Each voussoir is supported by the two pressures against its surfaces at the *joints* and its weight: fig. 48.* Let $G_1 G_2 \ldots\ldots$ be the centres of gravity of the voussoirs, all in the plane of the paper, since we suppose the arch to be symmetrical on each side of that plane. The weights of the voussoirs act in the vertical lines $G_1 b$, $G_2 d \ldots\ldots$ Now suppose, that the pressure of the abutment A against the first voussoir acts at a in the line ab, cutting $G_1 b$ in b: the resultant of this pressure and the weight of the voussoir acts through b in some direction bc, and is counteracted by the pressure of the next voussoir at c. In this manner it may be shewn, that the pressures act in a broken line, of which ab, bcd, def,... are the portions. This line is called *the line of pressure*. And when the voussoirs are smooth, the condition of equilibrium is, that wherever this line crosses a joint it must cut the joint at right angles: otherwise the voussoirs would slide on each other. We shall now put this condition into an analytical form.

* Fig. 48 ought to have been made larger, and the line of pressure more broken at b, d, f...... The reader may easily draw a larger and clearer figure for himself by following the description in the text.

Let $W_1 W_2 W_3$...be the weights of the voussoirs beginning with that most to the left; and $\theta_1 \theta_2 \theta_3$...the angles which the planes of the joints make with a vertical line, beginning from the abutment A. Let P be the mutual pressure of W_1 and W_2 at c. Then, by Art. 19,

$$\frac{W_1}{P} = \frac{\sin(\theta_1 - \theta_2)}{\cos\theta_1}; \text{ similarly } \frac{W_2}{P} = \frac{\sin(\theta_2 - \theta_3)}{\cos\theta_3};$$

$$\therefore \frac{W_1}{W_2} = \frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_2 - \theta_3)} \frac{\cos\theta_3}{\cos\theta_1} = \frac{\tan\theta_1 - \tan\theta_2}{\tan\theta_2 - \tan\theta_3}:$$

$$\text{similarly } \frac{W_2}{W_3} = \frac{\tan\theta_2 - \tan\theta_3}{\tan\theta_3 - \tan\theta_4}; \text{ \&c...}$$

these are the ratios, which the weights must bear to each other, that equilibrium may subsist when there is no friction. From these conditions we may either determine the weights when the inclinations of the joints are given; or *vice versâ*.

PROP. *To shew that we may conceive the number of joints to be increased without limit without disturbing the equilibrium: and to find the equation to the curve of pressure in that case.*

127. We can divide W_1 into two parts, W and $W_1 - W$, by a joint of which ϕ is the inclination: ϕ being determined by the equation

$$\frac{W}{W_1 - W} = \frac{\tan\theta_1 - \tan\phi}{\tan\phi - \tan\theta_2};$$

ϕ evidently lies between θ_1 and θ_2 in magnitude. In the same manner we may subdivide the voussoirs to any extent. When the number of joints is increased without limit the line of pressure becomes a curve, to which the portions of the line of pressure, when the voussoirs are of finite magnitude, are tangents. We shall call this curve *the curve of pressure*. It is necessary for the equilibrium of an arch, that it should lie wholly within the masonry. We shall in the following calculations suppose, that the highest point of the curve coincides

with the highest point of the intrados; because if in this case the curve of pressure falls within the intrados it will if its highest point have any other position.

To find the differential equation to the curve of pressure, let the axis of x be vertical, the highest point of the intrados being the origin of co-ordinates: let s be the distance of any one of the imaginary thin voussoirs, into which the arch may be conceived to be divided: s being measured along the curve of pressure. Let $w\,ds$ be the weight of this thin voussoir, ds being its thickness where it is cut by the curve. Then, if we bear in mind that the joints are normals to the curve, we have by Art. 126 this condition of equilibrium, the weights of the successive voussoirs vary as the change in the tangent of inclination of the joints bounding them; hence

$$w\,ds \propto d\,.\frac{dx}{dy}, \text{ or } w\frac{ds}{dy} = c\,\frac{d^2x}{dy^2}:$$

c being an arbitrary constant.

If the arch is loaded, (as in the case of a bridge with the fillings-up between the arches), we must include in w the effect of the pressure of the superincumbent mass.

Prop. *To explain how a bridge may be constructed, so that the line of pressure in each arch may cut the joints of the voussoirs at right angles, the intrados and extrados being of a given form.*

128. Let us suppose (fig. 6), that the extrados is a horizontal line, for the road-way: and the intrados is a semi-ellipse, which is a curve well adapted for convenience of water-way, and is also elegant in its form. Let CQ be the curve of pressure; x, y the co-ordinates to Q; x', y the co-ordinates to P. We shall make the voussoirs increase in depth as they are farther from the crown of the arch; because the pressure, which they have to sustain on their joints, is greater the lower down they are from the crown. The filling-up between the arches does not generally consist of solid work; but of parallel walls of brick to support the road-way. This both saves

expense and diminishes the lateral pressure on the piers and abutments. We shall suppose the depth of the voussoirs and the filling-up of the spandrels to be so arranged, that the weight of the voussoir Pp (including the effect of the pressure of the superincumbent mass) shall equal the weight of a mass PE of the same material, resting on the same base, and having its height $PE = a + \frac{2}{3}x'$, where $BC = a$.

$$\text{Hence } w\,ds \propto (a + \tfrac{2}{3}x')\,dy.$$

$$\therefore\ a + \tfrac{2}{3}x' = Q\frac{d^2x}{dy^2},$$

Q being an unknown constant.

Let b and c be the semi-axes of the intrados:

$$\text{then } bx' = c\,(b - \sqrt{b^2 - y^2});$$

$$\therefore\ a + \tfrac{2}{3}c - \frac{2c}{3b}\sqrt{b^2 - y^2} = Q\frac{d^2x}{dy^2}:$$

to solve this put $x = Ay^2 + By^4 + \ldots\ldots$and by equating powers of y^2, after expanding the radical, we shall have the values of A, B, C, &c. in terms of known quantities and Q: and by assuming a value for Q we can calculate the numerical values of A, B, C,......If after calculating these coefficients for any assumed value of Q, we see reason for changing that value, we have but to alter A, B, C,......in the same ratio in which we alter Q.

As a guide to form some idea of a fit value for Q we may make the curvature of the curve of pressure at the vertex the same as that of the intrados. Thus rad. of curv. of ellipse at $C = \frac{b^2}{c}$: rad. of curv. of curve of pressure

$$= \left(1 + \frac{dx^2}{dy^2}\right)^{\frac{3}{2}} \div \frac{d^2x}{dy^2} = \frac{Q}{a} \text{ at } C;$$

$$\therefore\ Q = \frac{ab^2}{c}.$$

If however it be thought desirable (in any particular case) to make the curve flatter, we must take a smaller value of Q; or, if the curve is to be more curved, a larger value of Q.

It should be observed, that in selecting a value of Q, we should, on the one hand, make it as large as possible, in order that the point H (where the curve of pressure cuts the vertical through the lowest part of the arch) may be brought as low, and therefore the lateral pressure be as small, as possible. On the other hand, Q must not be made so large as to cause the curve of pressure to fall outside the intrados.

When a fit value is found for Q we must tabulate the values of x for different values of y: we shall then be able to construct the curve of pressure; and to assign the proper directions to the joints of the voussoirs, since they are to be normals to this curve: and, by accommodating their lengths and the filling-up of the spandrels, so as to satisfy the condition laid down at the outset, we shall be able to construct the bridge as proposed. For numerical calculations, and indeed for a fuller development of the theory than we have room for in these pages, we refer the reader to a work entitled *Observations on the Re-building of London Bridge*; by John Seaward, Civil Engineer.

In the preceding Articles we have supposed the voussoirs smooth. Now in matter of fact there is an enormous quantity of friction between the different parts of a bridge, owing to the roughness of the materials, and the greatness of the pressures. But the preceding theory is useful; because if we can construct a convenient bridge, which will stand in the most disadvantageous circumstances, it will certainly stand when causes exist which add to its stability. We shall now shew how important friction is in giving stability to a bridge.

PROP. *To explain in what manner friction tends to preserve the equilibrium of an arch.*

129. When two bodies with *smooth* surfaces in contact are pressing against each other, then, in order that they may not slide upon each other, the mutual pressure must act in a line, which is perpendicular to the two surfaces. But if the surfaces in contact be *rough*, the mutual pressure need not act in a direction perpendicular to the surfaces to prevent sliding, but may act in any line making an angle with the perpendi-

cular less than a certain finite angle, the magnitude of which depends on the degree of roughness of the surfaces. See Art. 119.

Suppose fig. 48. represents an arch in equilibrium, the extreme voussoirs resting on the piers A and B: $G_1 G_2 G_3 \ldots\ldots$ are the centres of gravity of the voussoirs: the weights of the voussoirs act in the vertical lines $G_1 b$, $G_1 d$, $f G_3 \ldots\ldots$

Now if the surfaces of the voussoirs were *smooth* it would be necessary for the equilibrium, that this line of pressure should be perpendicular to the joints at the points a, c, e, g, i, k, m, o. (Art. 126). But when the surfaces of the voussoirs are *rough*, the only conditions for the equilibrium are, that the angles which the line of pressure makes with the perpendiculars to the joints at the points a, c, e, g, i, k, m, o should not exceed a certain finite angle, which can be made as large as we please by having the stones *rough-hewn* or by *joggling* them (Art. 125).

We see, then, that in consequence of friction the weights of the voussoirs and the angles of inclination of their joints need not fulfil those exact relations, which would be necessary if the surfaces were smooth.

But the great advantage of friction in the support of an arch is yet to appear. For in order that an arch in a bridge may be of service, it must be able to sustain weights (not immoderate in their magnitude) placed on different parts, or moving over the bridge, without breaking the arch.

Let us now suppose a weight W to be placed on the voussoir G_2. This weight adds to the weight of G_2, and consequently disturbs the line of pressure and shifts it to some new position $a'b'd'f'h'j'l'n'o'$, as represented in fig. 49. But the arch will still stand if the angles at a', c', e', g', i', k', m', o' do not exceed the finite limit. Whereas the equilibrium of the arch would certainly be disturbed by W if the surfaces were smooth.

In this way we see, then, the important aid that friction affords in the support of an arch.

Prop. *To explain the conditions of equilibrium of an arch that it may not break.*

130. If we suppose the friction of the voussoirs against each other to be so great, that they cannot slide upon each other, it follows that the arch can fall only in consequence of its breaking at the upper or lower extremities of some of the joints. And, since we suppose the piers A and B to be immoveable, simple geometrical considerations shew, that if the arch break it must break in at least three pieces, four joints at least opening, the points where they open being alternately in the extrados and intrados of the arch. So long as the line of pressure cuts all the joints (as is represented in figures 48, 49.) the arch must stand, because in that case the joints are prevented from opening by the pressure acting along that line.

But by continually loading the arch in the same part, we may gradually shift the line of pressure, till it passes through the extremity of one of the joints, as g' in fig. 50: and now the pressure acting through g' will not prevent the joint $g'H$ opening at H, although other circumstances may.

From what we have already said about the arch not falling till two joints at least in the extrados and two joints at least in the intrados open, it follows, that the arch will certainly stand as we continually load it, till the line of pressure passes through the extremities of at least four joints, the extremities being alternately in the intrados and extrados.

If, then, our arch be such, that by loading it we cannot shift the line of pressure into this position, the arch will sustain any load without falling. Nevertheless when the arch is much loaded, and the line of pressure passes through the extremity of any joint, there will be a great strain at that point. This explains the fact observed by Professor Robison, who constructed some chalk models and found that chips fell off from three or four of the extremities of the joints.

If, however, the arch be of such a form, that we can place on it a sufficient load to cause the line of pressure to pass through at least four extremities of joints, situated alternately in the intrados and extrados, there is a possibility of the arch breaking and falling by the opening of the joints.

Let D, K, L, M (fig. 51.) be the points through which the line of pressure passes in this case: join them by straight lines. Then the arch may be supposed to be a system of heavy beams DK, KL, LM. In order to determine the conditions, that the equilibrium of the arch shall be stable, suppose the joints are forcibly opened through very small angles, the parts of the arch being sustained in the position represented in fig. 51. That the equilibrium of the arch may be stable, the joints, when the arch thus sustained in a broken form is left to itself, ought to collapse and not open wider; a condition which is satisfied if the system of beams DK, KL, LM be such that when left to themselves the point K shall ascend and the point L descend.

Hence to ascertain whether an arch of certain dimensions and figure will sustain any weight placed on it, we must consider all the ways in which the arch can break and find whether in each case the system of beams is of the nature just described. If this be the case we may be assured that the arch will support any weight.

PROP. *To prove that an arch, in which a tangent line drawn at the highest point of the intrados and produced to the abutments lies wholly within the voussoirs, will sustain any weight placed on any part of the extrados without breaking: also if the arch be of greater span than this, it will bear any weight placed on those parts of the extrados from which straight lines can be drawn through the voussoirs to both abutments.*

131. We will take the arch of greatest length under the first conditions mentioned in the enunciation. Let F be the highest point of the intrados (fig 52.); then the tangent at F passes through the highest point C and C' of the extreme joints. Hence from any point G in the extrados a straight line can be drawn to each pier lying wholly within the mass of the voussoirs. This would not be the case if the arch were the least portion longer without having the voussoirs proportionably lengthened. For suppose the left hand abutment (in the figure) had the position of the dotted line, then no

straight line can be drawn from a point c through the voussoirs to the right-hand abutment.

It appears, then, that if two straight lines Ga, Gb can be drawn from every point G of the extrados through the voussoirs to the piers, the tangent at F must be wholly in the voussoirs: and, this being the case, any weight placed on G will be sustained, since the portions of the arch on the right and left of G will act like beams Gb, Ga, of which the points b and a cannot slip, because we suppose the friction of the voussoirs, or at any rate the joggling, sufficient to prevent sliding.

The second part of the Proposition is evidently true after what has been already written.

Cor. 1. The principle of this Proposition seems to have been used by Mylne in the construction of Blackfriars Bridge, London, one of the arches and piers of which we have represented in fig. 53. AVY is a circular arc of which C is the centre, and the radius 56 feet: OV height above low-water $= 40$ feet: $VK = 6$ feet 7 inches. AB, YE are circular arcs of radius 35 feet: $ab = 19$ feet: $YI =$ about 8 or 9 feet. All the joints are joggled: and a line from K to the middle point of the joint YI lies wholly within the masonry; and does not even pass near the extremities of the joints, so that chipping of the voussoirs cannot take place. Therefore the portion YVL cannot break however great the load on or near the crown of the bridge, except by the crushing of the materials: and it would require an enormous pressure on the *haunches* (near Y) to raise the crown, since the weight of KY is about 2000 tons.

The tangent at Y falls within the foot of the pier F: and the pier itself is like one solid mass by having the stones and oaken planks below ab (low-water mark) well joggled, and by having each of the voussoirs between Y and a projecting over the one below it, and so giving each a firm hold of the rubble-work in the centre of the pier (as represented in the figure). The rubble-work itself is held down in its place by the small inverted arch IG. Since, then, the tangent at Y falls within the foot F of the pier, and the pier is as one solid mass, the arch $BAVE$ would stand of itself even were the other

arches to fall, since if KY were on the point of falling the pressure would act through Y. This gives additional security to the bridge.

COR. 2. From this Proposition we learn how it is that the Gothic Arch will sustain such enormous weights upon its crown, as we see is the case in many of our ecclesiastical buildings. The stone steeple of St Dunstan's in the East, London, is supported by four semi-pointed arches. In fact, it is a principle, that a pointed arch must have a great pressure upon its crown to prevent its falling; for we may consider it as consisting of the two extreme portions of a very large circular arch brought together, so that the pressure on the crown must at least equal the pressure of the portion of the circular arch which is removed. Flying buttresses always have a great pressure upon their highest part.

But besides this the pointed arch, for the reason explained in the Proposition, will sustain almost any weight on its crown provided the lowest stones do not give way: and consequently the Gothic arch is stronger for lofty buildings than the circular: but the circular arch is far better adapted than the Gothic arch for bridges, since the pressure of weights passing over will act in succession upon every part of the arch, not only on the crown.

An arch built in a wall is almost sure to stand of whatever form it be, so long as its foundation is firm: for suppose the haunches were about to fall in, the crown rising; then, in order that the crown may rise, the whole of the masonry or brick-work above the black line (in fig. 54.) must be moved upwards, a weight sufficient to prevent the crown from rising. Moreover, suppose the crown would rise; then directly it had risen and thrust the masonry above it through a small space, the pressure which caused the haunches to sink will cease to act in consequence of the *dove-tailing* together (so to speak) of the stones or bricks, which lie above the haunches. In the same manner we see that the crown could not sink.

PROP. *To explain the manner in which a dome is supported.*

132. A Dome or Cupola is an assemblage of stones, bricks, or other materials in equilibrium, of which the intrados and extrados are surfaces of revolution having a common vertical axis.

If we consider any given horizontal course of stones, it is evident that this course cannot fall inwards, since all the stones tend equally towards the centre, and consequently wedge each other in. But the form of the dome might be such that the weight of the superincumbent courses should thrust out the course under consideration and the courses below. In this way, and in this way only, can the dome fall.

It is very easily seen that a conical dome is secure, and will bear any weight on its upper course, provided the lowest course is kept from bursting outwards. A dome with its convexity inwards would be still more secure: for every stone is pressed inwards, since it forms part of an arch with its convexity inwards, and extremities in the highest and lowest horizontal courses: consequently the stones of each horizontal course are more firmly held together than in the conical dome.

133. The stone lanthern on the top of St. Paul's Cathedral weighs several hundred tons, and is supported by a brick cone, which is concealed between the outer and inner domes. The lowest course of this cone is above the stone gallery at the bottom of the outer dome, and is held from bursting outwards by an iron chain.

The pyramids, which form the steeples of Gothic architecture, are for the same reasons as the cone stable in their equilibrium. The enormous weight of these steeples is supported by very pointed arches, which spring from the square tower at a considerable distance below the top, and are of such a slight curvature that a straight line can be drawn from the key-stone through all the stones of each leg (Art. 131.): and the thrust down these arches is counteracted by the massive masonry of the tower and the buttresses (as explained in Art. 122).

In the walls and tower of Salisbury Cathedral are to be

seen some fearful cracks which seem to indicate a want of sufficient support for the stupendous steeple which forms so striking a feature of this edifice. The foundation remains firm. Sir Christopher Wren examined these defects, and found that the steeple was braced in different parts with iron bars: and added more for greater security. A little less than a century ago Price, the author of the *British Carpenter*, narrowly inspected the whole, and seems to have proved that the cathedral was erected under two architects, one completing the work that the other commenced; but that the first architect never contemplated the erection of the steeple nor so lofty a tower as the second architect was bold enough to add to the low tower built by his predecessor; and in consequence of the insufficiency of the supports the cracks now to be seen warned the architect to resort to the expedient of bracers to hold the base of the steeple from spreading. For a very interesting account of this building we refer the reader to Price's *Series of Observations upon the Cathedral church of Salisbury*. We have mentioned these particulars because this building is a very remarkable illustration of the necessity of attending to the connexion between the weights and pressures in a building, and the walls arches and buttresses by which they are to be sustained.

134. As an example of the support of a stone vaulting, we shall explain the manner in which the roof of King's College Chapel, Cambridge, is supported.

Figure 55 represents a projection upon a horizontal plane of one compartment of the roof included between the four buttresses f, g, h, k; and figure 56 represents the projection of half this compartment upon the vertical plane of one of the windows on the south side: the same letters in the two figures refer to the same points.

The rib bc runs from the east to the west end of the Chapel, the stones which form it lie in the same horizontal line and at a greater elevation from the ground than any other part of the roof: K is the central stone of the compartment, and is the upper part of one of the ornamented drops seen

hanging from the roof of the interior. The stones in aKd lie in an arch of which K is the key-stone: it is clear that the *tendency* of this arch is to sink at the crown K, and thrust down the walls at a and d. We shall proceed, then, to explain how the stones in this arch are supported; and also the stones in the rib bc: and in the course of the explanation it will be seen, that we shew how every stone in the compartment $fghk$ is supported.

On examining the roof carefully it will be found that the stones are placed in semi-arches in vertical planes through the buttresses; the spring of all the semi-arches in the space ba being at f, and their crowns or key-stones in the courses bK or Ka: this is best seen in figure 56. Now any stone s in the arch aKd is the key-stone of the two semi-arches sf and sg: and the thrust of the stones in Ks is propagated down the semi-arches sf and sg, and ultimately acts upon the buttresses at f and g; the same is true of every stone in Ka: likewise on the other side of bc the stones in Kd are supported by the semi-arches, of which they are the key-stones, and which spring from the buttresses h and k. Again, any stone r in Kb is the key-stone of two semi-arches rk and rf, and is held in its place by the thrust of the stones in Kr; and this thrust is propagated down the semi-arches rk and rf, and acts ultimately upon the buttresses k and f: the masonry of the rib bc is sufficiently heavy to prevent these semi-arches from sinking by their key-stone rising. It will be clearly seen, then, how every stone in bc and aKd is supported: it will also be seen that every other stone in the roof is sustained by being a member of a semi-arch springing from one of the buttresses, and having its key-stone in bc or aKd. The pressure of the compartment $fghk$ upon the buttresses acts obliquely: for instance, that on f will act downwards in a line whose projection on the horizontal plane will lie towards the south-east. But the compartment east of $fghk$ will press upon the buttress f in a line whose horizontal projection lies towards the south-west: and consequently the resultant of these pressures will act in a line whose horizontal projection runs due south: let fF be this line (fig. 57.); this figure represents one of the buttresses. The dimensions of the buttress are so arranged that

fF shall lie within the masonry and pass into the foundation within the foot of the buttress.

The resultant pressure of the roof on the walls at each of the four angles acts obliquely; consequently instead of buttresses of the ordinary form at the four angles of the building, towers crowned with lofty turrets are erected of such a weight as to deflect the line of pressure of the roof, and cause it to pass into the ground through the masonry.

135. We proceed now to find the position of equilibrium of a chain suspended from two fixed points, and briefly to explain the construction of *Suspension Bridges.*

A chain is an assemblage of rigid pieces of iron linked together, or connected by pivots, as in the chains of suspension bridges. We may therefore apply the principles of Chapter III. to determine the position of equilibrium of the chain. The length of the chain is generally so great in comparison with the length of each link, that we shall suppose the polygonal figure in which the chain hangs to be a continuous curve. Also we suppose that the motion of the links about their points of connexion is perfectly free; or, in other words, that the mutual action of any two links acts in a tangent line to the curve in which the chain hangs. The curve in which the chain hangs when in equilibrium is called the *Catenary.*

PROP. *A chain of uniform density and thickness is suspended from two given points: required to find the equation to the curve in which the chain hangs when it is in equilibrium**.

* We may calculate the form of the curve in the following manner.

Let us suppose the chain to consist of an infinitely great number of rigid and straight portions, each equal to δs in length: and let r of these portions lie between C and P: fig. 58. then $s = r\delta s$: also α_r and α_{r-1} being the angles which these portions make with the axis of x, we have by Art. 123,

$$\tan \alpha_r = \frac{r + \frac{3}{2}}{r + \frac{1}{2}} \tan \alpha_{r-1}.$$

Hence $\delta \,.\, \tan PTM = \tan \alpha_r - \tan \alpha_{r-1}$

$$= \frac{1}{r + \frac{3}{2}} \tan \alpha_r,$$

or

136. Let A and B (in the plane of the paper, which is supposed to be vertical) be the two points of support: fig. 58. After the chain has ceased oscillating and has attained its position of permanent rest, suppose ACB is the curve which it forms, C being the lowest point: take this as the origin of co-ordinates, CM vertical $= x$; MP horizontal $= y$; $CP = s$; P being any point in the curve.

Now the equilibrium of any portion CP will not be disturbed if we suppose this part of the chain to become rigid: this appears from Art. 69. Cor. Let c and t be the lengths of portions of the chain of which the weights equal the tensions at C and P. Then CP is a rigid body acted on by three forces which are proportional to c, t, s, and act respectively in the directions Cc, Pt, Gs.

Draw PT the tangent at P cutting the axis of x in T. Then the forces holding CP in equilibrium have their directions parallel to the sides of the triangle PMT, and therefore bear the same proportion one to another that these sides do; (see Art 18.)

$$\therefore \frac{PM}{MT} = \frac{\text{tension at lowest point}}{\text{weight of the portion } CP}, \text{ or } \frac{dy}{dx} = \frac{c}{s};$$

$$\therefore \frac{dx}{ds} = \left\{1 + \frac{dy^2}{dx^2}\right\}^{-\frac{1}{2}} = \frac{s}{\sqrt{c^2 + s^2}};$$

$$\therefore x + c = \sqrt{c^2 + s^2} \text{.......................} (1),$$

the constant added being such that when $x = 0$ then $s = 0$, since the origin of co-ordinates is taken on the curve at C;

$$\therefore s^2 = x^2 + 2cx \text{.......................} (2).$$

$$\text{or } \delta . \frac{dy}{dx} = \frac{\frac{dy}{dx}\delta s}{s + \frac{1}{2}\delta s};$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{s}\frac{dy}{dx}\frac{ds}{dx}; \quad \therefore \log_e \frac{dy}{dx} = \log_e \frac{s}{c};$$

$$\therefore \frac{dy}{dx} = \frac{s}{c}, \text{ as in the text.}$$

Also $\dfrac{dy}{dx} = \dfrac{c}{s} = \dfrac{c}{\sqrt{x^2 + 2cx}}$;

$$\therefore\ y = c \log_e \left\{\frac{x + c + \sqrt{x^2 + 2cx}}{c}\right\} \dots\dots\dots (3),$$

the constant being so chosen that x and y vanish together.

This last equation may be put under another form,

$$ce^{\frac{y}{c}} = x + c + \sqrt{(x + c)^2 - c^2};$$

then transposing $x + c$ and squaring both sides of the equation

$$c^2 e^{\frac{2y}{c}} - 2ce^{\frac{y}{c}}(x + c) = -c^2;$$

$$\therefore\ x + c = \tfrac{1}{2}c\left\{e^{\frac{y}{c}} + e^{-\frac{y}{c}}\right\} \dots\dots\dots (4).$$

Also $s = \sqrt{(x + c)^2 - c^2}$ by equation (2),

$$= \tfrac{1}{2}c\left\{e^{\frac{y}{c}} - e^{-\frac{y}{c}}\right\} \dots\dots\dots (5).$$

Any one of these five equations may be taken as the equation to the curve.

When the chain is uniform in density and thickness, (as in the present instance) the curve is called the *Common Catenary*.

137. Cor. 1. Of all curves of a given length drawn between two fixed points in a horizontal line, the common catenary is that which has its centre of gravity furthest from the line joining the points.

For since the chain is in equilibrium the depth of its centre of gravity from the horizontal line is a maximum or minimum (Art. 79.), and it is clear that it is a maximum and not a minimum, because if you displace the chain slightly it will return to its position of equilibrium, or its equilibrium is *stable* (Art. 80). Hence in any other position of the chain than that of equilibrium the centre of gravity will be nearer the given

horizontal line. But the chain which hangs in the common catenary is of uniform density and thickness, and therefore its centre of gravity coincides with that of the curve: and consequently the common catenary is the curve of the nature described.

Cor. 2. By means of the formulæ of Art. 87. Ex. 2. we shall find that the co-ordinates to the centre of gravity from the lowest point are

$$\bar{x} = \frac{cy}{2s} + \frac{x-c}{2}, \quad \bar{y} = y - \frac{cx}{s}.$$

138. Cor. 3. We might have taken the origin of co-ordinates at any other point than the lowest; as C', fig. 59.

Let the tangent at C' make an angle α with the vertical. We shall then readily get, if c' be used instead of c in the Proposition,

$$\frac{s}{c'} = \frac{C'T}{C'R} = \frac{\sin(\alpha - PTM)}{\sin PTM} = \sin\alpha \cot PTM - \cos\alpha$$

$$= \sin\alpha \frac{dx}{dy} - \cos\alpha;$$

$$\therefore \frac{dx}{ds} = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{s + c'\cos\alpha.}{\sqrt{(s + c'\cos\alpha)^2 + (c'\sin\alpha)^2}},$$

$$x + c' = \sqrt{s^2 + 2sc'\cos\alpha + c'^2}.$$

We shall also find that

$$y = c'\sin\alpha \log_e \left\{\frac{x + c' + \sqrt{x^2 + 2c'x + c'^2\cos^2\alpha}}{c'(1 + \cos\alpha)}\right\}.$$

Prop. *To find the tension of the chain at any point.*

139. Let t be the tension at P acting in the direction of the tangent at P, and estimated in terms of the length of chain of which the weight equals the tension: then, by what was mentioned in the last Proposition, (fig. 58.)

$$\frac{\text{tension at } P}{\text{weight of } CP} = \frac{PT}{MT}; \quad \therefore \frac{t}{s} = \frac{ds}{dx}.$$

But $s^2 = x^2 + 2cx$, by equation (2) of Art. 136;

$$\therefore t = x + c.$$

This shews that the lengths of chain of which the weights equal the tensions at the various points of the common catenary are such, that if they were suspended from those points their lower extremities would lie in a horizontal line.

For draw CE and PQ vertically downwards and equal to c and $x + c$ respectively: these then are the lengths of chain, which measure the tensions at C and P. But $PQ = x + c = MC + CE$, and PM is horizontal: therefore Q and E are in the same horizontal line.

Cor. 1. If a uniform chain hang freely over any two points, the extremities of the chain will lie in the same horizontal line when the chain is in equilibrium.

Prop. *A chain of variable thickness, but of the same material throughout, is suspended from two points: required to find the law of the thickness that the tension at different parts of the chain may vary as the strength of the chain at those parts.*

140. Let S be the length of a uniform chain of which the thickness equals that at the lowest point, and weight equals the weight of the length s of the chain to be suspended.

Let, as before, C be the lowest point (fig. 58.): $CM = x$, $MP = y$, $CP = s$: c the length of uniform chain of the thickness at C, of which the weight equals the tension at C. The portion CP when it has assumed its form of equilibrium may be supposed to become rigid. The forces which retain it in equilibrium are its weight and the tensions at C and P, and these are parallel to the sides of the triangle MTP;

$$\text{and} \because PT = \sqrt{PM^2 + MT^2};$$

$$\therefore \text{tension at } P = \frac{\sqrt{c^2 + S^2}}{c} \text{ tension at } C.$$

But the thickness of the chain at P varies ultimately as the quantity of material in a given short length δs of the chain, since the density is constant: it therefore varies as $\frac{dS}{ds}$. But by the hypothesis the tension must vary as the thickness of the chain;

$$\therefore \frac{dS}{ds} \text{ varies as } \sqrt{c^2 + S^2} \text{ or } \frac{dS}{ds} = \frac{\sqrt{c^2 + S^2}}{c},$$

since S and s are ultimately equal;

$$\therefore s = c \log_e \left(\frac{S + \sqrt{S^2 + c^2}}{c}\right) \quad \dots\dots\dots (6).$$

$$\text{Also } \frac{c}{S} = \frac{MP}{MT} = \frac{dy}{dx}; \quad \therefore \frac{ds}{dx} = \frac{\sqrt{c^2 + S^2}}{S}.$$

$$\text{But } \frac{dS}{ds} = \frac{\sqrt{c^2 + S^2}}{c};$$

$$\therefore \frac{dS}{dx} = \frac{c^2 + S^2}{cS} \text{ or } \frac{dx}{dS} = \frac{cS}{c^2 + S^2};$$

$$\therefore x = c \log_e \frac{\sqrt{c^2 + S^2}}{c} \quad \dots\dots\dots\dots\dots\dots (7).$$

$$\text{Also } \frac{dy}{dS} = \frac{dy}{dx}\frac{dx}{dS} = \frac{c^2}{c^2 + S^2};$$

$$\therefore y = c \tan^{-1}\frac{S}{c} \text{ or } S = c \tan \frac{y}{c} \quad \dots\dots\dots (8).$$

141. These formulæ have been reduced to Tables by Sir Davies Gilbert in the Philosophical Transactions for 1826. We give the following extracts from them to elucidate the application of the equations to the construction of Suspension Bridges.

TABLE I. The Common Catenary: $y = 100$.

c	x	s	t	Angle.
1000	5.004	100.166	1005.004	84° 16′ 48″
...	....	......	....	.. .. ..
420	11.961	100.947	431.961	76 29 6
400	12.565	101.045	412.565	75 49 22
380	13.234	101.158	393.234	75 5 35
...	.....	... ...	... ...	.. .. ..

TABLE II. The Common Catenary: $c = 100$.

y	x	s	t	Angle.
1	.005	1.000	100.005	89° 25′ 39″
2	.020	2.000	100.020	88 51 15
.	...	.. ...	......	.. .. ..
20	2.007	20.134	102.007	78 36 59
21	2.213	21.155	102.213	78 3 19
..	....	.....	....	.. .. ..

Table III. The Catenary of equal strength: $y = 100$.					
c	x	s	S	t	Angle.
1000	5.008	100.167	100.334	1005.021	84° 16′ 13″
...	....	......	......	......	.. .. ..
420	12.019	100.958	101.933	432.193	76 21 29
400	12.631	101.057	102.137	412.832	75 40 33
...	.....	......	......	......	.. .. ..

Table IV. The Catenary of equal strength: $c = 100$.					
y	x	s	S	t	Angle.
1	.005	1.000	1.000	100.005	89° 25′ 37″
2	.020	2.000	2.000	100.020	88 51 14
.	...	....	....	......	.. .. ..
20	2.013	20.135	20.271	102.034	78 32 23
21	2.221	21.156	21.314	102.246	77 58 4
.	....	.....	.....	......	.. .. ..

To explain the use of these Tables we shall take an example of each species of Catenary.

Ex. 1. *Let the span proposed for a Suspension Bridge be* 800 *feet, and let the adjunct weight of suspension rods, road-way .... be taken at one half of the weight of the chains: and let it be determined to load the chains at the point of their*

greatest strain, that is, at the points of suspension, with one-sixth part of the weight they are theoretically capable of sustaining.

The modulus which measures the full tenacity of iron is shewn by numerous experiments to be 14800 feet: this being the greatest length of iron bar which another iron bar of equal transverse dimensions will support without sensibly stretching.

Now this modulus must be reduced in the ratio 3 : 2, since we have supposed the weight of the rods, road-way...to be equal to half the weight of the chains, and consequently we add to the weight of the chains without adding to their strength. The virtual modulus is therefore 9867 feet: and the tension of the chain at the points of support is by hypothesis to $= 9867 \div 6$ feet $= 1644.5$ feet.

The semi-span is 400 feet. In Table I. y is taken $= 100$ measures; therefore each of these measures is 4 feet: and the tension at the points of support expressed in these measures $= 1644.5 \div 4 = 411.124$. But by Table I. when $t = 412$,

$$c = 400 \text{ measures} = 1600 \text{ feet},$$

$$x = 12.565 \ldots\ldots\ldots = 50.260 \ldots.$$

$$s = 101.045 \ldots\ldots\ldots = 404.180 \ldots.$$

The angle of suspension $= 75^{\circ}\,49'$.

Having found the value of c we may make use of Table II. to find the lengths of the rods for the different ordinates of the curve. In this Table c is taken at 100 measures, consequently each measure equals 16 feet.

Each gradation of y in that Table will therefore be 16 feet; and the second column gives the number of measures by which the suspending rods corresponding to the respective values of y must exceed the length of the suspending rod at the apex or centre of the bridge.

Let the following Table be formed from Table II. by taking the successive differences of the values of s:

1st measure of y.	length of arc of catenary = 1.000 measures.
2nd	.... = 1.000 ....
....	....
....	....
21st..............	.... = 1.021 ....
....	....

The last column of numbers gives the proportional part of the adjunct weights which must be suspended from the successive portions of the catenary, in order to distribute them equally throughout.

In this example we have supposed the adjunct weights to be equally distributed along the chain, so as virtually merely to increase its uniform thickness. We shall now in

Ex. 2. *Suppose the catenary to be one of equal strength:* i. e. *the tension at every part proportional to the strength: the other data the same as before.*

In this case c represents the uniform tension on each portion of iron throughout the chains whose transverse section equals that at the lowest point. In the uniform catenary the greatest tension (that at the points of support) was found equal 411.125 measures of 4 feet each: we shall take this then for the value of c in the case of a catenary of equal strength.

Turning then to Table III. (in which, as before, each measure is 4 feet) and taking the proportional part between 400 and 420, we have

$x =$ 12.290 measures or 49.161 feet,

$s =$ 101.002 .... 404.008 ....

$S =$ 102.024 .... 408.096 ....

$t =$ 423.602 .... 1694.408 ....

angle $= 76^\circ\ 3'\ 17''$.

We have taken c at 411.125 measures or 1644.5 feet, but Table IV. is calculated for $c = 100$: and therefore each measure of this table is 16.445 feet: and the second column determines the excess of length of the respective rods over that at the apex for every gradation of y.

Let us form a Table, as before, of the differences of s and S.

	Differences of s.	Differences of S.	Ratios of these Dif.
1st measure of y	1.000	1.000	1.000
2nd.............	1.001	1.000	.999
....	......	......	......
....	......	......	......
21st	1.021	1.043	1.002
....	......	......	......

The fourth column gives the quantity of matter of which the chain must be composed at the various ordinates of which the values are in the first column. Also the adjunct weights of rods, road-way... should be distributed in portions proportional to the numbers of the third column.

CHAPTER VII.

PROBLEMS.

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142. In the last two Chapters we have illustrated the principles of equilibrium by applying them to the solution of various questions. Our object in the present Chapter is to make some general remarks upon the solution of Statical Problems, and to give a few more applications.

143. The conditions of equilibrium of a single particle acted upon by forces which act in any directions, are three in number,

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

X, Y, Z being the resolved parts parallel to three rectangular co-ordinate axes of any one of the forces: Art. 23.

If the directions of these forces all lie in the same plane, and this plane be taken for that of xy, then the third equation becomes identical and there are only two conditions. If the forces all act in the same line, and this line be taken for the axis of x, then the last two equations are identical and there is only one condition.

The conditions of equilibrium of a rigid body, or of a system of rigid bodies, acted on by forces which act in any directions, are six in number,

$$\Sigma . X = 0, \quad \Sigma . Y = 0, \quad \Sigma . Z = 0,$$

$$\Sigma . (Zy - Yz) = 0, \quad \Sigma . (Xz - Zx) = 0, \quad \Sigma . (Yx - Xy) = 0,$$

X, Y, Z being the resolved parts parallel to three rectangular co-ordinate axes of any one of the forces, and xyz the co-ordinates to the point of application of that force.

If the forces all act in the same plane and this plane be taken for the plane of xy, the third, fifth, and sixth equations become identical, and there are only three equations of condition. Arts. 56, 49.

144. When we wish to solve a statical problem, we must consider what forces act upon the body that is to be in equilibrium: for unknown pressures and reactions we must substitute unknown forces, which we shall call *mechanical quantities*: also for unknown distances, angles of position, and so on, we must use unknown quantities; these we shall term *geometrical quantities*. After this we must write down the equations of equilibrium, the number of which will depend upon the nature of the problem, as mentioned in the last article. We must next write down the equations (if there be any) which connect the geometrical quantities. Lastly, we must count the unknown quantities involved in the equations; and if their number exceed the number of equations, it shews that the problem is indeterminate, or else that we have not written down all the equations of condition: we must therefore search for more; they must be equations connecting the geometrical quantities, since we know, by the principles of equilibrium, that there cannot be any more mechanical equations.

If in the end the number of equations be less than the number of unknown quantities, then equilibrium will subsist under several circumstances, and is said to be indeterminate; it does not follow that *all* the unknown quantities are indeterminate. If the number of unknown quantities equal the number of equations, then equilibrium will subsist in one way only. If it be found that there are more equations than unknown quantities, then the equilibrium will not subsist unless the known quantities fulfil the conditions at which we arrive by eliminating the unknown quantities from the equations.

145. It will often happen that we can materially diminish the labour of solving the equations by properly choosing the centre of moments, and the lines parallel to which we resolve the forces. Also by having regard to the object of the pro-

blem, whether it be to find the position of equilibrium of a body, the magnitude and direction of an unknown pressure, and so on, we may frequently set aside some of the equations as having no reference to the particular point of enquiry. Thus in Art. 121. the object is to find T, the tension of the tie-beam. Upon examining the four equations we see immediately that (2) may be set aside, because it contains an unknown quantity R, which does not enter any of the other equations, and therefore (2) is of use solely to determine R, a quantity which it is not the immediate object of the problem to discover. Equation (1) gives T when P and θ are known, and these are found from (3) and (4). Again, Art. 122. gives a good illustration of an indeterminate problem. For (1) (2) (3) are the only mechanical equations that can possibly exist, and these contain only one unknown geometrical quantity x, and consequently a fourth equation does not exist, or the problem is indeterminate: as we might easily have foreseen from the nature of the case. It does not follow that every unknown quantity in the equations is indeterminate, as we see in this instance.

146. We shall now add a few Problems.

Prob. 1. A given weight W is held at rest on a known curve AP lying in a vertical plane by means of a given weight Q acting over the pully B: required the position of rest: fig. 60.

The vertical BM through B is the axis of x, B the origin, $BM = x$, $MP = y$, P being the position of the weight; angle $B = \theta$. Now the weight is held in equilibrium by Q acting in PB, W in PW, and the reaction of the curve, or R, acting in GR a normal to the curve at P: hence, resolving these forces vertically and horizontally, Art. 23. gives

$$W - Q\cos\theta - R\cos PGB = 0,$$

$$Q\sin\theta - R\sin PGB = 0;$$

$$\text{or, since } \tan PGB = \frac{dx}{dy},$$

$$W - Q\cos\theta - R\frac{dy}{ds} = 0 \ldots\ldots (1),$$

$$Q \sin\theta - R\frac{dx}{ds} = 0 \ldots\ldots (2),$$

two equations and five unknown quantities R, θ, x, y, s: since equations (1) (2) are the only conditions of equilibrium, the other three equations must be among the quantities θ, x, y, s: they are

$$\tan\theta = \frac{y}{x} \ldots\ldots (3), \quad \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} \ldots\ldots (4),$$

$$\text{and } \phi(x, y) = 0 \ldots\ldots (5),$$

the equation to the curve. These five equations will solve the problem when we select any particular curve.

The elimination of R from (1) and (2) gives

$$W - Q\left(\cos\theta + \frac{dy}{dx}\sin\theta\right) = 0,$$

$$\text{or } W - Q\left(\frac{x}{r} + \frac{y}{r}\frac{dy}{dx}\right) = 0 \text{ by (3)}: \; r^2 = x^2 + y^2,$$

$$\text{or } W dx - Q dr = 0;$$

the equation of virtual velocities which we should have obtained from Art. 24.

Suppose the curve is a circle, the centre being at a vertical distance c from the point B: then a being the radius

$$y^2 + (x - c)^2 = a^2;$$

$$\therefore\; r^2 = x^2 + y^2 = a^2 - c^2 + 2cx; \quad \therefore\; \frac{dr}{dx} = \frac{c}{r};$$

$$\therefore\; \frac{r}{c} = \frac{Q}{W}, \quad \frac{r^2}{c^2} = \frac{Q^2}{W^2};$$

$$\therefore\; x = \frac{r^2 - a^2 + c^2}{2c} = \frac{(Q^2 + W^2)c^2 - W^2 a^2}{2cW^2},$$

and the position of W is known.

PROB. 2. A cord $AA_1A_2\ldots a$ is held at rest by forces acting at its extremities and at the knots $A_1A_2A_3\ldots$in given directions: having given the form of the polygonal figure of the cord, required to find the relations of the forces; also to find the tensions of the portions of cord: fig. 61.

The portions of cord need not be in the same plane; but the force which acts at any knot, as P_1 at A_1, must have its direction in the plane of the portions of cord which join in A_1. Let $P_1P_2\ldots$be the forces acting at the knots $A_1A_2\ldots$: $TT_1T_2\ldots T_n$ the tensions of the portions of cord: $\alpha_1\beta_1$, $\alpha_2\beta_2$,...the angles which the directions of $P_1P_2\ldots$make respectively with the portions of cord at the knots.

Then A_1 is held at rest by the three forces P_1T_1T; hence, resolving these forces in the direction of P_1 and at right angles to this, we have by Art. 23,

$$P_1 - T\cos\alpha_1 - T_1\cos\beta_1 = 0 \ldots\ldots\ldots\ldots (1),$$

$$T\sin\alpha_1 - T_1\sin\beta_1 = 0 \ldots\ldots\ldots\ldots (2).$$

Again, A_2 is held at rest by $T_1P_2T_2$; hence

$$P_2 - T_1\cos\alpha_2 - T_2\cos\beta_2 = 0 \ldots\ldots\ldots (3),$$

$$T_1\sin\alpha_2 - T_2\sin\beta_2 = 0 \ldots\ldots\ldots (4),$$

and so on: if there be n knots we shall have $2n$ equations, involving $2n+1$ unknown forces $P_1P_2\ldots P_n\ TT_1\ldots\ldots T_n$: we shall therefore have an equation of condition connecting these forces, we shall suppose T to be known.

By equations (2) (4) ...... we have

$$\frac{T_1}{T} = \frac{\sin\alpha_1}{\sin\beta_1}, \quad \frac{T_2}{T_1} = \frac{\sin\alpha_2}{\sin\beta_2}, \ldots\ldots\ldots \frac{T_n}{T_{n-1}} = \frac{\sin\alpha_n}{\sin\beta_n};$$

$$\therefore\ T_1 = \frac{\sin\alpha_1}{\sin\beta_1}T, \quad T_2 = \frac{\sin\alpha_1\sin\alpha_2}{\sin\beta_1\sin\beta_2}T, \text{ and so on,}$$

and the tensions are all known in terms of T.

Also by (1) (2), eliminating T_1,

$$P_1 = \frac{T\sin\beta_1}{\sin(\alpha_1+\beta_1)}, \text{ and in like manner}$$

$$P_2 = \frac{T_1 \sin \beta_2}{\sin (\alpha_2 + \beta_2)} = \frac{T \sin \alpha_1 \sin \beta_2}{\sin \beta_1 \sin (\alpha_2 + \beta_2)}, \text{ and so on.}$$

Hence all the forces $P_1\ P_2$ ... are known in terms of T.

We shall now solve a few problems of forces acting on a rigid body in the same plane: see Arts. 46, 47. When the system consists of more than one rigid body, we shall consider each body separately.

PROB. 3. A uniform beam passing freely through a hole H in a wall rests with one end on an inclined plane: find the position of equilibrium: fig. 62.

AH horizontal $= h$, $\angle A = \alpha$, $PH = x$, $PG = a$: $\angle AHP = \theta$, pressure at $P = R$ perpendicular to the plane, pressure at H perpendicular to beam and $= Q$: resolving the forces vertically and horizontally,

$$W - R \cos \alpha - Q \cos \theta = 0 \text{ .... } (1),$$

$$R \sin \alpha - Q \sin \theta = 0 \text{ .... } (2),$$

taking the centre of moments at P,

$$Wa \cos \theta - Qx = 0 \text{ .... } (3),$$

these equations involve four unknown quantities R, Q, θ, x, we must search for a relation between x and θ: this is

$$\frac{x}{h} = \frac{\sin \alpha}{\sin (\alpha + \theta)} \text{ .... } (4).$$

Our object is to determine the position of equilibrium: that is, to find x and θ: we have one equation (4), we must therefore obtain another between x and θ by eliminating R and Q from (1) (2) (3).

By (1) (2), elim^g. R, $\frac{W}{Q} = \frac{\sin (\alpha + \theta)}{\sin \alpha}$: by (3) $\frac{W}{Q} = \frac{x}{a \cos \theta}$;

$$\therefore \frac{\sin (\alpha + \theta)}{\sin \alpha} = \frac{x}{a \cos \theta}.$$

Eliminating x from this by (4), we have

$$a \cos \theta \sin^2 (\alpha + \theta) = h \sin^2 \alpha,$$

from which θ, and therefore the position of the beam, is to be determined.

$$\text{If } \alpha = 90^\circ, \quad \cos \theta = \sqrt[3]{\frac{h}{a}}.$$

This gives two equal values of θ, but with opposite signs: the negative value corresponds to the case in which the end P is the *highest* point of the beam, and is *constrained* to remain in contact with the plane: equations (1) (2) shew that R is in this case negative.

Prob. 4. A sphere and cone in contact rest, as in fig. 63, on two inclined planes, the intersection of which is a horizontal line: required the angle of the cone and the position of equilibrium.

W, W' the weights of the sphere and cone: R the reaction at B: P the mutual action at E: the resultant of the reactions of the plane on the base of the cone must act at some point D, let Q be this resultant; $CD = x$: G the centre of gravity of the cone: rad. of sphere $= a$, $Gc = z$, c being the point where the normal at E cuts the axis of the cone: $2\theta =$ the angle of the cone: α, β the angles the planes make with the horizon.

$$\text{For the sphere, } W - R \cos \beta + P \sin (\alpha - \theta) = 0 \text{ ...... (1)},$$

$$R \sin \beta - P \cos (\alpha - \theta) = 0 \text{ ...... (2)}.$$

The equation of moments is an identical equation.

$$\text{For the cone, } W' - Q \cos \alpha - P \sin (\alpha - \theta) = 0 \text{ ...... (3)},$$

$$Q \sin \alpha - P \cos (\alpha - \theta) = 0 \text{ ...... (4)},$$

$$\text{moments about } G, \; Qx - Pz \cos \theta = 0 \text{ ...... (5)}.$$

These five equations involve six unknown quantities: if there be a sixth equation it must be a relation connecting the *geometrical* quantities involved in these five equations: but a little consideration will shew us that no necessary connexion exists between any two of x, θ, z: hence the problem is indeterminate. By examining the equations we perceive that the first four involve only the four unknown quantities P, R, Q, θ: hence

these are determinate; but x and z are indeterminate since they are connected only by (5): for any given position, however, of the bodies z is known by geometry, and consequently x becomes known by (5).

We learn from this that if θ be chosen so as to satisfy equations (1) (2) (3) (4), the bodies will remain at rest in whatever position they are placed, their centres of gravity remaining in the plane of the paper: and as we give the bodies different positions z varies, consequently x and therefore the point of application of Q changes.

By (1) (2) $\dfrac{P}{W} = \dfrac{\sin\beta}{\cos(\alpha + \beta - \theta)}$; by (3) (4) $\dfrac{P}{W'} = \dfrac{\sin\alpha}{\cos\theta}$;

$$\therefore \frac{W}{W'} = \frac{\sin\alpha\cos(\alpha+\beta-\theta)}{\sin\beta\cos\theta} = \frac{\sin\alpha}{\sin\beta}\{\cos(\alpha+\beta)+\sin(\alpha+\beta)\tan\theta\};$$

$$\therefore \tan\theta = \frac{W\sin\beta - W'\sin\alpha\cos(\alpha+\beta)}{W'\sin\alpha\sin(\alpha+\beta)}$$

$$= \frac{(W+W')\sin\beta}{W'\sin\alpha\sin(\alpha+\beta)} - \frac{\sin(2\alpha+\beta)}{\sin\alpha\sin(\alpha+\beta)}.$$

By (4) (5) $\dfrac{x}{z} = \dfrac{\sin\alpha\cos\theta}{\cos(\alpha-\theta)} = \dfrac{\sin\alpha}{\cos\alpha + \sin\alpha\tan\theta}$

$$= \frac{W'\sin\alpha\sin(\alpha+\beta)}{W'\{\cos\alpha\sin(\alpha+\beta) - \sin\alpha\cos(\alpha+\beta)\} + W\sin\beta}$$

$$= \frac{W'\sin\alpha\sin(\alpha+\beta)}{(W+W')\sin\beta}.$$

The value of $\tan\theta$ gives the angle of the cone necessary for equilibrium, and the value of x gives the point of application of Q for any given position of the bodies.

PROB. 5. A person suspended in a balance of which the arms are equal thrusts his centre of gravity out of the vertical by means of a rod fixed to the furthest extremity of the beam of the balance, the direction of the rod passing through his centre of gravity: given that the rod and the line from the nearer end of the beam of the balance to his centre of gravity

make angles a, β with the vertical, shew that his apparent and true weights are in the ratio $\sin(a + \beta) : \sin(a - \beta)$.

PROB. 6. A uniform beam placed in a hemispherical bowl is in equilibrium, find its position.

PROB. 7. A cylinder with its axis horizontal is supported on an inclined plane by a beam which rests upon it and has its lower extremity fastened to the plane by a hinge: find the conditions of equilibrium.

PROB. 8. Two uniform beams of equal length are loosely connected, each by one extremity, to the extremities of another uniform beam, they are then placed on a sphere; find the pressures on the sphere at the three points of contact, the length of the middle beam being less than the diameter of the sphere.

PROB. 9. To determine the conditions of equilibrium on Roberval's Balance; see Art. 99. and fig. 35.

This machine consists of five rigid bodies; and since the forces all act in the same (the vertical) plane we shall have fifteen equations: the figure will point out the meaning of the various unknown forces, the description of which we omit here to save room: the angles measure the inclinations of these to the vertical.

The equilibrium of the part supporting Q gives

$$Q - R\cos\theta - R'\cos\theta' = 0 \ldots\ldots\ldots\ldots (1),$$

$$R\sin\theta - R'\sin\theta' = 0 \ldots\ldots\ldots\ldots (2),$$

$$Qr - Rb\sin\theta = 0 \ldots\ldots\ldots\ldots (3).$$

The equilibrium of the bar CC' gives

$$V\cos\psi - R\cos\theta - S\cos\phi = 0 \ldots\ldots\ldots\ldots (4),$$

$$V\sin\psi + R\sin\theta - S\sin\phi = 0 \ldots\ldots\ldots\ldots (5),$$

$$Ra\sin(a + \theta) - Sa'\sin(a - \phi) = 0 \ldots\ldots\ldots\ldots (6).$$

The equilibrium of the bar DD' gives

$$V'\cos\psi' - R'\cos\theta' - S'\cos\phi' = 0 \ldots\ldots\ldots\ldots (7),$$

$$V'\sin\psi' + R'\sin\theta' - S'\sin\phi' = 0 \ldots\ldots\ldots\ldots (8),$$

$$R'a\sin(a - \theta') - S'a'\sin(a + \phi') = 0 \ldots\ldots\ldots\ldots (9).$$

The equilibrium of the part supporting P gives

$$P - S\cos\phi - S'\cos\phi' = 0 \text{ .... } (10),$$

$$S\sin\phi - S'\sin\phi' = 0 \text{ .... } (11),$$

$$Ps - Sb\sin\phi = 0 \text{ .... } (12).$$

The equilibrium of the stem and stand gives

$$W - T + V\cos\psi + V'\cos\psi' = 0 \text{ .... } (13),$$

$$V\sin\psi - V'\sin\psi' = 0 \text{ .... } (14),$$

$$Tx - V'(h+b)\sin\psi' + Vh\sin\psi = 0 \text{ .... } (15).$$

These equations contain 15 unknown quantities, namely, RR' VV' SS' $T\theta\theta'$ $\psi\psi'$ $\phi\phi' x$ and the ratio of Q to P. Some of these must be indeterminate since (as we might have foreseen) (14) is a consequence of (2) (5) (8) (11).

To obtain the ratio $\dfrac{Q}{P}$.

By (1) (2) $\dfrac{Q}{R} = \dfrac{\sin(\theta+\theta')}{\sin\theta'}$, by (10) (11) $\dfrac{P}{S} = \dfrac{\sin(\phi+\phi')}{\sin\phi'}$;

$$\therefore \frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')} = \frac{R\sin\phi'}{S\sin\theta'} = \frac{a'\sin(a-\phi)\sin\phi'}{a\sin(a+\theta)\sin\theta'} \text{ by (6).}$$

If we had eliminated R and S first and then R' S',

$$\frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')} = \frac{R'\sin\phi}{S'\sin\theta} = \frac{a'\sin(a+\phi')\sin\phi}{a\sin(a-\theta')\sin\theta} \text{ by (9).}$$

Adding these equations after multiplying them respectively by the denominators of the right-hand sides, we have

$$\{\sin(a+\theta)\sin\theta' + \sin(a-\theta')\sin\theta\}\frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')}$$

$$= \frac{a'}{a}\{\sin(a-\phi)\sin\phi' + \sin(a+\phi')\sin\phi\};$$

$$\therefore \sin a\sin(\theta+\theta')\cdot\frac{Q\sin(\phi+\phi')}{P\sin(\theta+\theta')} = \frac{a'}{a}\sin a\sin(\phi+\phi');$$

$$\therefore \frac{Q}{P} = \frac{a'}{a},$$

that is, the weights must always be inversely as the arms DE, $D'E'$, and do not depend on r and s.

To find T. Add together (1) (4) (7) (10) (13) after changing the signs in (4) (7), we have

$$T = W + P + Q.$$

To find x. By (14) (15) $Tx = Vb \sin\psi$

$$= Ps - Qr \text{ by (5) (3) (12)} = P\left(s - \frac{a'}{a}r\right);$$

$$\therefore x = \frac{P}{W+P+Q}\left(s - \frac{a'}{a}r\right).$$

This shews that as we shift the weights P and Q the point B, at which the reaction and consequently the resultant downward-pressure acts, shifts also. If the ratio of r and s be such that B is at C, then if P be shifted outwards or Q inwards the balance will fall moving about the point C. If the stem be fixed of course the balance will not fall; but then the *strain* upon the stem will change as we shift P and Q. The strains at the pivots are indeterminate, nevertheless they alter as P and Q are shifted.

In this way the paradoxical character of the balance is explained.

We shall illustrate the Principle of Virtual Velocities in the solution of the following problem.

Prob. 10. A beam in a vertical plane rests on a post B and against a wall at A, as represented in fig. 64: required the circumstances of equilibrium.

Distance of B from the wall $= b$: $AG = a$: $\angle GAD = \theta$. The reaction (P) of the post at B is perpendicular to the surfaces in contact, and therefore to the beam: the reaction (R) of the wall is perpendicular to the wall for the same reason: W the weight of the beam. We may consider the beam in equilibrium under the action of P, R, W, and suppose the post and wall removed.

Now the object of the problem might be solely to determine the position of equilibrium, or also to determine P and not R, or R and not P, or to determine both P and R and also the position of equilibrium. We shall solve the problem by the Principle of Virtual Velocities under these four suppositions, in order to explain the method of proceeding so as to avoid as much trouble as possible according to the nature of the question.

1. Suppose the position of equilibrium only required. We must then give the beam a small arbitrary geometric motion such that the unknown pressures P and R shall not occur in the equation of virtual velocities: the beam must therefore remain in contact with the wall and the post: as in fig. 64.

Let $\delta\theta$ be the increase of θ owing to the displacement. Then height of G above the horizontal through B (or h)

$$= GB \cos\theta = (a - b \operatorname{cosec}\theta)\cos\theta = a\cos\theta - b\cot\theta;$$

$$\therefore \text{ vertical space described by } G = \delta h = \left(\frac{b}{\sin^2\theta} - a\sin\theta\right)\delta\theta,$$

$$\text{and by virtual velocities } W\delta h = 0;$$

$$\therefore b - a\sin^3\theta = 0, \quad \sin\theta = \sqrt[3]{\frac{b}{a}},$$

and this determines the *position of equilibrium*.

2. But suppose we wished to find the pressure P as well as the position of equilibrium.

We ought in this case to have moved the beam off the post, as in fig. 65, in order that the virtual velocity of B with respect to P may not vanish, and consequently P not disappear as in case (1).

Let $AA' = c$, and let, as before, $\delta\theta$ be the change of θ.

Then the space described by B in direction of P's action, (since BP is perpendicular to AB) equals the difference of the resolved parts of AA' and $A'B'$ in the direction of P

$$= AA'\sin\theta - A'B'\cos(90^0 - \delta\theta),\ A'B' = AB = b\operatorname{cosec}\theta$$

$$= c\sin\theta - b\operatorname{cosec}\theta\,\delta\theta.$$

Also space described by G in direction of W

$$= AG\cos\theta - AA' - A'G'\cos(\theta + \delta\theta)$$

$$= a\cos\theta - c - a\cos\theta + a\sin\theta\,\delta\theta = a\sin\theta\,\delta\theta - c;$$

therefore by the equation of virtual velocities,

$$W(a\sin\theta\,\delta\theta - c) + P(c\sin\theta - b\operatorname{cosec}\theta\,\delta\theta) = 0;$$

$$\therefore\ \delta\theta\,(Wa\sin\theta - Pb\operatorname{cosec}\theta) - c\,(W - P\sin\theta) = 0;$$

and since c and $\delta\theta$ may be any independent small quantities

$$Wa\sin\theta - Pb\operatorname{cosec}\theta = 0,\quad W - P\sin\theta = 0;$$

$$\therefore\ \sin\theta = \sqrt[3]{\frac{b}{a}} \text{ and } \frac{P}{W} = \sqrt[3]{\frac{a}{b}}.$$

3. Suppose we wished to know R and the position of equilibrium, and not P.

Then we should give the beam such an arbitrary motion (fig. 66.) as to give A a virtual velocity with respect to R, but not one to B with respect to P. Let $AA' = c$, $BAA' = \alpha$;

$$\therefore\ \delta\theta = \frac{c\sin\alpha}{AB - c\cos\alpha} = \frac{c}{b}\sin\alpha\sin\theta; \text{ and the virtual vel. of } G$$

$$= AG\cos\theta - c\cos(\theta - \alpha) - A'G'\cos(\theta + \delta\theta)$$

$$= \left(\frac{a}{b}\sin^2\theta - \sin\theta\right) c\sin\alpha - c\cos\alpha\cos\theta;$$

and virtual velocity of $A = c\sin(\theta - \alpha)$;

$$\therefore\ W\left\{\left(\frac{a}{b}\sin^2\theta - \sin\theta\right) c\sin\alpha - c\cos\alpha\cos\theta\right\}$$

$$+ R\,(c\cos\alpha\sin\theta - c\sin\alpha\cos\theta) = 0;$$

$$\therefore\ W\left(\frac{a}{b}\sin^2\theta - \sin\theta\right) - R\cos\theta = 0,\qquad W\cos\theta - R\sin\theta = 0;$$

$$\therefore\ \sin\theta = \sqrt{\frac{b}{a}} \text{ and } \frac{R}{W} = \frac{\sqrt{a^{\frac{2}{3}} - b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}.$$

4. Lastly, suppose we wished to determine P and R and the position of equilibrium.

Then we must give the beam the most general disturbance possible in the plane of the forces: fig. 67.

$$AA' = c: \; BAA' = \alpha: \text{ and } \delta\theta \text{ the increase of } \theta;$$

$$\therefore \text{ vir. vel. of } A \text{ with respect to } R = c \sin(\theta - \alpha),$$

$$\ldots\ldots\ldots B \ldots\ldots\ldots\ldots\ldots P = c\sin\alpha - \operatorname{cosec}\theta b\delta\theta$$

$$\ldots\ldots\ldots G \ldots\ldots\ldots\ldots\ldots W = a\sin\theta.\delta\theta - c\cos(\theta - \alpha);$$

$$\therefore W\{a\sin\theta.\delta\theta - c\cos(\theta-\alpha)\}$$
$$+ P(c\sin\alpha - \operatorname{cosec}\theta b\delta\theta) + Rc\sin(\theta - \alpha) = 0;$$

$$\therefore c\sin\alpha(P - W\sin\theta - R\cos\theta) - c\cos\alpha(W\cos\theta - R\sin\theta)$$
$$- \delta\theta(Wa\sin\theta - Pb\operatorname{cosec}\theta) = 0;$$

and $c\sin\alpha$, $c\cos\alpha$, and $\delta\theta$ are independent;

$$\therefore P - W\sin\theta - R\cos\theta = 0 \ldots\ldots(1),$$

$$W\cos\theta - R\sin\theta = 0 \ldots\ldots(2),$$

$$Wa\sin\theta - Pb\operatorname{cosec}\theta = 0 \ldots\ldots(3).$$

These three equations are the equations which we should have obtained by the principles of Art. 49; they give by elimination

$$\sin\theta = \sqrt[3]{\frac{b}{a}}; \quad \frac{P}{W} = \left(\frac{a}{b}\right)^{\frac{1}{3}}, \quad \frac{R}{W} = \frac{\sqrt{a^{\frac{2}{3}} - b^{\frac{2}{3}}}}{b^{\frac{1}{3}}}.$$

We have thus illustrated the method of application of this principle: and we observe, in general, that when the object of the problem does not require certain unknown forces we must give the body the most arbitrary geometrical motion possible without giving the points of application of these forces any motion in their direction.

The first case of the four just solved is an application of the principle proved in Art. 79. and which was deduced from the principle of virtual velocities. We may determine whether

the equilibrium be stable or unstable (Art. 80.) by differentiating h a second time:

$$\frac{dh}{d\theta} = \frac{b}{\sin^2\theta} - a\sin\theta\,; \quad \therefore\ \frac{d^2h}{d\theta^2} = -\left(\frac{2b}{\sin^3\theta} + a\right)\cos\theta,$$

which is negative when $\dfrac{dh}{d\theta} = 0$: hence h is a maximum and the equilibrium is unstable.

We may frequently make use of this method to discover the nature of the equilibrium.

PROB. 11. A body with a convex surface rests on a fixed body with a convex surface: required whether the equilibrium is stable or unstable: fig. 68.

Let CAO be a normal to the two surfaces at the point of contact A of the two bodies when the upper body is at rest: then the centre of gravity of the upper body is in that line: let c be its distance from O the centre of curvature at A: let a and b be the radii of curvature at A of the curves in which the plane of the paper (supposed vertical) cuts the bodies: displace the upper body through a very small angle as in the figure: angle $C = \theta$:

$\therefore\ h$ = dist. of cen. of grav. from horizontal through C,

$$= (a+b)\cos\theta - c\cos(\theta + A'O'B),\ A'O'B = \frac{A'B}{b} = \frac{a\theta}{b}$$

$$= (a+b)\cos\theta - c\cos\left(1 + \frac{a}{b}\right)\theta$$

$$= (a+b)\left(1 - \frac{c}{b}\right) - \left\{(a+b) - \frac{c}{b^2}(a+b)^2\right\}\frac{\theta^2}{2} + \dots$$

Hence h is a maximum or minimum, or the equilibrium is unstable or stable, according as c is $<$ or $>$ $\dfrac{b^2}{a+b}$,

$$\text{or as } AG \text{ is} > \text{or} < (b - c \text{ or})\ \frac{ab}{a+b}.$$

We shall close this Chapter with a few examples of Problems in which Friction is considered. The only change

will be that we must substitute some unknown force for the friction acting at right angles to the pressure; if we suppose the parts acted on by friction to be on the point of slipping, this force $= \mu . P$, where P is the pressure of the rough surfaces and μ a constant known by experiment: see Art. 118.

PROB. 12. A cylinder with its axis horizontal is held at rest on an inclined plane by a string coiled round its middle and then fastened on the plane; fig. 69: find the conditions of equilibrium, friction being considered. The forces act as drawn in the figure.

The conditions of equilibrium are

$$W - R \sin a - F \cos a - T \cos (\theta + a) = 0 \ldots\ldots (1),$$

$$R \cos a - F \sin a - T \sin (\theta + a) = 0 \ldots\ldots (2),$$

$$\text{moments about the axis, } Ta - Fa = 0 \ldots\ldots (3),$$

these are the only equations; and they contain four unknown quantities, R, T, F, θ; but we know that F cannot be greater than $\mu . R$: this limits the indeterminateness of the problem.

$$\text{Eliminate } T \text{ from (2) (3)}; \therefore \frac{F}{R} = \frac{\cos a}{\sin a + \sin (\theta + a)};$$

$$\therefore \sin a + \sin (\theta + a) \text{ cannot be less than } \frac{\cos a}{\mu},$$

$$\theta + a \text{ cannot be less than } \sin^{-1} \left(\frac{\cos a - \mu \sin a}{\mu}\right),$$

but it may be greater.

PROB. 13. A cylinder lies upon two equal cylinders all in contact and having their axes parallel: and the lower cylinders rest on a horizontal plane: $\mu\mu'$ the coefficients of friction respectively between the cylinders and each cylinder and the plane: find the conditions of equilibrium, and the relation of μ and μ' that all the points of contact may begin to slip at the same instant. fig. 70.

The forces as in the figure.

The upper cylinder, $W - 2R \cos a - 2F \sin a = 0 \ldots\ldots (1)$,
the other two equations of this cylinder are identical.

One of the lower cylinders,

$$W' - R' + R\cos\alpha + F\sin\alpha = 0 \ldots\ldots (2),$$
$$F' - R\sin\alpha + F\cos\alpha = 0 \ldots\ldots. (3),$$
$$F' - F = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

these are all the equations.

By (3) (4) $\dfrac{F}{R} = \dfrac{\sin\alpha}{1+\cos\alpha} = \tan\dfrac{\alpha}{2}$, not greater than μ.

By (1) (2) $2R' = 2W' + W$,

by (1) (3) (4) $F' = \dfrac{W\sin\alpha}{2(1+\cos\alpha)} = \dfrac{W}{2}\tan\dfrac{\alpha}{2}$;

$$\therefore \frac{F'}{R'} = \frac{W\tan\dfrac{\alpha}{2}}{2W' + W}, \text{ not greater than } \mu'.$$

If $\mu = \mu'$ then since W is less than $W + 2W'$ the lower cylinders will slip first as we continually increase the weight of the upper cylinder. In order that the points of contact may all slip together, we must have

$$\tan\frac{\alpha}{2} = \mu \text{ and } \frac{W\tan\dfrac{\alpha}{2}}{2W' + W} = \mu';$$

$$\therefore \frac{W'}{W} = \frac{\mu - \mu'}{2\mu'}.$$

PROB. 14. Three equal rough rods are loosely connected together by one extremity of each, and placed on a rough horizontal plane. Shew how to graduate one of the rods so that by noting the position of a smooth ring resting in a horizontal position on the rods and *just* in equilibrium we may know the coefficient of friction between the rods and the plane.

CHAPTER VIII.

ATTRACTIONS.

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147. The phenomena of the motion of the heavenly bodies lead us to conjecture, as we shall hereafter perceive, that the various particles of matter in the universe attract each other with a force which varies directly as the mass of the attracting particle and inversely as the square of the distance of the attracted from the attracting particle. Now in anticipation of this it will be an interesting and useful enquiry to calculate the resultant attraction of an assemblage of molecules which constitute a mass such as the Earth, the Sun, or any of the heavenly bodies. We shall commence with the calculations of the attraction of homogeneous bodies bounded by surfaces of the second order, and then of any homogeneous bodies differing but little in figure from a sphere, and lastly of heterogeneous bodies consisting of homogeneous strata all differing but little from spherical shells in their form. Also in the course of these calculations we shall introduce a few Propositions which we shall find of use hereafter.

Prop. *To find the resultant attraction of an assemblage of particles constituting a homogeneous spherical shell of very small thickness upon a particle outside the shell: the law of attraction of the particles being that of the inverse square of the distance.*

148. Let O be the centre of the shell (fig. 71), P any particle of it, dr its thickness: C the attracted particle, $OC = c$, $\angle POC = \theta$, $OP = r$: $mPMn$ a plane perpendicular to OC, ϕ the angle which the plane POC makes with the plane of the paper, $PC = y$.

The attraction of the whole shell C acts in CO.

Let OP revolve about O through a small angle $d\theta$ in the plane MOP: then $rd\theta$ is the space described by P. Again, let OPM revolve about OC through a small angle $d\phi$, then $r\sin\theta d\phi$ is the space described by P. Likewise the thickness of the shell equals dr. Hence the volume of the elementary solid at P equals $dr\,rd\theta\,r\sin\theta d\phi$ ultimately, since its sides are ultimately at right angles to each other.

Then, if the unit of attraction be chosen to be the attraction of a unit of mass at a unit of distance, the attraction of the elementary mass at P on C in the direction CP

$$=\frac{\rho r^2 \sin\theta dr d\theta d\phi}{y^2}, \quad \rho \text{ the density of the shell};$$

$$\therefore \text{ attraction of } P \text{ on } C \text{ in } CO = \frac{\rho r^2 \sin\theta dr d\theta d\phi}{y^2}\,\frac{c - r\cos\theta}{y}.$$

We shall eliminate θ from this equation by means of

$$y^2 = c^2 + r^2 - 2cr\cos\theta,$$

$$\therefore \sin\theta\frac{d\theta}{dy} = \frac{y}{cr}, \quad c - r\cos\theta = \frac{y^2 + c^2 - r^2}{2c};$$

$$\therefore \text{ attrac. of } P \text{ on } C \text{ in direct. } CO = \frac{\rho r dr}{2c^2}\left(1 + \frac{c^2 - r^2}{y^2}\right)dy\,d\phi.$$

To obtain the attraction of all the particles of the shell we integrate this with respect to ϕ and y, the limits of ϕ being 0 and 2π, those of y being $c - r$ and $c + r$;

$$\therefore \text{ att}^{\text{n}}\text{. of shell on } C \text{ in } CO = \frac{\rho r dr}{2c^2}\int_{c-r}^{c+r}\int_0^{2\pi}\left(1 + \frac{c^2 - r^2}{y^2}\right)dy\,d\phi$$

$$= \frac{\pi\rho r dr}{c^2}\int_{c-r}^{c+r}\left(1 + \frac{c^2 - r^2}{y^2}\right)dy = \frac{\pi\rho r dr}{c^2}(2r + 2r)$$

$$= \frac{4\pi\rho r^2 dr}{c^2} = \frac{\text{mass of the shell}}{c^2}.$$

This result shews that the shell attracts the particle at C in the same manner as if the mass of the shell were condensed into its centre.

149. It follows also that a sphere which is either homogeneous or consists of concentric spherical shells of uniform density will attract the particle at C in the same manner as if the whole mass were collected at its centre.

PROP. *To find the attraction of a homogeneous spherical shell of small thickness on a particle placed within it.*

150. We must proceed as in the last Proposition: but the limits of y are in this case $r - c$ and $r + c$: hence

$$\text{attraction of shell} = \frac{\pi\rho r\,dr}{c^2}\int_{r-c}^{r+c}\left(1 - \frac{r^2 - c^2}{y^2}\right)dy$$

$$= \frac{\pi\rho r\,dr}{c^2}(2c - 2c) = 0;$$

therefore a particle within the shell is equally attracted in every direction.

PROP. *To find the attraction of a homogeneous spherical shell on a particle without it; the law of attraction being represented by ϕ (y), y being the distance.*

151. The calculation is exactly analogous to that of Art. 148: we have only to alter the law of attraction: then attraction on C in CO

$$= \frac{\pi\rho r\,dr}{c^2}\int_{c-r}^{c+r}(y^2 + c^2 - r^2)\,\phi(y)\,dy, \text{ (integrated by parts)}$$

$$= \frac{\pi\rho r\,dr}{c^2}\left\{(y^2 + c^2 - r^2)\int\phi(y)\,dy - 2\int\left[y\int\phi(y)\,dy\right]dy\right\}$$

$$= \frac{\pi\rho r\,dr}{c^2}\left\{(y^2 + c^2 - r^2)\,\phi_1(y) - 2\,\psi(y) + \text{const.}\right\} \text{ suppose}$$

between the specified limits

$$= 2\pi\rho r\,dr\left\{\frac{c+r}{c}\phi_1(c+r) - \frac{1}{c^2}\psi(c+r) - \frac{c-r}{c}\phi_1(c-r) + \frac{1}{c^2}\psi(c-r)\right\}$$

$$= 2\pi\rho r\,dr\,\frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\},$$

this latter form being introduced merely as an analytical artifice to simplify the expression.

PROP. *To find the attraction of the shell on an internal particle.*

152. The calculation is the same as in the last article except that the limits of y are $r - c$ and $r + c$;

$$\therefore \text{ attraction} = 2\pi\rho r dr \left\{\frac{r+c}{c}\phi_1(r+c) - \frac{1}{c^2}\psi(r+c) + \frac{r-c}{c}\phi_1(r-c) + \frac{1}{c^2}\psi(r-c)\right\}$$

$$= 2\pi\rho r dr\,\frac{d}{dc}\left\{\frac{\psi(r+c) - \psi(r-c)}{c}\right\}.$$

The formulæ of these two Articles will give the attraction when the law of attraction is known.

Ex. 1. Let $\phi(r) = \dfrac{1}{r^2}$; $\quad\therefore\ \phi_1(r) = -\dfrac{1}{r} + A,$

$\psi(r) = -r + \frac{1}{2}Ar^2 + B$: A and B arbitrary constants;

therefore attraction on an external particle

$$= 2\pi\rho r dr\,\frac{d}{dc}\left\{\frac{-4r + A\{(c+r)^2 - (c-r)^2\}}{2c}\right\}$$

$$= 2\pi\rho r dr\,\frac{d}{dc}\left\{\frac{-2r}{c} + 2Ar\right\} = \frac{4\pi\rho r^2 dr}{c^2} \text{ (Art. 148).}$$

Attraction on an internal particle

$$= 2\pi\rho r dr\,\frac{d}{dc}\left\{\frac{-4c + A\{(r+c)^2 - (r-c)^2\}}{2c}\right\}$$

$$= 2\pi\rho r dr\,\frac{d}{dc}\{-2 + 2Ar\} = 0, \text{ (Art. 150).}$$

Ex. 2. Let $\phi(r) = r$;

$$\therefore\ \phi_1(r) = \tfrac{1}{2}r^2 + A,\ \ \psi(r) = \tfrac{1}{8}r^4 + \tfrac{1}{2}Ar^2 + B.$$

Attraction on an external particle

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{(c+r)^4 - (c-r)^4 + 4A\{(c+r)^2 - (c-r)^2\}}{8c}\right\}$$

$$= 2\pi\rho r dr \frac{d}{dc}\{c^2 r + r^3 + 2Ar\} = 4\pi\rho r^2 drc = \text{mass} \times c.$$

The attraction is the same as if the shell were collected at its centre. This property we discovered for the law of the inverse square. We shall now ascertain whether there are any other laws which give the same property.

Prop. *To find what laws of attraction allow us to suppose a spherical shell condensed into its centre when attracting an external particle.*

153. Let $\phi(r)$ be the law of force: then if c be the distance of the centre of the shell from the attracted point and r the radius of the shell, and $\psi(r) = \int\{r\int\phi(r)dr\}dr$, then the attraction of the shell

$$= 2\pi\rho r dr \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\}.$$

But if the shell be condensed into its centre this attraction

$$= 4\pi r^2 dr \rho\phi(c);$$

$$\therefore\ 2r\phi(c) = \frac{d}{dc}\left\{\frac{\psi(c+r) - \psi(c-r)}{c}\right\};$$

$$\therefore\ 2r\phi(c) = 2\frac{d}{dc}\left\{\frac{d\psi c}{dc}\frac{r}{c} + \frac{d^3\psi c}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \dots\right\}$$

$$= 2r\phi(c) + 2\frac{d}{dc}\left\{\frac{d^3\psi(c)}{dc^3}\frac{r^3}{c}\frac{1}{1.2.3} + \dots\right\};$$

$$\therefore\ \frac{d}{dc}\left\{\frac{1}{c}\frac{d^3\psi(c)}{dc^3} + \dots\right\} = 0 \text{ whatever } r \text{ be};$$

$$\therefore\ \frac{d}{dc}\left\{\frac{1}{c}\frac{d^3\psi c}{dc^3}\right\} = 0,\quad \frac{d}{dc}\left\{\frac{1}{c}\frac{d^5\psi c}{dc^5}\right\} = 0\dots$$

But $\dfrac{d\psi c}{dc} = c\int\phi(c)\,dc,\quad \dfrac{d^2\psi c}{dc^2} = \int\phi(c)\,dc + c\phi c,$

$$\frac{d^3\psi c}{dc^3} = 2\phi c + c\frac{d\phi c}{dc};$$

therefore by the first of the above equations of condition for $\psi(c)$

$$\frac{2}{c}\phi c + \frac{d\phi c}{dc} = 3A,$$

and multiplying by c^2 and integrating

$$c^2\phi(c) = Ac^3 + B: \; A \text{ and } B \text{ being independent of } c$$

$$\phi(c) = Ac + \frac{B}{c^2},$$

and this satisfies all the other equations of condition for $\psi(c)$; therefore the required laws of attraction are those of the direct distance, the inverse square, and a law compounded of these.

Prop. *To find for what laws the shell attracts an internal point equally in every direction.*

154. When this is the case

$$\frac{d}{dc}\left\{\frac{\psi(r+c) - \psi(r-c)}{c}\right\} = 0,$$

$$\frac{d\psi(r)}{dr} + \frac{d^3\psi(r)}{dr^3}\cdot\frac{c^2}{1\,.\,2\,.\,3} + \ldots = -A$$

whatever c is, A being a constant independent of c;

$$\therefore\ \frac{d\psi(r)}{dr} = -A,\quad \frac{d^3\psi(r)}{dr^3} = 0\ \ldots\ldots$$

These conditions are all satisfied if the first is: this gives

$$r\int\phi(r)\,dr = -A,\quad \phi(r) = \frac{A}{r^2},$$

and therefore the inverse square is the only law which satisfies the condition.

PROP. *To find the attraction of a homogeneous oblate spheroid of small ellipticity on a particle at its pole: the law being the inverse square of the distance.*

155. Let $APBp$, $AQBq$ be sections of the spheroid and the sphere touching it, made by a plane through the axis of the spheroid: fig. 72. $AM = x$, $MP = y$, $AC = c$, $CD = a$, $c = a(1 - \epsilon)$, ϵ very small. The mass of the annulus Pp between the sphere and spheroid and of thickness dx

$$= \pi\rho y^2 dx\left(1 - \frac{c^2}{a^2}\right): \text{ also } AQ = \sqrt{2cx}\,;\ y^2 = \frac{a^2}{c^2}(2cx - x^2),$$

and if we consider every particle of the annulus Pp equidistant from A, the attraction of this annulus on A in direction AB

$$= \pi\rho y^2 dx\left(1 - \frac{c^2}{a^2}\right)\frac{1}{2cx}\frac{x}{\sqrt{2cx}} = \frac{2\pi\rho\epsilon}{(2c)^{\frac{3}{2}}}(2cx^{\frac{1}{2}} - x^{\frac{3}{2}})\,dx\,;$$

therefore attraction of whole difference of sphere and spheroid

$$= \frac{2\pi\rho\epsilon}{(2c)^{\frac{3}{2}}}\int_0^{2c}(2cx^{\frac{1}{2}} - x^{\frac{3}{2}})\,dx = 2\pi\rho\epsilon\left\{\frac{4c}{3} - \frac{4c}{5}\right\} = \frac{16\pi\rho\epsilon c}{15}$$

but attraction of sphere on $A = \frac{4}{3}\pi\rho c$; (Art. 149)

$$\therefore \text{ attraction of spheroid on } A = \tfrac{4}{3}\pi\rho c\,(1 + \tfrac{4}{5}\epsilon)\,c.$$

PROP. *To find the attraction on a particle at the equator.*

156. Let DC be the axis of revolution (fig. 73), $APBp$ and $AQBq$ sections of the spheroid and circumscribing sphere by the plane of the paper passing through the axis of revolution: $AM = x$, $MP = y$, $AC = a$, $CD = c$, $y^2 = \frac{c^2}{a^2}(2ax - x^2)$.

Let an elementary slice of the spheroid and sphere be made by planes perpendicular to the axis of x, one passing through P and the other at a distance dx from it; therefore mass of the part of this slice between the sphere and spheroid

$$= \pi\rho(QM^2 - MN \cdot PM)\, dx = \pi\rho \left\{\frac{a^2}{c^2} y^2 - \frac{a}{c} y^2\right\} dx$$

$$= \pi\rho \left(1 - \frac{c}{a}\right) (2ax - x^2)\, dx.$$

Now the distance of each portion of this from A nearly $= AQ = \sqrt{2ax}$; therefore attraction of the part between the sphere and spheroid in the direction AC

$$= \pi\rho\epsilon \int_0^{2a} (2ax - x^2) \frac{x\,dx}{(2ax)^{\frac{3}{2}}} = \pi\rho\epsilon \int_0^{2a} \frac{1}{(2a)^{\frac{3}{2}}} (2ax^{\frac{1}{2}} - x^{\frac{3}{2}})\, dx$$

$$= \pi\rho\epsilon \left(\tfrac{4}{3} - \tfrac{4}{5}\right) a = \tfrac{8}{15} \pi\rho a\epsilon;$$

and the attraction of the sphere $= \frac{4}{3} \pi\rho a$;

$\therefore$ attraction of spheroid $= \frac{4}{3} \pi\rho \left(1 - \frac{2}{5}\epsilon\right) a = \frac{4}{3} \pi\rho \left(1 + \frac{3}{5}\epsilon\right) c.$

157. In the same manner it might be shewn that the attractions of a homogeneous prolate spheroid of small ellipticity on particles at the pole and equator are respectively

$$\tfrac{4}{3}\pi\rho \left(1 - \tfrac{4}{5}\epsilon\right) c \text{ and } \tfrac{4}{3}\pi\rho\left(1 - \tfrac{3}{5}\epsilon\right) c,$$

$2c$ being the axis of revolution of the spheroid.

PROP. *To find the attraction of a homogeneous oblate spheroid upon a particle within its mass: the law of attraction being that of the inverse square of the distance.*

158. Let a, c, be the semi-axes, the minor axis of $2c$ coinciding with the axis of z: then the equation to the spheroid from the centre is

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1,$$

fgh the co-ordinates to the attracted particle: we shall take this as the origin of polar co-ordinates, A in fig. 30.

r = radius vector of any particle of the attracting mass:

θ = angle which r makes with a line parallel to z:

ϕ = angle which the plane rz makes with the plane xz:

$$\therefore\ x = f + r\sin\theta\cos\phi,\quad y = g + r\sin\theta\sin\phi,\quad z = h + r\cos\theta,$$

and the equation to the spheroid becomes

$$\frac{(f + r\sin\theta\cos\phi)^2 + (g + r\sin\theta\sin\phi)^2}{a^2} + \frac{(h + r\cos\theta)^2}{c^2} = 1,$$

$$\text{or } r^2\left\{\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2}\right\} + 2r\left\{\frac{f\sin\theta\cos\phi + g\sin\theta\sin\phi}{a^2} + \frac{h\cos\theta}{c^2}\right\}$$

$$= 1 - \frac{f^2 + g^2}{a^2} - \frac{h^2}{c^2};$$

$$\text{put } \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{c^2} = K,\ \frac{f\sin\theta\cos\phi + g\sin\theta\sin\phi}{a^2} + \frac{h\cos\theta}{c^2} = F,$$

$$\text{and } F^2 + K\left(1 - \frac{f^2 + g^2}{a^2} - \frac{h^2}{c^2}\right) = H, \text{ then}$$

$$K^2r^2 + 2KFr + F^2 = H,$$

and the values of r are

$$r' = \frac{-F + \sqrt{H}}{K} \text{ and } r'' = \frac{-F - \sqrt{H}}{K}.$$

Volume of element at $P = r^2\sin\theta\, dr\, d\theta\, d\phi$ as in Art. 148: let ρ be the density of the spheroid: then the attraction of this element on the attracted particle is $\rho\sin\theta\, dr\, d\theta\, d\phi$; and the resolved parts of this parallel to the axes of xyz are

$$\rho\sin^2\theta\cos\phi\, dr\, d\theta\, d\phi,\quad \rho\sin^2\theta\sin\phi\, dr\, d\theta\, d\phi$$

$$\text{and } \rho\sin\theta\cos\theta\, dr\, d\theta\, d\phi.$$

Let A, B, C be the attractions of the whole spheroid in the directions of the axes estimated positive towards the centre of the spheroid: then these equal the integrals of the attractions of the element; the limits of r being $-r'$ and r'', of θ being 0 and π, and of ϕ being 0 and π: hence

$$A = -\int_{-r'}^{r''}\int_0^{\pi}\int_0^{\pi}\rho\sin^2\theta\cos\phi\, dr\, d\theta\, d\phi,$$

$$B = -\int_{-r'}^{r''}\int_0^{\pi}\int_0^{\pi}\rho\sin^2\theta\sin\phi\, dr\, d\theta\, d\phi,$$

$$\text{and } C = -\int_{-r'}^{r''}\int_0^{\pi}\int_0^{\pi} \rho \sin\theta\cos\theta\, dr d\theta d\phi.$$

$$\text{Then } A = -\rho\int_0^{\pi}\int_0^{\pi}(r''+r')\sin^2\theta\cos\phi\, d\theta d\phi$$

$$= 2\rho\int_0^{\pi}\int_0^{\pi}\frac{F}{K}\sin^2\theta\cos\phi\, d\theta d\phi.$$

Now it is easily seen that if R ($\sin\alpha$, $\cos^2\alpha$) be a rational function of $\sin\alpha$ and $\cos^2\alpha$, then $\int_0^{\pi} R\,(\sin\alpha,\ \cos^2\alpha)\cos\alpha\, d\alpha = 0$. Wherefore by substituting for F and K we have

$$A = 2f\rho c^2\int_0^{\pi}\int_0^{\pi}\frac{\sin^3\theta\cos^2\phi\, d\theta d\phi}{c^2\sin^2\theta + a^2\cos^2\theta}$$

$$= \pi f\rho c^2\int_0^{\pi}\frac{\sin^3\theta\, d\theta}{c^2\sin^2\theta + a^2\cos^2\theta}$$

$$= \pi f\rho c^2\int_0^{\pi}\frac{(1-\cos^2\theta)\sin\theta\, d\theta^*}{c^2+(a^2-c^2)\cos^2\theta}$$

$$= \pi f\rho\frac{c^2}{a^2-c^2}\int_0^{\pi}\left\{\frac{a^2\sin\theta}{c^2+(a^2-c^2)\cos^2\theta} - \sin\theta\right\}d\theta$$

$$= \pi f\rho\frac{c^2}{a^2-c^2}\left\{-\frac{a^2}{c\sqrt{a^2-c^2}}\tan^{-1}\left(\frac{\sqrt{a^2-c^2}}{c}\cos\theta\right) + \cos\theta + C\right\}$$

between specified limits,

$$= 2\pi f\rho\frac{c^2}{a^2-c^2}\left\{\frac{a^2}{c\sqrt{a^2-c^2}}\tan^{-1}\frac{\sqrt{a^2-c^2}}{c} - 1\right\},\quad \frac{c^2}{a^2} = 1-e^2$$

$$= 2\pi f\rho\left\{\frac{\sqrt{1-e^2}}{e^3}\tan^{-1}\frac{e}{\sqrt{1-e^2}} - \frac{1-e^2}{e^2}\right\}$$

$$= 2\pi f\rho\left\{\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right\}.$$

* If the spheroid be *prolate* c is $> a$ and the denominator of this must be written $c^2-(c^2-a^2)\cos^2\theta$, and the integral would involve logarithms instead of circular arcs.

In the same manner we should find that

$$B = 2\pi g\rho\left\{\frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - \frac{1-e^2}{e^2}\right\}.$$

$$\text{Also } C = 2\rho\int_0^\pi\int_0^\pi \frac{F}{K}\sin\theta\cos\theta\, d\theta\, d\phi$$

$$= 2\rho h a^2\int_0^\pi\int_0^\pi \frac{\sin\theta\cos^2\theta\, d\theta\, d\phi}{c^2\sin^2\theta + a^2\cos^2\theta}$$

$$= 2\pi\rho h\frac{a^2}{a^2-c^2}\int_0^\pi\left\{\sin\theta - \frac{c^2\sin\theta}{c^2+(a^2-c^2)\cos^2\theta}\right\}d\theta$$

$$= 4\pi\rho h\frac{a^2}{a^2-c^2}\left\{1 - \frac{c}{\sqrt{a^2-c^2}}\tan^{-1}\frac{\sqrt{a^2-c^2}}{c}\right\}$$

$$= 4\pi\rho h\left\{\frac{1}{e^2} - \frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e\right\}.$$

159. Cor. 1. We see from these expressions that the attraction is independent of the magnitude of the spheroid, and depends solely upon the eccentricity.

Hence the attraction of the spheroid similar to the given one and passing through the attracted particle is the same as of any other similar concentric spheroid comprising the attracted particle in its mass. Hence a spheroidal shell the surfaces of which are similar and concentric, attracts a point within it equally in all directions.

This may be proved also in the following manner. Suppose the shell is divided into a great number of very thin similar shells. From any point inside the given shell as vertex describe a double cone of a very small vertical angle, and cutting the given shell in two parts lying on opposite sides of the point. The double cone cuts out of any one of the thin shells two small masses, which may very easily be shewn to be ultimately in the ratio of the squares of the distances from the common vertex, or the chosen point within the shell: but the law of attraction is the inverse square: therefore the

attractions of these small masses are ultimately equal and opposite on the chosen point: and the whole shell consists of such pairs of small masses. Hence the attractions on any internal point are all equal and opposite.

160. Cor. 2. If we put the ellipticity of the spheroid $=\epsilon$, and suppose ϵ very small so that we may neglect its square, we have $e^2 = 1 - \dfrac{c^2}{a^2} = 1 - (1-\epsilon)^2 = 2\epsilon$;

$$\therefore\ A = \tfrac{4}{3}\pi\rho\,(1 - \tfrac{2}{5}\epsilon)f,\ B = \tfrac{4}{3}\pi\rho\,(1 - \tfrac{2}{5}\epsilon)g,$$

$$C = \tfrac{4}{3}\pi\rho\epsilon\,(1 + \tfrac{4}{5}\epsilon)\,h.$$

161. Cor. 3. By the values of A, B, C after integrating with respect to r we have

$$\frac{A}{f} + \frac{B}{g} + \frac{C}{h} = 2\rho\int_0^\pi\int_0^\pi \frac{(c^2\sin^3\theta + a^2\sin\theta\cos^2\theta)\,d\theta d\phi}{c^2\sin^2\theta + a^2\cos^2\theta}$$

$$= 2\rho\int_0^\pi\int_0^\pi \sin\theta d\theta d\phi = 2\pi\rho\int_0^\pi \sin\theta d\theta = 4\pi\rho.$$

$$\text{But if } V = \int\frac{\text{element of mass}}{\text{distance from attracted point}}$$

$$= \int\frac{dm}{\{(x-f)^2 + (y-g)^2 + (z-h)^2\}^{\frac{1}{2}}};$$

$$\therefore\ -\frac{dV}{df} = -\int\frac{dm\,(x-f)}{\{(x-f)^2 + (y-g)^2 + (z-h)^2\}^{\frac{3}{2}}} = A;$$

$$\therefore\ -\frac{d^2V}{df^2} = \frac{dA}{df} = \frac{A}{f} \text{ by the form of } A.$$

$$\text{In the same manner } -\frac{d^2V}{dg^2} = \frac{B}{g},\quad -\frac{d^2V}{dh^2} = \frac{C}{h}.$$

Hence for an internal particle

$$\frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = -4\pi\rho.$$

162. Cor. 4. If we had taken an ellipsoid instead of a spheroid we should have had

$$A = \frac{3fM}{a^3} L, \quad B = \frac{3gM}{a^3} \frac{d(\lambda L)}{d\lambda}, \quad C = \frac{3hM}{a^3} \frac{d(\lambda' L)}{d\lambda'},$$

where M is the mass of the ellipsoid, $\lambda^2 = \frac{a^2 - b^2}{a^2}$, $\lambda'^2 = \frac{a^2 - c^2}{a^2}$,

$$\text{and } L = \int_0^1 \frac{x^2 dx}{\sqrt{1 - \lambda^2 x^2}\sqrt{1 - \lambda'^2 x^2}},$$

the integration of this depends upon the properties of elliptic transcendants: see Legendre's *Traité des Fonctions Elliptiques*, Tome I, p. 545.

163. Cor. 5. If we wished to find the attraction on an external particle we should have the same integrals for A, B, C as in the Proposition, but the limits of r would be r' and r'' (and not $-r'$ and r''), since the point from which r is measured, the attracted particle, is outside the spheroid;

$$\begin{aligned} \therefore A &= -\rho \int_{r'}^{r''} \int_0^{\pi} \int_0^{\pi} \sin^2\theta \cos\phi \, dr \, d\theta \, d\phi \\ &= \rho \int_0^{\pi} \int_0^{\pi} (r' - r'') \sin^2\theta \cos\phi \, d\theta \, d\phi \\ &= 2\rho \int_0^{\pi} \int_0^{\pi} \frac{\sqrt{H}}{K} \sin^2\theta \cos\phi \, d\theta \, d\phi, \end{aligned}$$

and this cannot be integrated by any known method.

Mr Ivory has, however, discovered a relation between the attractions of ellipsoids on external and internal particles: so that by means of this relation we can calculate the attraction on an external particle.

Prop. *To enunciate and prove Ivory's Theorem.*

164. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$

be the equations to the surfaces bounding two homogeneous ellipsoids having the same centre and foci: then

$$a^2 - b^2 = \alpha^2 - \beta^2, \quad a^2 - c^2 = \alpha^2 - \gamma^2 \ldots\ldots\ldots\ldots (1).$$

Let fgh, $f'g'h'$ be the co-ordinates to two particles so situated on the surfaces of these ellipsoids that

$$\frac{f}{f'} = \frac{a}{\alpha}, \quad \frac{g}{g'} = \frac{b}{\beta}, \quad \frac{h}{h'} = \frac{c}{\gamma} \ldots\ldots\ldots (2).$$

Also since (fgh) and $(f'g'h')$ are points in the surfaces of the first and second ellipsoids respectively, we have

$$\frac{f^2}{a^2} + \frac{g^2}{b^2} + \frac{h^2}{c^2} = 1, \quad \frac{f'^2}{\alpha^2} + \frac{g'^2}{\beta^2} + \frac{h'^2}{\gamma^2} = 1 \ldots\ldots (3).$$

Then *the attraction of the first ellipsoid parallel to the axis of* z *on the particle situated at the point* (f′g′h′) *on the surface of the second is to the attraction of the second ellipsoid on the particle situated at the point* (f g h) *on the surface of the first in the same direction as* a b : *a*β *the law of attraction being any function of the distance: and similarly with respect to the axes of* y *and* x. This is Ivory's Theorem. We shall, for convenience, represent the law of attraction by the function $r\phi(r^2)$, r being the distance.

The attraction of the first ellipsoid on the particle $(f'g'h')$ parallel to the axis of z

$$= \rho \iiint (h' - z)\, \phi \left\{(f' - x)^2 + (g' - y)^2 + (h' - z)^2\right\} dx\,dy\,dz,$$

the limits of z are $-c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ and $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$,

the limits of y are $-b\sqrt{1 - \frac{x^2}{a^2}}$ and $b\sqrt{1 - \frac{x^2}{a^2}}$, the limits of x are $-a$ and a

$$= \rho \iint \left\{\psi\left[(f' - x)^2 + (g' - y)^2 + (h' + z)^2\right] - \psi\left[(f' - x)^2 + (g' - y)^2 + (h' - z)^2\right]\right\} dx\,dy$$

between the specified limits: $\psi(r) = \frac{1}{2}\int \phi(r)\, dr$: it must be remembered that in this expression $z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$, but we do not substitute this value merely for preserving the func-

tions under as simple a form as possible. Now put $x = ar$, $y = bs$, $z = ct$, then the attraction

$$= \rho ab \iint \{\psi[(f' - ar)^2 + (g' - bs)^2 + (h' + ct)^2]$$
$$- \psi[(f' - ar)^2 + (g' - bs)^2 + (h' - ct)^2]\} \, drds,$$

the limits of s being $-\sqrt{1 - r^2}$ and $\sqrt{1 - r^2}$, and those of r being -1 and 1: also $t = \sqrt{1 - r^2 - s^2}$.

$$\text{Now } (f' - ar)^2 + (g' - bs)^2 + (h' \pm ct)^2$$
$$= f'^2 + g'^2 + h'^2 - 2(f'ar + g'bs \pm h'ct) + a^2r^2 + b^2s^2 + c^2t^2,$$

substituting for h'^2 by (3) and putting $1 - r^2 - s^2$ for t^2

$$= f'^2\left(1 - \frac{\gamma^2}{\alpha^2}\right) + g'^2\left(1 - \frac{\gamma^2}{\beta^2}\right) + \gamma^2 - 2(f'ar + g'bs \pm h'ct)$$
$$+ (a^2 - c^2)r^2 + (b^2 - c^2)s^2 + c^2;$$

eliminating $f'g'h'$ by (2) and making use of (1)

$$= \frac{f^2}{a^2}(a^2 - c^2) + \frac{g^2}{b^2}(b^2 - c^2) + c^2 - 2(f\alpha r + g\beta s \pm h\gamma t)$$
$$+ (\alpha^2 - \gamma^2)r^2 + (\beta^2 - \gamma^2)s^2 + \gamma^2$$
$$= f^2 + g^2 + h^2 - 2(f\alpha r + g\beta s \pm h\gamma t) + \alpha^2r^2 + \beta^2s^2 + \gamma^2t^2 \text{ by (3)}$$
$$= (f - \alpha r)^2 + (g - \beta s)^2 + (h \pm \gamma t)^2.$$

Hence the attraction of the first ellipsoid on $(f'g'h')$ parallel to x

$$= \rho ab \iint \{\psi[(f - \alpha r)^2 + (g - \beta s)^2 + (h + \gamma t)^2]$$
$$- \psi[(f - \alpha r)^2 + (g - \beta s)^2 + (h - \gamma t)^2]\} \, drds,$$

the limits of s being $-\sqrt{1 - r^2}$, $\sqrt{1 - r^2}$; of r being -1, 1

$= \dfrac{ab}{\alpha\beta} \times$ attraction of second ellipsoid on (fgh) parallel to x:

the same may be proved for the attractions parallel to the other axes: and consequently the Theorem, as enunciated, is true.

We observe that one of these ellipsoids lies wholly within the other: for if not the points in which they cut each other lie in the line of which the equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ and } \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1.$$

We shall suppose a less than α: the points of intersection must therefore satisfy the equation

$$x^2\left(\frac{1}{a^2} - \frac{1}{\alpha^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{\beta^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{\gamma^2}\right) = 0,$$

and this by (1) becomes $\left(\frac{x}{a\alpha}\right)^2 + \left(\frac{y}{b\beta}\right)^2 + \left(\frac{z}{c\gamma}\right)^2 = 0$, an equation which can be satisfied solely by $x = 0$, $y = 0$, $z = 0$: but these do not satisfy the equations above, and therefore the surfaces do not intersect in any point.

Hence to find the attraction of an ellipsoid of which the semi-axes are a, b, c on an external particle of which the coordinates are $f'g'h'$, we must first calculate the attraction of an ellipsoid of which the semi-axes are α, β, γ parallel to the axes on an internal particle of which the co-ordinates are f, g, h, these six quantities being determined by the equations

$$\alpha^2 - \beta^2 = a^2 - b^2, \quad \alpha^2 - \gamma^2 = a^2 - c^2, \quad \frac{f'^2}{\alpha^2} + \frac{g'^2}{\beta^2} + \frac{h'^2}{\gamma^2} = 1,$$

$$f = \frac{af'}{\alpha}, \quad g = \frac{bg'}{\beta}, \quad h = \frac{ch'}{\gamma},$$

and then the attractions required will be these three calculated attractions multiplied respectively by

$$\frac{bc}{\beta\gamma}, \quad \frac{ac}{\alpha\gamma}, \quad \frac{ab}{\alpha\beta}.$$

The following Proposition we shall find of use in a subsequent part of this work.

Prop. *To prove that the resultant attraction of the particles of a body of any figure upon a body of which the distance is very great in comparison with the greatest diameter of the attracting body, is very nearly the same, as if the particles*

were condensed into their centre of gravity and attracted according to the same law, whatever that law be.

165. Let the origin of co-ordinates be taken at the centre of gravity of the attracting body, the axis of x through the attracted particle; let c be its abscissa and xyz the co-ordinates of any particle of the body, ρ the density of that particle.

Then the distance between these two particles, or r, $= \sqrt{(c-x)^2 + y^2 + z^2}$.

Let $r\phi\,(r^2)$ be the law of attraction: then the whole attraction parallel to the axis of x

$$= \iiint \rho\,(c-x)\,\phi\,(c^2 - 2cx + x^2 + y^2 + z^2)\,dxdydz,$$

the limits being obtained from the equation to the surface of the body

$$= \iiint \rho\,(c-x)\,\{\phi\,(c^2) - (2cx - x^2 - y^2 - z^2)\,\phi'\,(c^2) + \dots\}\,dxdydz$$

$$= c\phi(c^2)\iiint \rho\left\{1 - \frac{x}{c}\left(1 + \frac{2c^2\phi'(c^2)}{\phi\,(c^2)}\right) + (y^2+z^2+3x^2)\frac{\phi'\,(c^2)}{\phi(c^2)} + \dots\right\}dxdydz$$

$$= Mc\phi\,(c^2) + c^3\,\phi'\,(c^2)\iiint \rho\,\frac{y^2+z^2+3x^2}{c^2}\,dxdydz + \dots\dots$$

M the mass of the body: $\iiint \rho x\,dx\,dy\,dz = 0$ since x is measured from the centre of gravity of the body (Art. 87. Ex. 25).

Now suppose xyz to be exceedingly small in comparison with c; then all the terms of A after the first are extremely small in comparison with that term, it being observed that $c^3\,\phi'\,(c^2)$ is of the same order as $c\,\phi\,(c^2)$ in terms of c. Hence the Proposition is true.

Cor. It appears also that to produce a given resultant law, the law of attraction of the constituent molecules must be the same.

166. We shall now proceed to the calculation of the attraction of bodies differing but little from a sphere in figure. The object of these calculations will be seen when we come to the higher branches of Physical Astronomy. The reader may therefore, if he please, omit the remainder of this Chapter till

he enters upon those investigations. We shall suppose that the law of attraction is that of the inverse square of the distance.

PROP. *To obtain formulæ for the calculation of the attraction of a heterogeneous mass upon any particle.*

167. Let ρ be the density of the body at the point (xyz): fgh the co-ordinates of the attracted particle: and, as before, suppose A, B, C are the attractions parallel to the axes of x, y, z, estimated positive *towards* the origin. Then

$$A = \iiint \frac{\rho (f - x)\, dx dy dz}{\{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{\frac{3}{2}}},$$

$$B = \iiint \frac{\rho (g - y)\, dx dy dz}{\{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{\frac{3}{2}}},$$

$$C = \iiint \frac{\rho (h - z)\, dx dy dz}{\{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{\frac{3}{2}}},$$

the limits being determined by the equation to the surface of the body.

Let $$V = \iiint \frac{\rho\, dx dy dz}{\{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{\frac{1}{2}}};$$

$$\therefore\ A = -\frac{dV}{df}, \quad B = -\frac{dV}{dg}, \quad C = -\frac{dV}{dh}.$$

It follows, then, that the calculation of the attractions A, B, C depends upon that of V. This function cannot be calculated except when expanded into a series. We proceed in the following articles to shew how this is to be done.

PROP. *To prove that* $\dfrac{d^2 R}{df^2} + \dfrac{d^2 R}{dg^2} + \dfrac{d^2 R}{dh^2} = 0$,

where $R = \{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{-\frac{1}{2}}$.

168. Since $R = \{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{-\frac{1}{2}}$,

$$\therefore\ \frac{dR}{df} = \frac{-(f - x)}{\{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{-\frac{3}{2}}};$$

$$\therefore\ \frac{d^2 R}{df^2} = \frac{2(f - x)^2 - (g - y)^2 - (h - z)^2}{\{(f - x)^2 + (g - y)^2 + (h - z)^2\}^{\frac{5}{2}}}.$$

$$\text{Similarly } \frac{d^2R}{dg^2} = \frac{2(g-y)^2 - (f-x)^2 - (h-z)^2}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{5}{2}}},$$

$$\text{and } \frac{d^2R}{dh^2} = \frac{2(h-z)^2 - (f-x)^2 - (g-y)^2}{\{(f-x)^2 + (g-y)^2 + (h-z)^2\}^{\frac{5}{2}}};$$

$$\therefore \frac{d^2R}{df^2} + \frac{d^2R}{dg^2} + \frac{d^2R}{dh^2} = 0^*.$$

* The function V will also satisfy this equation, when the attracted particle is not part of the attracting mass. For, as above,

$$\frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = \iiint \frac{OX\,dx.dy.dz}{\{(f-x)^2+(g-y)^2+(h-z)^2\}^{\frac{5}{2}}}.$$

When the attracted particle is not a portion of the mass, then xyz will never equal fgh respectively; and consequently the expression under the signs of integration vanishes for every particle of the mass:

$$\therefore \frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = 0.$$

This equation was first given by Laplace: and Poisson was the first who shewed, that it was not true when the attracted particle is part of the attracting mass. The error arises in consequence of the expression under the signs of integration not vanishing for all values of xyz; since it equals $\frac{0}{0}$ when $x=f$, $y=g$, $z=h$.

To determine the value of $\frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2}$ in this case, suppose a sphere described in the body so that it shall include the attracted particle: and let $V = U + U'$, U referring to the sphere, and U' to the excess of the body over the sphere. Then, by what is already proved,

$$\frac{d^2U'}{df^2} + \frac{d^2U'}{dg^2} + \frac{d^2U'}{dh^2} = 0.$$

$$\text{Hence } \frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = \frac{d^2U}{df^2} + \frac{d^2U}{dg^2} + \frac{d^2U}{dh^2}.$$

The centre of the sphere may be chosen as near the attracted particle as we please, and therefore the radius of the sphere may be taken so small that its density may be considered ultimately uniform and equal to that at the particle (fgh), which we shall call ρ'.

Let $f'g'h'$ be the co-ordinates to the centre of the sphere; then the attractions of the sphere on the attracted particle parallel to the axes are, by Art. 149, 150,

$$\frac{4}{3}\pi\rho'(f-f'),\quad \frac{4}{3}\pi\rho'(g-g'),\quad \frac{4}{3}\pi\rho'(h-h'),$$

$$\text{or } -\frac{dU}{df},\quad -\frac{dU}{dg},\quad -\frac{dU}{dh}, \text{ by Art. 167;}$$

$$\therefore \frac{d^2U}{df^2} + \frac{d^2U}{dg^2} + \frac{d^2U}{dh^2} = -4\pi\rho';$$

$$\therefore \frac{d^2V}{df^2} + \frac{d^2V}{dg^2} + \frac{d^2V}{dh^2} = -4\pi\rho',$$

when the attracted particle is within the attracting mass.

Since the attracting body is nearly spherical we shall find it most convenient to transform our rectangular to polar co-ordinates, the origin of the radius vector of the surface being near the centre.

PROP. *To transform* R *and the partial differential equation in* R *to polar co-ordinates.*

169. Let r, θ, ω be the co-ordinates to the point (fgh),

r', θ', ω' .... (xyz),

the angles θ and θ' being measured from the axis of z: ω and ω' being the angles which the planes on which θ and θ' are measured make with the plane zx; as in fig. 30 and in Art. 87. Ex. 25 ϕ being replaced by ω;

$$\therefore\ f = r\sin\theta\cos\omega,\quad g = r\sin\theta\sin\omega,\quad h = r\cos\theta,$$

$$x = r'\sin\theta'\cos\omega',\quad y = r'\sin\theta'\sin\omega',\quad z = r'\cos\theta'.$$

These are the same as

$$r^2 = f^2 + g^2 + h^2,\quad \cos\theta = \frac{h}{\sqrt{f^2+g^2+h^2}},\ \tan\omega = \frac{g}{f}\ldots(1);$$

$$\therefore\ \frac{dR}{df} = \frac{dR}{dr}\frac{dr}{df} + \frac{dR}{d\theta}\frac{d\theta}{df} + \frac{dR}{d\omega}\frac{d\omega}{df},$$

$$\frac{d^2R}{df^2} = \frac{d}{df}\frac{dR}{dr}\frac{dr}{df} + \frac{d}{df}\frac{dR}{d\theta}\frac{d\theta}{df} + \frac{d}{df}\frac{dR}{d\omega}\frac{d\omega}{df}$$

$$+ \frac{dR}{dr}\frac{d^2r}{df^2} + \frac{dR}{d\theta}\frac{d^2\theta}{df^2} + \frac{dR}{d\omega}\frac{d^2\omega}{df^2}$$

$$= \frac{d^2R}{dr^2}\frac{dr^2}{df^2} + \frac{d^2R}{d\theta^2}\frac{d\theta^2}{df^2} + \frac{d^2R}{d\omega^2}\frac{d\omega^2}{df^2}$$

$$+ 2\frac{d^2R}{dr\,d\theta}\frac{dr}{df}\frac{d\theta}{df} + 2\frac{d^2R}{dr\,d\omega}\frac{dr}{df}\frac{d\omega}{df} + 2\frac{d^2R}{d\theta\,d\omega}\frac{d\theta}{df}\frac{d\omega}{df}$$

$$+ \frac{dR}{dr}\frac{d^2r}{df^2} + \frac{dR}{d\theta}\frac{d^2\theta}{df^2} + \frac{dR}{d\omega}\frac{d^2\omega}{df^2}.$$

The expressions for $\frac{d^2R}{dg^2}$ and $\frac{d^2R}{dh^2}$ are of the same form.

These must all be added together and equated to zero. When this is effected the formulæ (1) make

the coefficient of $\frac{d^2R}{dr^2} = \frac{dr^2}{df^2} + \frac{dr^2}{dg^2} + \frac{dr^2}{dh^2} = 1$,

the coefficient of $\frac{d^2R}{d\theta^2} = \frac{d\theta^2}{df^2} + \frac{d\theta^2}{dg^2} + \frac{d\theta^2}{dh^2} = \frac{1}{r^2}$,

the coefficient of $\frac{d^2R}{d\omega^2} = \frac{d\omega^2}{df^2} + \frac{d\omega^2}{dg^2} + \frac{d\omega^2}{dh^2} = \frac{1}{r^2\sin^2\theta}$,

the coefficient of $\frac{d^2R}{drd\theta} = 2\frac{dr}{df}\frac{d\theta}{df} + 2\frac{dr}{dg}\frac{d\theta}{dg} + 2\frac{dr}{dh}\frac{d\theta}{dh} = 0$,

the coefficient of $\frac{d^2R}{drd\omega} = 2\frac{dr}{df}\frac{d\omega}{df} + 2\frac{dr}{dg}\frac{d\omega}{dg} + 2\frac{dr}{dh}\frac{d\omega}{dh} = 0$,

the coefficient of $\frac{d^2R}{d\theta d\omega} = 2\frac{d\theta}{df}\frac{d\omega}{df} + 2\frac{d\theta}{dg}\frac{d\omega}{dg} + 2\frac{d\theta}{dh}\frac{d\omega}{dh} = 0$,

the coefficient of $\frac{dR}{dr} = \frac{d^2r}{df^2} + \frac{d^2r}{dg^2} + \frac{d^2r}{dh^2} = \frac{2}{r}$,

the coefficient of $\frac{dR}{d\theta} = \frac{d^2\theta}{df^2} + \frac{d^2\theta}{dg^2} + \frac{d^2\theta}{dh^2} = \frac{\cos\theta}{r^2\sin\theta}$,

the coefficient of $\frac{dR}{d\omega} = \frac{d^2\omega}{df^2} + \frac{d^2\omega}{dg^2} + \frac{d^2\omega}{dh^2} = 0$.

Hence the equation in R becomes

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{1}{r^2}\frac{d^2R}{d\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{dR}{d\theta} + \frac{1}{r^2\sin^2\theta}\frac{d^2R}{d\omega^2} = 0;$$

$$\therefore\ r\frac{d^2rR}{dr^2} + \frac{d^2R}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dR}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2R}{d\omega^2} = 0.$$

Put $\cos\theta = \mu$ and $\cos\theta' = \mu'$: then

$$r\frac{d^2rR}{dr^2} + \frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2R}{d\omega^2} = 0.$$

Prop. *To explain the method of expanding R in a series.*

170. The expression for R becomes

$$\{r^2 + r'^2 - 2rr'[\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega')]\}^{-\frac{1}{2}},$$

and this may be expanded into either of the series

$$\left.\begin{array}{l} P_0\dfrac{1}{r'} + P_1\dfrac{r}{r'^2} + \dots\dots\dots\dots + P_i\dfrac{r^i}{r'^{i+1}} + \dots\dots \\ \text{or } P_0\dfrac{1}{r} + P_1\dfrac{r'}{r^2} + \dots\dots\dots\dots + P_i\dfrac{r'^i}{r^{i+1}} + \dots\dots \end{array}\right\}\dots\dots(1),$$

where $P_0, P_1, \dots\dots\dots P_i, \dots\dots\dots$ are all rational and entire functions of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$, and the same functions of μ', $\sqrt{1-\mu'^2}\cos\omega'$, and $\sqrt{1-\mu'^2}\sin\omega'$: the general coefficient P_i is of i dimensions in μ, $\sqrt{1-\mu^2}\cos\omega$ and $\sqrt{1-\mu^2}\sin\omega$.

The greatest value of P_i (disregarding its sign) is unity.

For if we put

$$\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\omega-\omega') = \cos\phi = \tfrac{1}{2}\left(x + \frac{1}{x}\right),$$

then P_i = coefficient of c^i in

$$(1 + c^2 - 2c\cos\phi)^{-\frac{1}{2}}, \text{ or } (1-cx)^{-\frac{1}{2}}\left(1 - \frac{c}{x}\right)^{-\frac{1}{2}}$$

$$= \text{coefficient of } c^i \text{ in}$$

$$\left\{1 + \tfrac{1}{2}cx + \frac{1.3}{2.4}c^2x^2 + \dots\dots\right\}\left\{1 + \tfrac{1}{2}\frac{c}{x} + \frac{1.3}{2.4}\frac{c^2}{x^2} + \dots\dots\right\}$$

$$= A\left(x^i + \frac{1}{x^i}\right) + B\left(x^{i-2} + \frac{1}{x^{i-2}}\right) + \dots\dots$$

$$= 2A\cos i\phi + 2B\cos(i-2)\phi + \dots\dots$$

A, B …… being all positive and finite: the greatest value of this is when $\phi = 0$: hence P_i is greatest when $\phi = 0$.

But then P_i = coefficient of c^i in $(1 + c^2 - 2c)^{-\frac{1}{2}}$ or $(1 - c)^{-1}$

$$= \dots\dots\dots\dots\dots\dots (1 + c + c^2 + \dots + c^i \dots)$$

$$= 1.$$

Hence 1 is the greatest value of P_i. It follows that the first or second of the series (1) will be convergent according as r is less than or greater than r'.

To obtain equations for calculating the coefficients P_0, $P_1, \dots P_i \dots$ substitute either of the series (1) in the differential equation for R in the last Art., and equate the powers of r to zero: the general term gives the following equation,

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\frac{d^2P_i}{d\omega^2} + i(i+1)P_i = 0,$$

from which P_i should be calculated. The series for R would then be known.

This equation we meet with very frequently in the higher branches of Physical Science. It has never yet been integrated; but it enables us to derive certain properties of the function P_i. Laplace was the first who introduced this function and its properties into mathematical calculations. The functions P_0, P_1, P_2, ... P_i... are accordingly called *Laplace's Coefficients* of the order 0, 1, 2,... i ... respectively.

171. We are now able to write down the series from which V is to be calculated. By Arts. 167 and 170 we have $V = \iiint \rho R\, dx\, dy\, dz$: substitute for x, y, z, the values introduced in Art. 169; and take the proper limits; and we have

$$V = \int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi} \rho\left\{P_0\frac{r'^2}{r} + P_1\frac{r'^3}{r^2} + \dots + P_i\frac{r'^{i+2}}{r^{i+1}} + \dots\right\} dr'\, d\mu'\, d\omega'$$

$$\text{or} \int_0^{\mathrm{r}}\int_{-1}^{1}\int_0^{2\pi} \rho\left\{P_0 r' + P_1 r + P_2\frac{r^2}{r'} + \dots + P_i\frac{r^i}{r'^{i-1}} + \dots\right\} dr'\, d\mu'\, d\omega'$$

according as r is greater or less than unity: r is the value of r' at the surface of the body.

We shall now proceed to reduce these series to the various cases we shall have to consider; and afterwards shall deduce

the properties of Laplace's coefficients, which enable us to perform the integrations.

We must first calculate V for a sphere by actual integration.

Prop. *To calculate the value of* V *for a homogeneous sphere.*

172. Let the sphere be referred to polar co-ordinates, the centre being the pole (fig. 71): C the attracted particle; $OC = r$; P a particle in a shell of the sphere of which the radius $OP = r_1$, $\angle POC = \theta$, $\angle PMm = \omega$; a the radius of the sphere: then $PC = \sqrt{r^2 + r_1^2 - 2rr_1 \cos\theta}$; and the mass of the element at $P = \rho r_1^2 \sin\theta dr_1 d\theta d\omega$, the limits of ω are 0 and 2π; of θ are 0 and π; of r_1 are 0 and a;

$$\therefore\ V = \int_0^a \int_0^\pi \int_0^{2\pi} \frac{\rho r_1^2 \sin\theta\, dr_1\, d\theta d\omega}{\sqrt{r^2 + r_1^2 - 2rr_1 \cos\theta}}$$

$$= 2\pi\rho \int_0^a \int_0^\pi \frac{r_1^2 \sin\theta\, dr_1\, d\theta}{\sqrt{r^2 + r_1^2 - 2rr_1\cos\theta}}$$

$$= 2\pi\rho \int_0^a \frac{r_1}{r} \{\sqrt{r^2 + r_1^2 - 2rr_1\cos\theta} + \text{const.}\} dr_1$$

$$= 2\pi\rho \int_0^a \frac{r_1}{r} \{(r + r_1) \mp (r - r_1)\}\, dr_1$$

$-$ when C is without, and $+$ when C is within the shell,

$$= \frac{4\pi\rho}{r} \int_0^a r_1^2 dr_1 = \frac{4\pi\rho a^3}{3r},$$

when C is without the sphere.

And when C is within the sphere, the part of V for the shells which enclose $C = 2\pi\rho \int_r^a 2r_1 dr_1 = 2\pi\rho\ (a^2 - r^2)$: and the part of V for the other shells of the sphere

$$= \frac{4\pi\rho}{r} \int_0^r r_1^2 dr_1 = \tfrac{4}{3}\pi\rho r^2.$$

Hence $V = \dfrac{4\pi\rho a^3}{3r}$ for an *external* particle,

$$V = 2\pi\rho a^2 - \tfrac{2}{3}\pi\rho r^2 \text{ for an } internal \text{ particle.}$$

We shall find the use of these in the next two Propositions.

Prop. *To find the attraction of a homogeneous body, differing little from a sphere in form, upon a particle without it.*

173. Since the attracted particle is without the attracting mass we must expand V in a descending series of powers of r: we shall therefore take the *first* series of Art. 171.

Let the mean radius of the body $= \mathrm{a}$: and let $\mathrm{a}\,(1 + \alpha y')$ be the variable radius, y' being a function of μ' and ω', and α being a very small numerical quantity of which the square and higher powers are to be neglected.

Then, for the excess of the attracting mass over the sphere of which the radius $= \mathrm{a}$, the value of V

$$= \alpha\rho \int_{-1}^{1}\int_{0}^{2\pi}\left\{\frac{\mathrm{a}^3}{r}P_0 + \frac{\mathrm{a}^4}{r^2}P_1 + \dots\dots + \frac{\mathrm{a}^{i+3}}{r^{i+1}}P_i + \dots\dots\right\} y'd\mu'd\omega'.$$

$$\text{Let } \int_{-1}^{1}\int_{0}^{2\pi} P_i y' d\mu' d\omega' = U_i,$$

hence for the excess over the sphere we have

$$V = \alpha\rho\left\{\frac{\mathrm{a}^3}{r}U_0 + \frac{\mathrm{a}^4}{r^2}U_1 + \dots\dots + \frac{\mathrm{a}^{i+3}}{r^{i+1}}U_i + \dots\dots\right\}.$$

But for the sphere of which the radius is a, $V = \dfrac{4\pi\rho\mathrm{a}^3}{3r}$ by Art. 172.

Hence for the whole mass

$$V = \frac{4\pi\rho\mathrm{a}^3}{3r} + \frac{\alpha\rho\mathrm{a}^3}{r}\left\{U_0 + \frac{\mathrm{a}}{r}U_1 + \dots\dots + \frac{\mathrm{a}^i}{r^i}U_i + \dots\right\}$$

and the attraction $= -\dfrac{dV}{dr}$, (see Art. 167),

$$= \frac{4\pi\rho a^3}{3r^2} + \frac{\alpha\rho a^3}{r^2}\left\{U_0 + \frac{2a}{r}U_1 + \ldots\ldots + \frac{(i+1)a^i}{r^i}U_i + \ldots\right\}.$$

PROP. *To find the attraction of a homogeneous body, differing but little from a sphere in form, upon an internal particle.*

174. We must in this case expand V in an ascending series of powers of r: we shall therefore take the *second* series of Art. 171. By proceeding as above we have the following value of V, as far as regards the excess of the attracting mass over the sphere of which the radius $=$ a;

$$\alpha\rho\int_{-1}^{1}\int_{0}^{2\pi}\left\{a^2 P_0 + ar P_1 + \ldots\ldots + \frac{r^i}{a^{i-2}}P_i + \ldots\right\} y' d\mu' d\omega'$$

$$\text{or } \alpha\rho a^2\left\{U_0 + \frac{r}{a}U_1 + \ldots\ldots + \frac{r^i}{a^i}U_i + \ldots\right\}.$$

Also for the sphere of which the radius $=$ a the value of V is $2\pi\rho a^2 - \frac{2}{3}\pi\rho r^2$ (Art. 172). Hence for the whole mass

$$V = 2\pi\rho a^2 - \tfrac{2}{3}\pi\rho r^2 + \alpha\rho a^2\left\{U_0 + \frac{r}{a}U_1 + \ldots + \frac{r^i}{a^i}U_i + \ldots\right\},$$

and the attraction $= -\dfrac{dV}{dr}$

$$= \tfrac{4}{3}\pi\rho r - \alpha\rho a\left\{U_1 + \frac{2r}{a}U_2 + \ldots + \frac{ir^{i-1}}{a^{i-1}}U_i + \ldots\right\}.$$

175. The calculation of the functions $U_0, U_1, \ldots U_i, \ldots$ can be effected without integration when we know the equation to the surface of the body. We proceed to demonstrate this in the three following Propositions.

PROP. *To prove that if* Q_i *and* $R_{i'}$ *be two of Laplace's Coefficients,* then $\int_{-1}^{1}\int_{0}^{2\pi} Q_i R_{i'} d\mu\, d\omega = 0$, i *and* i′ *being different integers.*

176. By the equation of Laplace's Coefficients, (Art. 170.),

$$i(i+1)\,Q_i = -\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} - \frac{1}{1-\mu^2}\,\frac{d^2Q^i}{d\omega^2}\,;$$

$$\therefore \int_{-1}^{1}\int_{0}^{2\pi} Q_i R_{i'}\,d\mu\,d\omega$$

$$= -\frac{1}{i(i+1)}\int_{-1}^{1}\int_{0}^{2\pi}\left(\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\} + \frac{1}{1-\mu^2}\,\frac{d^2Q_i}{d\omega^2}\right)R_{i'}\,d\mu\,d\omega.$$

Now by a double integration by parts

$$\int\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\}R_{i'}\,d\mu = (1-\mu^2)\frac{dQ_i}{d\mu}R_{i'} - (1-\mu^2)\frac{dR_{i'}}{d\mu}Q_i$$

$$+\int\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\}Q_i\,d\mu\,;$$

$$\therefore \int_{-1}^{1}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dQ_i}{d\mu}\right\}R_{i'}\,d\mu = \int_{-1}^{1}\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\}Q_i\,d\mu.$$

Again, $\displaystyle\int R_{i'}\frac{d^2Q_i}{d\omega^2}\,d\omega = R_{i'}\frac{dQ_i}{d\omega} - Q_i\frac{dR_{i}}{d\omega} + \int Q_i\frac{d^2R_{i'}}{d\omega^2}\,d\omega\,;$

$$\therefore \int_{0}^{2\pi} R_{i'}\,\frac{d^2Q_i}{d\omega^2}\,d\omega = \int_{0}^{2\pi} Q_i\frac{d^2R_{i'}}{d\omega^2}\,d\omega,$$

since when $\omega = 0$ and 2π, each of the functions Q_i, $R_{i'}$, $\dfrac{dQ_i}{d\omega}$, $\dfrac{dR_{i'}}{d\omega}$ has the same values, because they are functions of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$.

$$\text{Hence}\ \int_{-1}^{1}\int_{0}^{2\pi} Q_i R_{i'}\,d\mu\,d\omega$$

$$= -\frac{1}{i(i+1)}\int_{-1}^{1}\int_{0}^{2\pi}\left(\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dR_{i'}}{d\mu}\right\} + \frac{1}{1-\mu^2}\,\frac{d^2R_{i'}}{d\omega^2}\right)Q_i\,d\mu\,d\omega$$

$$= \frac{i'(i'+1)}{i(i+1)}\int_{-1}^{1}\int_{0}^{2\pi} Q_i R_{i'}\,d\mu\,d\omega,$$

by the equation of Laplace's Coefficients.

Hence $\int_{-1}^{1}\int_{0}^{2\pi} Q_i R_{i'}\, d\mu\, d\omega = 0$, when i and i' are unequal.

If $i = i'$ the above equation becomes identical and therefore gives no condition.

PROP. *To prove that a function of* μ, $\sqrt{1-\mu^2}\cos\omega$ *and* $\sqrt{1-\mu^2}\sin\omega$, *as* $F(\mu, \omega)$, *can be expanded in a series of Laplace's Coefficients: provided that* $F(\mu, \omega)$ *do not become infinite between the values* -1 *and* 1 *of* μ, *and* 0 *and* 2π *of* ω.

177. Let $\mu'\mu + \sqrt{1-\mu'^2}\sqrt{1-\mu^2}\cos(\omega' - \omega) = p$: then by Art. 170,

$$(1 + c^2 - 2cp)^{-\frac{1}{2}} = P_0 + P_1 c + P_2 c^2 + \ldots + P_i c^i + \ldots$$

c being any quantity not greater than unity.

Differentiating with respect to c,

$$\frac{p - c}{(1 + c^2 - 2cp)^{\frac{3}{2}}} = P_1 + 2P_2 c + \ldots + iP_i c^{i-1} + \ldots$$

Multiply the latter equation by $2c$ and add it to the former,

$$\therefore \frac{1 - c^2}{(1 + c^2 - 2cp)^{\frac{3}{2}}} = P_0 + 3P_1 c + 5P_2 c^2 + \ldots + (2i + 1) P_i c^i + \ldots$$

Now c being arbitrary we may put it $= 1$. Then the fraction on the left-hand side of this equation vanishes, except when $p = 1$, in which case the fraction equals $\frac{0}{0}$.

When $p = 1$, then

$$\cos(\omega' - \omega) = \frac{1 - \mu'\mu}{\sqrt{(1-\mu'^2)(1-\mu^2)}} = \sqrt{\frac{1 - 2\mu'\mu + \mu'^2\mu^2}{1 - \mu'^2 - \mu^2 + \mu'^2\mu^2}},$$

and that this may not be greater than unity we must take $\mu'^2 + \mu^2$ not greater than $2\mu'\mu$, or $(\mu' - \mu)^2$ not greater than

zero: hence $\mu' = \mu$, and therefore $\cos(\omega' - \omega) = 1$ and $\omega' = \omega$. These, then, are the values of μ' and ω' which make $p = 1$.

Hence the series

$$P_0 + 3P_1 + \ldots + (2i+1)P_i + \ldots$$

vanishes for all values of the variables μ, ω and μ', ω'; except when $\mu = \mu'$ and $\omega = \omega'$, in which case the series is indeterminate.

It follows, therefore, that for *any* limiting values of μ and ω, which include $\mu = \mu'$ and $\omega = \omega'$,

$$\iint (P_0 + 3P_1 + \ldots)\, d\mu\, d\omega = \int_{-1}^{1}\int_0^{2\pi} (P_0 + 3P_1 + \ldots)\, d\mu\, d\omega,$$

$$\text{and this} = \int_{-1}^{1}\int_0^{2\pi} P_0\, d\mu\, d\omega, \text{ by Art. 176,}$$

$$= 4\pi, \quad \text{since } P_0 = 1.$$

Now suppose in the following integral we take the *same* limiting values for the independent variables μ, ω and μ', ω' respectively; but keep the limits arbitrary: then

$$\iiiint F(\mu', \omega') \{P_0 + 3P_1 + \ldots + (2i+1)P_i + \ldots\}\, d\mu\, d\omega\, d\mu'\, d\omega'$$

$$= \iint \left(F(\mu', \omega') \iint \{P_0 + 3P_1 + \ldots\}\, d\mu\, d\omega \right) d\mu'\, d\omega'$$

$$= 4\pi \iint F(\mu', \omega')\, d\mu'\, d\omega'$$

$$= 4\pi \iint F(\mu, \omega)\, d\mu\, d\omega,$$

because the limits of μ, ω are, by hypothesis, the same as those of μ', ω';

$$\therefore \iint \left\{ 4\pi F(\mu,\omega) - \iint F(\mu',\omega') \{P_0 + 3P_1 + \ldots\}\, d\mu'\, d\omega' \right\} d\mu\, d\omega = 0;$$

and this being true for all values of μ and ω included within the limits -1 and 1, and 0 and 2π respectively, we have

$$F(\mu, \omega) = \frac{1}{4\pi}\iint F(\mu', \omega') \{P_0 + 3P_1 + \ldots\ldots + (2i+1)P_i + \ldots\} d\mu' d\omega'.$$

Now the general term of this, viz.

$$\frac{2i+1}{4\pi}\iint F(\mu', \omega') P_i d\mu' d\omega',$$

is a function of μ and ω which satisfies the equation of Laplace's coefficients in Art. 170; as may easily be seen, when it is remembered, that P_i satisfies that equation, and also that the equation is *linear*.

Hence $F(\mu, \omega)$ can be expanded is a series of Laplace's coefficients.

This property of these Coefficients we shall find of the greatest service as we proceed.

178. Scholium. In the First Edition of this Work the result of the last Article was obtained in a different way, which we shall now only describe: we shall at the same time point out the marks of difference between that and the one now given.

179. It will be observed, that $P_0 + 3P_1 + \ldots$ is a *discontinuous* function; because it equals zero for all values of μ and ω, except for those in particular when $\mu = \mu'$ and $\omega = \omega'$. This discontinuity has been introduced by making $c = 1$.

To avoid introducing functions of this character, we before followed Poisson's method, in his *Théorie Mathématique de la Chaleur*, in which c is not put $= 1$ till the *end* of the operation. The process was as follows:

It was borne in mind, that the ultimate intention was to put $c = 1$, and that in consequence of this the only values of the function which would have any effect on our *ultimate* result were those which correspond to values of μ and ω differing but slightly from μ' and ω'. We therefore calculated the value of the integral

$$\iint F(\mu', \omega') \{P_0 + 3cP_1 + \ldots + (2i+1)c^i P_i + \ldots\} \, d\mu' d\omega',$$

or rather, of its equivalent,

$$\iint \frac{(1-c^2) F(\mu', \omega') \, d\mu' d\omega'}{(1+c^2-2cp)^{\frac{3}{2}}},$$

by substituting for μ' and ω' values $\mu + \nu$ and $\omega + x$, where ν and x were supposed very small: and yet we chose *any* limits for the variables μ', ω' that we found most *convenient* for the integrations, bearing in mind, that all the redundant terms, that we thus introduced, would vanish in our ultimate result by making $c = 1$. This artifice of M. Poisson's in fact amounts to this; we substitute for one function in its general form, another and a very different function in its general form; but one which is far more easy of integration; and which, in the one particular case which we intend ultimately to use, coincides in form with the first function.

180. The demonstration given above, in Art. 177, resembles that given by Mr. O'Brien in Art. 29 of his *Mathematical Tracts*, and was indeed suggested to the Author by that work. Mr. O'Brien has given another very excellent proof of the same property in Art. 28.

PROP. *To prove that $F(\mu, \omega)$ can be expanded in* only one *series of Laplace's Coefficients.*

181. For if possible suppose the function can be expanded in two distinct forms: viz.

$$Q_0 + Q_1 + \ldots + Q_i + \ldots \text{ and } R_0 + R_1 + \ldots + R_i + \ldots$$

Then $(Q_0 - R_0) + (Q_1 - R_1) + \ldots + (Q_i - R_i) + \ldots = 0.$

Multiply by P_i, and integrate, and we have, by Art. 176,

$$\int_{-1}^{1} \int_0^{2\pi} P_i (Q_i - R_i) \, d\mu d\omega = 0:$$

and by the principle involved in Art. 177, $Q_i' - R_i'$,

$$= \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}(Q_i - R_i)\,\{P_0 + \dots + (2i+1)\,P_i + \dots\}\,d\mu\,d\omega$$

$$= \frac{2i+1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}(Q_i - R_i)\,P_i\,d\mu\,d\omega$$

$= 0$, by what is proved above;

$$\therefore\ Q' = R'_i, \text{ or } Q_i = R_i:$$

and the two expansions are term for term the same.

Cor. If we have two series of Laplace's Coefficients equal to each other, as,

$$Y_0 + Y_1 + \dots + Y_i + \dots = Z_0 + Z_1 + \dots + Z_i + \dots$$

then $Y_i = Z_i$. For multiply both sides by any coefficient Q_i of the order i, and integrate, and we easily see, as above, that $Y_i = Z_i$.

182. We are now able to shew, as we promised in Art. 175, that the functions $U_0\ U_1\ U_2 \dots\dots$ can be calculated without integration where the equation of the surface of the attracting body is known.

For y can be expanded in a series of Laplace's Coefficients,

$$\frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\{P_0 + 3P_1 + \dots\dots + (2i+1)\,P_i + \dots.\}\,y'\,d\mu'\,d\omega'$$
$$= Y_0 + Y_1 + \dots\dots + Y_i + \dots\dots$$

and this series admits of only one form by what has just been proved. Hence when the equation to the surface, and therefore y, is known the functions $Y_0, Y_1, \dots\dots Y_i, \dots$ are determinate, and we may equate terms of the same order in the two series for y written above: (see Art. 181. Cor.)

$$\therefore \int_{-1}^{1}\int_{0}^{2\pi}(2i+1)\,P_i y'\,d\mu'\,d\omega' = 4\pi Y_i;$$

$$\therefore\ U_i = \int_{-1}^{1}\int_{0}^{2\pi}P_i y'\,d\mu'\,d\omega' \text{ (Art. 173.)} = \frac{4\pi}{2i+1}\,Y_i,$$

and consequently when the equation to the surface is known $U_0 U_1 U_2 \ldots\ldots U_i, \ldots$ are also known, as was mentioned in Art. 175.

By substituting for $U_0\, U_1 \ldots$ in the expressions of Arts. 173, 174, we have for an *external* particle

$$V = \frac{4\pi\rho a^3}{3r} + \frac{4\pi\rho\alpha a^3}{r}\left\{Y_0 + \frac{a}{3r}Y_1 + \ldots\ldots + \frac{a^i}{(2i+1)r^i}Y_i + \ldots\right\}:$$

and for an *internal* particle

$$V = 2\pi\rho a^2 - \tfrac{2}{3}\pi\rho r^2 + 4\pi\rho\alpha a^2\left\{Y_0 + \frac{r}{3a}Y_1 + \ldots + \frac{r^i}{(2i+1)a^i}Y_i + \ldots\right\}.$$

The property of Laplace's Coefficients proved in Art. 176, enables us to prove that Y_0 and Y_1 may be made to disappear from the expression for y by properly choosing the value of (a) and the origin of the radius-vector of the surface.

Prop. *By choosing* a *equal to the radius of the sphere of which the mass equals that of the attracting body we cause* Y_0 *to vanish from the series* $Y_0 + Y_1 + \ldots + Y_i + \ldots$; *and by taking the centre of gravity of the body as the origin of the radius-vector we cause* Y_1 *to vanish.*

183. If r, θ, ω be the co-ordinates to any point in the body, an element of the mass

$$= \rho dr r d\theta r \sin\theta d\omega = -\rho r^2 dr d\mu d\omega;$$

therefore the mass of the body

$$= \rho\int_0^{r}\int_{-1}^{1}\int_0^{2\pi} r^2 dr d\mu d\omega = \tfrac{1}{3}\rho\int_{-1}^{1}\int_0^{2\pi} r^3 d\mu d\omega,$$

then putting $r = a(1+\alpha y)$ we have

mass of body $=$ mass of sphere (rad. $=$ a) $+ \rho a^3\alpha\int_{-1}^{1}\int_0^{2\pi} y\, d\mu d\omega$

$= \text{mass of sphere} + \rho a^3\alpha\int_{-1}^{1}\int_0^{2\pi} Y_0 d\mu d\omega$ by Art. 176,

(since the multiplier of y is constant, and $\therefore$ a Laplace's coef. of the form Z_0)

$=$ mass of sphere $+ 4\pi\rho a^3\alpha Y_0$, since Y_0 is constant.

If, then, a be taken equal to the radius of the sphere of which the mass equals the mass of the body $Y_0 = 0$, as was stated.

184. Again: let $\bar{x}\,\bar{y}\,\bar{z}$ be the co-ordinates to the centre of gravity of the body, M its mass: the co-ordinates to the element, of which the mass is $-\rho r^2 dr d\mu d\omega$, are

$$r\sqrt{1-\mu^2}\cos\omega,\quad r\sqrt{1-\mu^2}\sin\omega,\text{ and } r\mu;$$

$$\therefore\ M\bar{x} = \int_0^r\int_{-1}^1\int_0^{2\pi}\rho r^3\sqrt{1-\mu^2}\cos\omega\, dr d\mu d\omega$$

$$= \tfrac{1}{4}\int_{-1}^1\int_0^{2\pi}\rho \mathrm{r}^4\sqrt{1-\mu^2}\cos\omega\, d\mu d\omega,$$

$$M\bar{y} = \int_0^{\mathrm{r}}\int_{-1}^1\int_0^{2\pi}\rho \mathrm{r}^3\sqrt{1-\mu^2}\sin\omega\, dr d\mu d\omega$$

$$= \tfrac{1}{4}\int_{-1}^1\int_0^{2\pi}\rho r^4\sqrt{1-\mu^2}\sin\omega\, d\mu d\omega,$$

$$M\bar{z} = \int_0^{\mathrm{r}}\int_{-1}^1\int_0^{2\pi}\rho r^3\mu\, dr d\mu d\omega = \tfrac{1}{4}\int_{-1}^1\int_0^{2\pi}\rho \mathrm{r}^4\mu\, d\mu d\omega,$$

putting $\mathrm{r} = \mathrm{a}(1+\alpha y) = \mathrm{a}(1+\alpha Y_0 + \alpha Y_1 + \dots + \alpha Y_i + \dots)$, and observing that $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$, μ satisfy Laplace's Equation (Art. 170), and are of the first order (Art. 171), we have by Art. 176,

$$M\bar{x} = \rho \mathrm{a}^4\alpha\int_{-1}^1\int_0^{2\pi} Y_1\sqrt{1-\mu^2}\cos\omega\, d\mu d\omega,$$

$$M\bar{y} = \rho \mathrm{a}^4\alpha\int_{-1}^1\int_0^{2\pi} Y_1\sqrt{1-\mu^2}\sin\omega\, d\mu d\omega,$$

$$M\bar{z} = \rho \mathrm{a}^4\alpha\int_{-1}^1\int_0^{2\pi} Y_1\mu\, d\mu d\omega.$$

But Y_1, being a function of μ, $\sqrt{1-\mu^2}\cos\omega$, $\sqrt{1-\mu^2}\sin\omega$ of the first order, is of the form

$$A\sqrt{1-\mu^2}\cos\omega + B\sqrt{1-\mu^2}\sin\omega + C\mu;$$

$$\therefore\ M\bar{x} = \tfrac{4}{3}\pi\rho \mathrm{a}^4\alpha A,\quad M\bar{y} = \tfrac{4}{3}\pi\rho \mathrm{a}^4\alpha B,\quad M\bar{z} = \tfrac{4}{3}\pi\rho \mathrm{a}^4\alpha C.$$

Hence if we take the origin of co-ordinates at the centre of gravity $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$, and consequently $A = 0$, $B = 0$, $C = 0$ and therefore $Y_1 = 0$, as stated in the enunciation.

We shall in future parts of this work require to know the attraction of a body consisting of strata nearly spherical and varying in density according to any law. We shall therefore proceed to the calculation of these attractions.

PROP. *To find the attraction of a heterogeneous body upon a particle without it: the body consisting of thin strata nearly spherical, homogeneous in themselves, but differing one from another in density.*

185. Let $a'(1 + \alpha y')$ be the radius of the external surface of any stratum, a' being chosen so that

$$y' = Y_1' + Y_2' + \dots\dots\dots + Y_i' + \dots \text{ (Art. 183).}$$

Since the strata are supposed not to be similar to each other y' is a function of a' as well as of μ' and ω'. Let ρ' be the density of the stratum of which the mean radius is a'. Now the value of V for this stratum equals the difference between the values of V for two homogeneous bodies of the density ρ' and mean radii a' and $a' - da'$. But for the body of which the mean radius is a' (Art. 182.)

$$V = \frac{4\pi\rho' a'^3}{3r} + \frac{4\pi\alpha\rho' a'^3}{r}\left\{\frac{a'}{3r}Y_1' + \dots\dots + \frac{a'^i}{(2i+1)\,r^i}Y_i' + \dots\right\}$$

hence for the stratum of which the external mean radius is a',

$$V = \frac{4\pi\rho' a'^2}{r}da' + \frac{4\pi\alpha\rho'}{r}\frac{d}{da'}\left\{\frac{a'^4}{3r}Y_1' + \dots + \frac{a'^{i+3}}{(2i+1)r^i}Y_i' + \dots\right\}da',$$

and therefore for the whole body

$$V = \frac{4\pi}{r}\int_0^a \rho'\left\{a'^2 + \alpha\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1' + \dots + \frac{a'^{i+3}}{(2i+1)r^i}Y_i' + \dots\right)\right\}da'.$$

The attraction $= -\dfrac{dV}{dr}$, and is easily found by differentiation.

PROP. *To find the attraction of the same body on an internal particle.*

186. Let $r = a(1 + \alpha y)$ be the radius of the attracted particle. Then for the strata within the surface of which the radius is $a(1 + \alpha y)$ we have (Art. 185.)

$$V = \frac{4\pi}{r}\int_0^a \rho'\left\{a'^2 + a\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)r^i}Y_i' + \ldots\right)\right\}da'.$$

But for a stratum external to the attracted particle we obtain by Art. 182,

$$V = 4\pi\rho'a'da' + 4\pi\rho'a\frac{d}{da'}\left(\frac{ra'}{3}Y_1' + \ldots + \frac{r^i}{(2i+1)a'^{i-2}}Y_i' + \ldots\right)da',$$

and therefore for all the strata external to the particle

$$V = 4\pi\int_a^a \rho'\left\{a' + a\frac{d}{da'}\left(\frac{ra'}{3}Y_1' + \ldots + \frac{r^i}{(2i+1)a'^{i-2}}Y_i' + \ldots\right)\right\}da',$$

and consequently for the whole body

$$V = \frac{4\pi}{r}\int_0^a \rho'\left\{a'^2 + a\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)r^i}Y_i' + \ldots\right)\right\}da'$$

$$+ 4\pi\int_a^a \rho'\left\{a' + a\frac{d}{da'}\left(\frac{a'r}{3}Y_1' + \ldots + \frac{r^i}{(2i+1)a'^{i-2}}Y_i' + \ldots\right)\right\}da'.$$

From this the attraction, or $-\dfrac{dV}{dr}$, is easily obtained.

APPENDIX.

In the foregoing Articles the actual expansions of Laplace's Coefficients in series have not been given; because they are never made use of in this work. We will here state the steps of the method of expansion, and the intelligent student will be able to fill up the intermediate calculations.

By Art. 182, if $y = Y_0 + Y_1 + \ldots + Y_i + \ldots$ be any function of μ and ω, which is to be thus expanded in a series of Laplace's Coefficients, then

$$Y_i = \frac{2i+1}{4\pi}\int_{-1}^{1}\int_0^{2\pi} P_i y' \, d\mu' d\omega' \ldots\ldots(1).$$

To calculate Y_i by this formula, we must first calculate P_i. Now P_i is a rational and entire function of μ and $\sqrt{1-\mu^2}$

$\cos(\omega - \omega')$, Art. 170. The general term of P_i, viz. that involving $\cos n(\omega - \omega')$, can arise solely from the powers n, $n+2$, $n+4$,...of $\cos(\omega - \omega')$. Hence $(1-\mu^2)^{\frac{n}{2}}$ will occur as a factor of that term: and the other part of its coefficient will be a factor of the form $A_0\mu^{i-n} + A_1\mu^{i-n-2} + \ldots$ If the expansion of P_i, thus deduced in terms of arbitrary coefficients, be substituted in the equation of Laplace's Coefficients, and the coefficient of $\cos n(\omega - \omega')$ be equated to zero, we have

$$A_s = -\frac{(i-n-2s+2)(i-n-2s+1)}{2s(2i-2s+1)} A_{s-1};$$

by making s successively $= 1, 2, 3, \ldots$ we have $A_1, A_2, \ldots$ in terms of A_0: let these be substituted and we have the coefficient of $\cos n(\omega - \omega')$ equal

$$A_0(1-\mu^2)^{\frac{n}{2}} \left\{\mu^{i-n} - \frac{(i-n)(i-n-1)}{2(2i-1)}\mu^{i-n-2} + \ldots\right\}:$$

the part within the last brackets we will call $f(\mu)$. Now A_0 is a function of μ', but is independent of μ: and because P_i is the same function of μ' that it is of μ, if follows that

$$A_0 = a_n f(\mu'):$$

and therefore the coefficient of

$$\cos n(\omega - \omega') = a_n f(\mu') \cdot f(\mu),$$

where a_n is a numerical quantity. To find a_n we compare the first term of the ascending expansion of $a_n f(\mu') f(\mu)$ in powers of μ with the corresponding term in the coefficient of c_i in the actual expansion of

$$\left\{1 + c^2 - 2c\left[\mu\mu' + \sqrt{1-\mu'^2}\sqrt{1-\mu^2}\cos(\omega - \omega')\right]\right\}^{-\frac{1}{2}},$$

see Art. 170. This leads to the following result:

$$a_n = 2\left\{\frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i}\right\}^2 \frac{i(i-1)\ldots(i-n+1)}{(i+1)(i+2)\ldots(i+n)}:$$

this applies when $n = 1, 2, 3, \ldots$ but evidently not when $n = 0$: a_0 is found by equating coefficients to be

$$\left\{\frac{1.3.5\ldots\ldots(2i-1)}{1.2.3\ldots\ldots i}\right\}^2.$$

In this way the function P_i is calculated. We may then by equation (1) calculate Y_i the function y, and therefore y', being given.

When, however, we know that the number of coefficients in the series for y is finite, we may use another method. Let n be the number of coefficients. Then, by what has gone before, it appears, that the most general form of Y_n, an entire and rational function of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$, is

$$(1-\mu^2)^{\frac{n}{2}}\left\{A_n \cos n\omega + B_n \sin n\omega\right\}$$
$$+ (1-\mu^2)^{\frac{n-1}{2}}\mu\left\{A_{n-1}\cos(n-1)\omega + B_{n-1}\sin(n-1)\omega\right\}$$
$$+ \ldots\ldots$$

the most general form of Y_{n-1} is

$$(1-\mu^2)^{\frac{n-1}{2}}\left\{C_{n-1}\cos(n-1)\omega + D_{n-1}\sin(n-1)\omega\right\}$$
$$+ (1-\mu^2)^{\frac{n-2}{2}}\mu\left\{C_{n-2}\cos(n-2)\omega + D_{n-2}\sin(n-2)\omega\right\}$$
$$+ \ldots\ldots$$

Suppose F is the function, of the n^{th} order, which is to be expanded: subtract the above general form of Y_n from F, and assume A_n, B_n, ... so that the terms in $F - Y_n$ of the n^{th} order in μ, and $(1-\mu^2)^{\frac{1}{2}}$ may vanish. Then from $F - Y_n$ subtract the general form of Y_{n-1}, and determine C_{n-1}, D_{n-1}... in the same manner. In this way the values of Y_n, Y_{n-1}, ... Y_0 are determined, and F is expanded in a series of Laplace's coefficients.

This process proves, that (in the case of the number of terms being finite) there can be *only one* series: see Art. 181.

DYNAMICS.

CHAPTER I.

187. In this part of our Work we are engaged with the laws which regulate the motion of bodies. We shall proceed therefore to explain the means we use for measuring the motion of a body algebraically.

The *position* of a body in space, considering the body as a material particle, is determined at any instant by its distances from three fixed planes at right angles to each other: these distances are called the *co-ordinates* of the particle; and the position of a rigid body in space is determined at any instant by the co-ordinates of a given point of the body and the angles which three fixed lines in the body make with three fixed lines in space.

If the body be in motion the co-ordinates will be continually changing in magnitude: and one of the chief objects of the Science of Dynamics is to find the analytical relation between each co-ordinate and the time of motion.

188. We shall pause, however, a little to make a remark which cannot be too carefully remembered.

All our ideas of the magnitude of quantities (such as space, time, and so on) are ideas of *comparative* and not *absolute* magnitude: for the same quantity may be great when compared with one standard, and small when compared with another. In consequence of this it is necessary, in order to avoid ambiguity, to choose for quantities of the same kind a certain standard to which they may be referred. This standard is called the *unit* of these quantities. Thus we speak of the

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unit of time, and the unit of space; by which we mean the duration of time and the extent of space which we choose as standards to which all other quantities of these species are to be severally referred.

It is by this means that quantities are made the subjects of numerical calculation. For instance, when we say that a body describes a space x in a time t, we mean, that x and t represent the ratios which the space described and the time of describing it bear to their respective units: and so of all other quantities. We forbear choosing these units at once because it generally happens, as we shall see, that by a judicious selection our formulæ may be materially simplified. Before closing these remarks we will observe, that though in the same calculation we must have only one standard of quantities of the same kind, yet in different calculations we need not retain the same unit, so long as we bear in mind what unit is chosen in each calculation. Thus in one calculation we might take the length of the mean day as the unit to which we should refer all portions of time; while in another calculation we might take a year as the unit of time. We return now to the consideration of the means of measuring the motion of a body.

DEFINITION AND MEASURE OF VELOCITY.

189. *Velocity* is a term used to indicate the degree of quickness or slowness with which a body moves. Velocity is measured by the space the body describes in a given time: and may be uniform or variable.

Uniform Velocity. Velocity is said to be uniform when the body passes through equal spaces in equal times.

It appears, then, that the magnitude of the velocity of a body moving uniformly depends conjointly upon the space described and the time of describing the space; and is greater or less exactly in the proportion in which the space described in any given time is greater or less, and the time of describing any given space is less or greater.

Consequently when bodies move with different uniform velocities, these velocities are in the proportion of the ratios

which the spaces described in any times bear respectively to the times of describing them.

Suppose, then, that a body moving uniformly with the velocity v describes a space s in the time t: also suppose that a body moving uniformly with the *unit* of velocity describes a space S in the time T: then by what precedes

$$v : 1 :: \frac{s}{t} : \frac{S}{T};$$

$$\therefore\ v = \frac{T}{S}\frac{s}{t}.$$

In this formula the only arbitrary quantities are S and T; we shall choose them so as to simplify the formula as much as possible: in choosing their values we fix the unit of velocity.

We shall take $S = 1$ and $T = 1$; we then have

$$v = \frac{s}{t},$$

the unit of uniform velocity being the velocity of a body moving uniformly through the unit of space in the unit of time.

It will be seen that the units of space and time are as yet quite arbitrary.

190. *Variable Velocity.* Velocity is said to be variable when the body in motion does not describe equal spaces in equal times.

Suppose a body moves uniformly and at the time t we wish to estimate its velocity. Let s' be the space described in *any* portion of time t', this time either terminating or commencing with the instant of expiration of the time t. Then, by what precedes, the velocity will equal $\frac{s'}{t'}$, however large or small t' be taken.

But when the velocity is not uniform the ratio $\frac{s'}{t'}$ is not the same for different values of t', and therefore cannot be taken as a measure of the velocity at the time t: unless we select some

particular value of t' always to be used. It will be necessary to select such a value of t', that the peculiar circumstances of the motion before and after the time t shall not affect our measure of velocity: it is evident that this value of t' must be indefinitely small.

Now if the time t' terminate with the time t, then the space $s - s'$ is described by the body in the time $t - t'$; and therefore by Taylor's Theorem, since s is a function of t,

$$s - s' = s - \frac{ds}{dt}t' + \frac{d^2s}{dt^2}\frac{t'^2}{1\,.\,2} - \ldots\ldots\ldots$$

$$\therefore \frac{s'}{t'} = \frac{ds}{dt} - \frac{d^2s}{dt^2}\frac{t'}{1\,.\,2} + \ldots\ldots\ldots$$

If t' commence with the expiration of t, then $s + s'$ is the space described in the time $t + t'$, and

$$s + s' = s + \frac{ds}{dt}t' + \frac{d^2s}{dt^2}\frac{t'^2}{1\,.\,2} + \ldots\ldots\ldots$$

$$\therefore \frac{s'}{t'} = \frac{ds}{dt} + \frac{d^2s}{dt^2}\frac{t'}{1\,.\,2} + \ldots\ldots\ldots$$

When t' is taken indefinitely small the values of the ratio $\frac{s'}{t'}$ are the same, and each equal to $\frac{ds}{dt}$. We shall therefore adopt this as the measure of variable velocity. It is the *limit* of the ratio of the space described in a portion of time after t, and the time of describing it. This is sometimes expressed by saying, that variable velocity is measured by the ratio of the space, that *would be* described in any time if the velocity ceased varying at the time t, and the time of describing that space.

It will be observed that in selecting this as the measure of variable velocity we do not violate any conditions previously established in reference to uniform velocity: we only restrict those conditions, inasmuch as t' may be of any value in the case of uniform velocity, but we take it indefinitely small in that of variable velocity. But, notwithstanding this, the formula

$v = \frac{ds}{dt}$ includes the case of uniform motion: for if v be constant we have by integration $vt = s$ (the constant of integration vanishes, since when $t = 0$, $s = 0$), and this is the formula already adopted for uniform motion.

Hence, then, *if* v *be the velocity of a body moving uniformly or not at the time* t *and* s *be the space described in that time, the quantities* v, s, t *are connected by the formula*

$$\mathrm{v} = \frac{\mathrm{ds}}{\mathrm{dt}}.$$

Having thus explained the conventional terms and means used for measuring algebraically the motion of a body, we shall enter upon an enquiry into the laws which regulate this motion. Since, as far as we know, it might have pleased the Author of the Universe to endue matter with laws and properties different from those which He has chosen to impress, it is evident that these laws can be discovered by no process of abstract reasoning, but solely by an appeal to experiment.

FIRST LAW OF MOTION.

191. As the simplest case we shall first consider the motion of a body uninfluenced by external forces. We have already defined *force* to be any cause which produces or tends to produce motion in a body; see Art. 5.

Throughout the whole universe it is impossible to find a single spot free from the action of force. It is consequently beyond our power to determine by direct experiment the nature of the motion of a body uninfluenced by external causes. But by combining the results of various experiments we shall be able to eliminate, so to speak, the principles which are foreign to our enquiry, and in that way ascertain the laws we are seeking.

Experience teaches us, that the more external causes are removed the more nearly uniform is the motion of a body.

A bowl thrown along a bowling green is observed to move slower and slower till it finally stops: but the smoother the

green is made, the longer does the motion continue. If the bowl be thrown with the same velocity along a pavement the motion is of longer duration; and still longer when the motion takes place on a sheet of ice. One cause of the diminution of velocity is the friction of the body on the plane: this is inferred from the fact, that the retardation is less the smoother the plane on which the motion takes place. Also any change in the uniformity of the decrease of the velocity can always be attributed to some disturbing cause; as the greater roughness of the surface, and the deficiency in perfect horizontality.

The experiment shews likewise, that the motion is in a straight line, unless some assignable cause produce a deviation.

Steam-carriages moving on horizontal rail-roads, when once in motion, require a constant power of the engine to maintain a uniform velocity; and since, when the motion is uniform, the retarding effect of friction and the resistance of the air may be assumed to be constant, we infer (after what we have said in the case of the bowl) that the constant power of the engine exactly counterbalances the constant retarding force, and that therefore supposing them both removed the result would be a uniform motion.

The reader is referred to Desaguliers' *Course of Experimental Philosophy*, 4to. 1734, Vol. I. Lecture V. for more experiments upon the motion of bodies.

Philosophers have assumed, then, as a fundamental principle of the motion of matter that

A body in motion, not acted on by any external force, will move uniformly and in a straight line.

This is called the *First Law of Motion.*

192. It must not be imagined that these experiments *prove* the truth of the law here enunciated: for the law embraces an infinite variety of cases, and many in which it would be impracticable to make experiments. Also the roughness of the experiments prevents our supposing it proved even for the cases we have mentioned. The truth is, that the law is only *suggested* by the facts we have detailed; and it remains to be

seen whether or no this, in conjunction with other laws (which we shall soon consider), satisfies the tests we shall hereafter have to submit them to; whether, combined in endless variety, they will account for the numerous phenomena continually coming under our observation. It is found that they do lead to results which precisely accord with observation. Of the more obvious phenomena, the explanation of which depends on the truth of these laws, we may mention the prediction of the time of an eclipse and the certainty of its fulfilment. Results of this nature are the only satisfactory proofs.

193. It may be inferred from the First Law of Motion, that a body has no *internal* forces residing in it to influence its motion; for when all *external* forces are removed the motion is the same whatever be the nature or magnitude of the body. The state of the body when all external forces cease to act is quite independent of the body itself. In other words, matter has no inherent property of changing its state of motion. It is equally a result of experiment and observation, that matter has no inherent property of changing its state of rest (Art. 4). This property of matter, that when not acted on by any external force it continues in the same state, whether of rest or uniform rectilinear motion, is called its *Inertia.*

DYNAMICAL MEASURE OF FORCE.

194. The First Law of Motion enables us to extend the definition of force given in Arts. 5 and 191. We may now define it, not only as any cause, which produces or tends to produce motion in a body; but also as any cause, which changes the uniform and rectilinear motion of a body.

It appears then, that when a body moves with a variable velocity, force is acting on the body: and, conversely, when force acts upon a body, its velocity is continually changing. Now we take the magnitude of the change of velocity during a given time as the measure of the magnitude of the force which acts upon the body: and, for the sake of distinction, when force is measured in this manner it is termed *Accelerating*

Force.[*] We have already mentioned, that when force is measured statically, it is called *Pressure* (Art. 7).

Distinction between Finite and Impulsive Force.

195. Although the sources of force are very various, yet its effect in accelerating the motion is always measured in Dynamics by the change in velocity in a certain time. Thus when a body is dropped from the hand, the accelerating force of the Earth's attraction at any instant is estimated dynamically by the velocity generated in a given time after that instant. Suppose a body placed on a smooth horizontal table is drawn along by means of a thread passing over the edge of the table and attached to a falling body. The magnitude of the accelerating force, which causes the body to move on the table, is measured by the change in velocity in a given time. When a body is moved along a smooth horizontal table by means of a constrained spring, the accelerating force which causes the body to move, though differing in its source from the force mentioned in the last case, is measured in the same way. If a body resting on a smooth horizontal table be set in motion by the sudden blow of another body upon it, the accelerating force, which causes the motion is measured as before. When a ball is fired from a cannon the accelerating force, which causes the motion, is still measured by the velocity generated.

It will be seen in the first two of these cases (especially in the second if the descending body be small), that the motion is *gradually* communicated, the velocity increasing continuously. But in the last two cases it may perhaps be thought, that the motion is *instantaneously* communicated: this is not, however, true: for the time occupied in generating the velocity is of finite duration, although, to our senses, it is of inappreciable magnitude. That it is of finite duration appears, in

[*] When a force retards the velocity of a body, it is sometimes called a *retarding force*; but still it is of exactly the same nature as an accelerating force, since it is measured by the decrements, instead of the increments, of velocity in a given time. In short, a retarding force is an accelerating force when estimated in the direction of its action. Thus it will be seen that *retarding force* is merely a relative term, and is included in the term accelerating force.

the case of the collision of the bodies, from the fact, that if a small spot of ink be put upon the point of contact of either of the bodies before the motion takes place, then, after the collision, the ink is found spread over a larger surface than it occupied before, and on both bodies; shewing that the bodies suffered mutual compression and then separated, and this must have occupied time. In the case of the cannon ball, the expansive force of the ignited powder acts during the time, that the ball takes to move along the bore of the cannon. In both these instances, as well as in the others, the velocity of the body commences from zero and passes through successive and continuous gradations of magnitude, the only difference being, that the intensity of the force originating from the collision, and from the explosion, is very far greater than the intensity of the force arising from the Earth's attraction; and consequently the velocity which is generated in a falling body, in a few seconds by the attraction of the Earth may be generated by impact, or other such means, in an extremely short portion of time.

When a body moves under the action of a force a continual change of velocity takes place; and if the force cease to act the body will move uniformly in a straight line with its last acquired velocity, as the First Law of Motion teaches us. If the force act for a finite time, then our object is to discover such laws of nature and to establish such conventional rules as shall enable us to determine the velocity acquired and the space described by the body during any portion of the time that the force is in action. If, however, the force act for only an indefinitely short time, we are concerned only with the velocity and position after the action of the force ceases, since the changes that take place during the action of the force are so rapid, that the whole process of the action appears to our senses to be instantaneous*.

* We have a popular illustration of the effects of forces, which act for a finite time, and for an indefinitely short time, in the game of cricket. The bowler rotates his arm in order to give the ball velocity; he opens his hand and the ball flies from him with the velocity acquired; and (supposing he delivers the ball *full pitch*), after moving in a curve slightly deflected downwards by the Earth's attraction, is received upon the bat. Now this velocity was generated by the muscular effort of the bowler's arm acting on the ball during the finite time, that he retained it in his grasp. While this

196. We have made these remarks to shew, that it is necessary, in explaining the means of measuring force dynamically, to consider two cases: first, when the force is of such a nature as to require a finite time to generate a finite velocity; secondly, when the force is such as to generate a finite velocity in an indefinitely short time.

In the first case we shall call the force *Finite Accelerating Force:* in the second, *Impulsive Accelerating Force.* We shall, however, generally drop the term *Finite:* and it must therefore be remembered that when we speak of accelerating forces, we mean finite accelerating forces, and never impulsive accelerating forces, unless the term *impulsive* be prefixed.

We proceed now to explain more fully how accelerating forces, which require an appreciable duration of time to manifest their effects, are measured.

Measure of Finite Accelerating Force.

197. Finite accelerating force may be uniform or variable.

Uniform Accelerating Force. When equal velocities are generated in equal times the force is said to be uniform.

It appears, then, that the magnitude of the force depends conjointly upon the velocity generated by the action of the

this is going on the batter swings his bat, that it may acquire a great velocity; and the ball and bat come in collision: and what is the consequence? the ball flies back; not only is its original motion destroyed, but new motion is given to it, as if instantaneously, in an opposite direction.

We explain the phenomenon of this sudden recoil in the following manner. When the ball and bat come in contact their particles are moving in opposite directions, and tend to penetrate each other: but the molecular forces, by which the particles of each of the bodies are bound together, are too powerful to allow of this separation; nevertheless the relative positions of the particles are slightly changed by the yielding of the bodies, and in consequence of their unnatural restraint a mutual resultant pressure is exerted by the bat on the ball and by the ball on the bat, till their relative motion is destroyed: but the particles of the two bodies are still under restraint, when the motion is destroyed, and the mutual pressure of the bodies now acts to effect their separation, and new velocity is generated: this process, which we conceive represents the actual process in nature, goes on with inconceivable rapidity, in consequence of the great intensity of the molecular forces, which bind the particles of each body together. If the bat split, or the ball burst, then the molecular forces, which held together those particles which separate, were not powerful enough to resist the separation.

force, and the time in which this velocity is generated: and is greater or less exactly in the proportion in which the velocity generated in a given time is greater or less, and the time in which a given velocity is generated is less or greater.

Consequently when bodies are acted upon by different uniform accelerating forces, these forces are in the proportion of the ratios, which the velocities generated in any times bear respectively to the times in which they are generated.

Suppose, then, that a body acted on by the uniform accelerating force f has the velocity v generated in it in the time t: also suppose, that a body, acted on by the unit of uniform accelerating force, has a velocity V generated in the time T: then by what precedes

$$f : 1 :: \frac{v}{t} : \frac{V}{T}; \qquad \therefore f = \frac{T}{V}\frac{v}{t}.$$

In this formula the only arbitrary quantities are V and T: we shall choose them so as to simplify the formula as much as possible: in choosing their values we fix the unit of uniform accelerating force.

We shall take $V = 1$ and $T = 1$, we then have $f = \dfrac{v}{t}$,

the unit of uniform accelerating force being the force which generates in a body a unit of velocity in a unit of time.

We have already chosen the unit of velocity (Art. 191); we may consequently say, that the unit of uniform accelerating force is the force, that causes a body during each successive unit of time in its motion to describe a space greater by the unit of space than it did during the unit of time immediately preceding.

198. Hence, in uniformly accelerated motion, s the space described from rest, t the time of describing it, v the velocity acquired during that time, and f the constant force are connected by the equations

$$v = \frac{ds}{dt} \text{ and } f = \frac{v}{t};$$

the units of v and f being given in Arts. 191. and 197. By

means of these equations we can obtain four equations differing from each other, and each containing three of the quantities s, t, v, f. Thus, if we eliminate v we have

$$\frac{ds}{dt} = ft\,; \ \therefore\ s = \tfrac{1}{2}ft^2 \ldots\ldots\ldots\ldots (1), f \text{ is constant.}$$

By eliminating t we have

$$\frac{ds}{dv} = \frac{v}{f}\,; \ \therefore\ s = \frac{v^2}{2f} \ldots\ldots\ldots\ldots (2).$$

Also $v = ft \ldots\ldots\ldots (3)$, $2s = vt \ldots\ldots (4)$, by (2) (3).

199. *Variable Accelerating Force.* Accelerating force is said to be variable when equal degrees of velocity are not generated in equal times.

Suppose a body is moving under the action of a uniform accelerating force, and at the time t we wish to estimate the magnitude of the force. Let v' be the velocity generated in *any* portion of time t', this time either terminating or commencing with the instant of expiration of the time t. Then, by what precedes, the uniform force will equal $\frac{v'}{t'}$, however large or small t' be taken.

But when the force is not uniform the ratio $\frac{v'}{t'}$ is not the same for all values of t', and therefore cannot be taken as a measure of the force at the time t, unless we select some particular value of t' always to be taken. It will be necessary to select such a value of t', that the peculiar circumstances of the motion before and after the time t shall not affect our measure of accelerating force: it is evident, that this value of t' must be indefinitely small.

Now if the time t' terminate with the time t, then the velocity $v - v'$ is generated in the time $t - t'$, and therefore by Taylor's Theorem, since v is a function of t,

$$v - v' = v - \frac{dv}{dt}t' + \frac{d^2v}{dt^2}\frac{t'^2}{1\,.\,2} - \ldots\ldots$$

$$\therefore\ \frac{v'}{t'} = \frac{dv}{dt} - \frac{d^2v}{dt^2}\frac{t'}{2} + \ldots\ldots$$

If t' commence with the expiration of t then $v + v'$ is the velocity generated in the time $t + t'$;

$$\therefore\ v + v' = v + \frac{dv}{dt}t' + \frac{d^2v}{dt^2}\frac{t'^2}{1\,.\,2} + \ldots\ldots$$

$$\therefore\ \frac{v'}{t'} = \frac{dv}{dt} + \frac{d\ v}{dt^2}\frac{t'}{2} + \ldots\ldots$$

When t' is taken indefinitely small the values of the ratio $\frac{v'}{t'}$ are the same, and each equal to $\frac{dv}{dt}$. We shall therefore adopt this particular value as the measure of variable accelerating force. It is the *limit* of the ratio of the velocity generated in a portion of time after t, and the time of generating it. This is sometimes expressed by saying, that variable accelerating force is measured by the ratio of the velocity, that *would be* generated in any time, if the force ceased to vary, and the time of generating that velocity.

It will be observed (as in the case of variable velocity), that in selecting this as the measure of variable accelerating force we do not violate any conditions previously established in reference to uniform accelerating force: we only restrict these conditions, inasmuch as t' may be of any value in the case of uniform force, but we take it indefinitely small in the case of variable force. But, notwithstanding this, the formula $f = \frac{dv}{dt}$ includes the case of uniform motion: for if f be constant we have by integration $ft = v$ (the constant of integration vanishes since when $t = 0$, then $v = 0$), and this is the formula already adopted for uniform accelerating force. Hence, *if* f *be the accelerating force, uniform or variable, which generates the velocity* v *in a body in the time* t, then f, v, t *are connected by the equation* $\mathrm{f} = \frac{\mathrm{d\,v}}{\mathrm{d\,t}}$.

200. We have seen (Art. 190) that $v = \frac{ds.}{dt}$. Hence the equations connecting f, v, s, t are

$$v = \frac{ds}{dt}, \quad f = \frac{dv}{dt}\left(= \frac{d^2s}{dt^2}\right),$$

in which it must be observed, that the unit of velocity is the velocity of a body moving uniformly through a unit of space in a unit of time: and the unit of accelerating force is the uniform force which generates a unit of velocity in a unit of time.

Measure of Impulsive Accelerating Force.

201. Impulsive forces, as we have already stated, are such as generate a finite velocity in a body in an indefinitely short space of time. If such forces were to act for a portion of time of any sensible duration, they would generate an indefinitely great velocity, and so would carry the body beyond the reach of observation. But all the impulsive forces with which we are acquainted in nature act only for an indefinitely short time. And, since we have no means of determining the length of time during which they act, because it is inappreciably short, we cannot measure impulsive forces as we measure finite forces; viz. by the ratio of the velocity generated and the time of generating that velocity, or the limit of that ratio. Thus suppose a ball A originally at rest, is suddenly set in motion by the blow of a ball B. Again, suppose instead of this, that a third ball C of different material from B impinges on A, and generates the same velocity. Now owing to the peculiar circumstances of each case, such as the mass, the velocity, and the material of the impinging body, the times during which the bodies are in contact, that is, during which the impulsive forces in the two cases generate the same velocity, will in general be different in length; but both being inappreciably short we have no means of determining their ratio, and therefore no means of estimating the relative magnitudes of the forces, if we adopt the measure, which we use for finite forces.

We therefore measure impulsive forces by the whole velocity generated, irrespective of the time occupied in generating the velocity.

This removes all uncertainty and difficulty: nor is this measure less convenient than the other; for it is only with the

final velocity generated by impulsive forces, that we have to do; and never with the intermediate stages of velocity. In consequence, however, of this difference of measures of finite and impulsive forces we must have two distinct sets of equations for calculating their effects, as we shall see.

202. We find it convenient to divide impulsive forces into two classes; viz. those which generate velocity, and those which destroy velocity: the first we call *Impulsive Forces of the nature of Explosion*; the second *Impulsive Forces of the nature of Collision**.

203. One source of impulsive forces is the elasticity of bodies.

It is found that all rigid bodies rebound more or less when struck together: this property is termed their *elasticity:* no bodies are totally devoid of this property: yet some have it more eminently than others; balls of clay have little elasticity, but ivory balls and balls of glass are considerably elastic. The degree of elasticity is measured by the ratio which the velocity of rebound bears to the velocity at the first contact. The elasticity is perfect when these two velocities are the same, but this is a limit which no bodies actually attain. The cause of this property of matter is of course conjectural, and our conclusions as to its laws are deduced solely from experiment†.

* The following experimental fact seems to shew, that impulsive forces are of the same nature as finite forces, generating or destroying velocity by continuous gradations.

Robins' experiments on the velocity of bullets and cannon balls lead to the following result. If bullets of the same diameter and density impinge on the same solid substance with different velocities they will penetrate that substance to different depths, which will be in the duplicate ratio of those velocities nearly; Robins' *Mathematical Tracts*, edited by Wilson, Vol. I. p. 152.

This was proved by various experiments. Now a property of uniformly accelerating (or retarding) forces is, that the squares of the velocities generated (or destroyed) are proportional to the spaces described: Art. 198. Hence the retarding force of the solid substance used in each experiment was a uniform force. But the duration of its action was so short and its intensity so great, that although the changes effected by the force were continuous, yet they were so rapid, that the force comes under the denomination of what we term impulsive forces.

† Tables of the results of a series of experiments made by Mr. Hodgkinson, of Manchester, on the elasticity of bodies, will be found in Vol. III. p. 534. of the Reports of

SECOND LAW OF MOTION.

204. The next enquiry we shall make into the laws which regulate the motion of bodies is, how to calculate the combined effect of two or more causes acting simultaneously on a body. We must, as before, appeal to experimental facts for the solution of this question.

A ball rolled along the horizontal deck of a vessel moving equably will move on the deck as it would if the vessel were at rest; this is proved by experiment. Suppose S is the deck of a boat moving uniformly on a sheet of water, fig. 74: and in a given time suppose it moves to S'. Let A be the place of the ball at the beginning of the time of motion: and B' its place in space at the end. Draw AA' in the direction of the boat's motion, and equal to the distance through which the boat has moved; and join $A'B'$. Suppose AB is the space the ball would have described if the vessel had not moved. Now, as we have already stated, experiment shews that $A'B'$, the space actually described on the deck, is the same in reference to the vessel as if the vessel had been stationary. Hence $A'B'$ is equal and parallel to AB. From this we gather, that if two causes act simultaneously on a body to produce uniform motions, each cause will have its full effect in its own direction; and the body will be found at the extremity of the diagonal

of the British Association for the Advancement of Science. The following are the Conclusions deduced.

(1). All rigid bodies are possessed of some degree of elasticity: and among bodies of the same nature, the hardest are generally the most elastic.

(2). There are no perfectly hard inelastic bodies, as assumed by the earlier and some modern writers on Mechanics.

(3). The elasticity as measured by the velocity of recoil divided by the velocity of impact is a ratio, which, though decreasing as the velocity increases, is nearly constant, when the same rigid bodies are struck together with considerably different velocities.

(4). The elasticity as defined in (3), is the same whether the impinging bodies be great or small.

(5). The elasticity is the same, whatever be the relative weights of the impinging bodies.

(6). In impacts between bodies differing very much in hardness, the common elasticity is nearly that of the softer body.

(7). In impacts between bodies of which the hardness differs in any degree the resulting elasticity is made up of the elasticities of both; each body contributing a part of its own elasticity in proportion to its relative softness or compressibility.

of the parallelogram described on the linear spaces, which the body would have passed through under the action of the causes separately.

This principle is found to be true if one or both of the separate motions be not uniform. For a ball dropped from the top of the vertical mast of a vessel sailing uniformly, falls at the foot of the mast, although the vertical motion is not uniform.

The following experiment well illustrates this principle. Two balls are placed at the same height above the ground: one is projected horizontally, the other suffered to fall of itself: it is so contrived that the motions shall commence at the same instant. The result is that they are heard to strike the ground at the same time, although they describe very different paths, one ball having moved in a straight line, the other in a curve. This experiment shews that although one ball had a horizontal motion, still the attraction of the Earth produced the same effect on the two balls in a vertical direction.

The muscular efforts necessary to raise the arm, move the head, or raise the body, are the same on board a vessel sailing equably, or in a steam-carriage moving uniformly on a rail-road, as when the ship or carriage is at rest.

It can be proved independently of any mechanical principles that the Earth revolves round its axis from west to east; but the effort of moving a body from one place to another does not depend, *cæteris paribus*, on the point of the compass towards which the motion is directed. To bring to our aid, however, more delicate tests, it is found that the motion of a pendulum is precisely the same in whatever vertical plane it vibrates, whether east and west, or north and south, or in any other direction.

If a ball, more or less elastic, is dropt from the hand on the ground, it will rise to a certain height in consequence of the impulsive force, which is put in play during the restitution of its figure, after its compression ceases. But if the body have given to it a horizontal motion by the hand, it will descend in a curved line and ascend in a curved line; and experiment shews, that the height to which it rises is the same as in the first case. This shews, that the impulsive force has its own

effect on the body whatever other motion the body may have in consequence of other forces.

For more facts and experiments we refer again to Lecture V. of Desaguliers' *Experimental Philosophy*.

These facts point out to us the following general principle:

When a force acts upon a body in motion, the change of motion in magnitude and direction is the same as if the force acted on the body at rest.

This is called the *Second Law of Motion*.

For the full elucidation and proof of this Law we ought to make experiments with forces of all degrees of magnitude and motions combined in all directions; since, however, this can never be accomplished, we must have recourse to the expedient spoken of in Art. 192, to satisfy ourselves of the truth of this as well as the First Law.

CURVILINEAR MOTION.

205. We shall now shew the importance of this Second Law in enabling us to refer the curvilinear motion of a particle to three rectangular axes.

Since a particle, moving under the action of one or more forces, in the general case describes a curvilinear path, it is continually changing the direction of its motion. It becomes necessary, then, to devise some means of referring the motion to fixed axes in space. At any proposed instant of the motion the particle is moving with a definite velocity and in a definite direction. Now this motion may be supposed to be the result of three motions taking place simultaneously parallel to the three axes of co-ordinates. Imagine the particle, in the first place, to have only its motions parallel to the axes of y and z combined. Then, in the second place, by combining with these the motion parallel to the axis of x, we have the actual motion of the particle in space: and the *change* in the motion by this last step is, that the particle has moved to a distance x parallel to the axis of x in the time t. But by the Second Law of Motion this change is the same as if the other motions did not exist. Hence the velocity and accelerating force of the particle parallel to the axis of x are the same, as if the

particle described the space x in the time t; and we have proved in Arts. 190, 199, that these are $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$: and in a similar way it may be shewn, that those parallel to the axes of y and z are $\frac{dy}{dt}$, $\frac{d^2y}{dt^2}$ and $\frac{dz}{dt}$, $\frac{d^2z}{dt^2}$.

The same would be true if the axes were oblique. But oblique axes are seldom, if ever, used because of the complicated expressions which they introduce into the equations.

206. It follows then, that when a particle is moving in space, and xyz are its rectangular co-ordinates at the expiration of the time t, the velocities of the particle parallel to the axes are $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, and the accelerating forces parallel to the three axes are $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$.

207. Since $\frac{dx}{dt} = \frac{ds}{dt}\frac{dx}{ds}$, and $\frac{dx}{ds}$ is the cosine of the angle, which the curve at the point (xyz) makes with the axis of x, and similarly of the other axes, s being the arc; it follows, that velocities may be resolved and compounded in exactly the same way that we resolve and compound statical forces.

208. The force $\frac{d^2s}{dt^2}$ is not the complete resultant of the forces $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$; as may easily be seen, because its square does not equal the sum of the squares of those three forces: but it is only that part of their resultant which has any effect upon the *velocity*; in short $\frac{d^2s}{dt^2}$ is the sum of the resolved parts of $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ in the direction of the motion; as the following identical equation teaches us,

$$\frac{d^2s}{dt^2} = \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2}.$$

The other part of the resultant is at right angles to this, and shews its effect in changing the *direction* of the motion of the particle. The magnitude and direction of this part of the resultant will be given in Art. 255.

209. Before we proceed to explain how equations are to be formed for calculating the motion of a body or system of bodies, we must search for another law of motion.

We have chosen two independent and arbitrary measures of force: one, the magnitude of the *pressure* produced in a body; the other, the magnitude of the *velocity* generated in a body in a given time by the action of the force. We must discover the relation, that connects these measures; so that when we know the accelerating effect of a force, we may be able to determine the magnitude of the pressure, which the force will cause a body on which it acts to exert; and *vice versâ*. It is manifest, that some relation between these two measures of force must exist, since the cause of the pressure, and the cause of the change of motion are the same. But since causing pressure and causing motion are two properties of force, which, abstractedly speaking, have no common character, we cannot discover the relation they bear to each other by reasoning *à priori;* but must again appeal to experiment. And thus we see *the necessity of a Third Law of Motion.*

THIRD LAW OF MOTION.

1. *Finite Forces.*

210. We will begin with experiments on falling bodies. It is found by experiments made under a receiver exhausted of air, that bodies of every variety of weight and material fall downwards in exactly the same manner: even a guinea and a feather strike the plate of the air pump at the same instant, if they are set at liberty together and from the same altitude. These experiments shew, that the accelerating effect the Earth's attraction on all falling bodies at the same place is the same. In these cases, then, while the force of the Earth's attraction measured *statically* is different for different bodies, when measured *dynamically* it is the same for all bodies, if the experiments be made in the same place.

We can explain why this is the case. We find, that the aggregate weight of two bodies is the same whether they are weighed separately, or one enclosed in the other. This proves, that the Earth's attraction is not affected by having to act upon matter *through* matter, or by penetration, as it is termed. And this leads us to infer, that the Earth attracts *every particle* of a body; and therefore that one body is heavier than another, because it contains more matter. Thus the *statical* measure will depend upon the nature of the body. But, on the other hand, since all the particles, if unconnected, would describe the same spaces in the same times, it appears, that when connected into one mass they would fall in the same manner: and therefore the *dynamical* measure of the Earth's attraction is the same for different bodies. This is the explanation of the experimental fact. We could not reason in this manner *à priori*, because we could not assume, that the motion of the particles is the same when connected as when unconnected.

211. In the case, then, of bodies of the same homogeneous material, the weight will evidently be in proportion to the quantity of matter which they contain. But since we are unacquainted with the ultimate constitution of bodies, we are unable to compare the quantities of matter in two bodies of different material by examining their structure. We therefore *assume*, as a definition and means of measuring the quantity of matter in a body, (what we have proved by experiment in the case of homogeneous bodies), that *the quantities of matter, or the masses, of bodies in the same place vary as their weights.* Suppose, then, that M is the mass of a body of which the weight is W; then $W = Mg$; g being some quantity, which is constant for the same place, and depends upon the units of weight and mass. Hence, in the same place, the *statical* measure varies as the mass of the body: and the *dynamical* measure is constant.

212. It remains to be determined what change the weight undergoes, if the body be removed to a place where the accelerating force is different: or if by any contrivance the accelerating force of a body be changed without changing the place

of experiment. Such a contrivance is Atwood's machine, described below*.

We will give the results of experiments made with this machine, referring to the note for the meaning of the quantities we make use of.

Suppose P is placed with its lowest surface level with the zero point of the scale, and set at liberty at any tick of the pendulum: it is always found, however much P and Q are altered, that, in each experiment, the spaces described by P in successive seconds form an arithmetic progression, and therefore that the accelerating force in each case is uniform. Also it is found, that the common differences of the series in the various experiments are proportional to the respective values of the ratio $\dfrac{P-Q}{P+Q+W}$. This is proved by numerous experiments, for the details of which we refer to the work of Atwood.

We gather, then, from this that the accelerating force varies as $\dfrac{P-Q}{P+Q+W}$ in the different experiments, and therefore the pressure producing motion (or $P-Q$) varies as the product of the accelerating force and the weight moved (or $P+Q+W$),

* Four wheels, two of which A and B are represented in figure 75, the other two being hid by these, are placed parallel to each other, their centres being fixed so as to allow of rotation with as little friction as possible: A and B are placed as near as possible without touching: and so are the other two wheels. Upon these four rests the axle of another wheel C placed midway between A and B and the other two wheels: a fine string as flexible and inextensible as possible is passed over the circumference of C and two weights P and Q are attached to its extremities. When P and Q are left to themselves the heavier will descend and draw up the lighter of the two. It will be readily understood that the object of the four wheels is to diminish the friction on the axle of C; which it does very considerably, since the friction of *rolling* is far less than that of *rubbing*. Suppose that P descends, then $P-Q$ is the weight or pressure which causes the motion and $P+Q$ is the weight put in motion. It is found by experiment that the inertia of the wheels produces the effect of adding to the weight moved without adding to the pressure producing motion. Atwood determines by experiment what this weight is, we shall call it W. Hence $P-Q$ is the weight causing the motion and $P+Q+W$ is the weight put in motion. A graduated scale of inches is placed behind the thread supporting P in order to mark the motion of P. The excellence of this machine consists in this, that we can have bodies falling with various degrees of acceleration and as slowly as we please by altering P and Q. The time of motion is marked by a seconds pendulum: see Atwood on *Rectilinear Motion* for a full explanation.

and therefore as the product of the accelerating force and the mass moved, since the weight of a body at the same place varies as the mass (Art. 211), and these experiments were made at the same place.

Hence, for all uniform accelerating forces, the *statical* measure of the force varies conjointly as the mass of the body upon which it acts and the *dynamical* measure of the force.

213. This will evidently be equally true in the case of variable forces: since variable forces may be considered constant for an indefinitely short time. We shall, however, give an experiment for variable forces.

Let two balls be suspended from two points by strings in such a manner, that when hanging at rest they just touch each other, and have their centres in a horizontal line. The strings may be of any length, the same or different. Let one of the balls, as A, be drawn aside through an arc having any vertical ver-sine, and left to impinge on B: observe the arc through which B rises. Now change the length of the string of A by changing the point of suspension; and it is found, that if A be raised through an arc of which the ver-sine is the same as before, B will move exactly as before: and therefore the velocity of A at its lowest point depends only upon the ver-sine of the arc through which it falls, and not upon the length of the string.

Now let us see whether this result will follow, if we assume the relation between pressure and motion established for uniform forces.

Let θ be the angle which the string makes with the vertical at the time t; W the weight of A; l the length of the string; a the value of θ at the beginning of the motion of A: then $l(a - \theta)$ is the space, which the centre of A has described at the time t: and therefore $-l\dfrac{d^2\theta}{dt^2}$ is the accelerating force at the time t: $W\sin\theta$ is the part of the weight, or pressure, acting in the direction of the motion: and, by hypothesis, these vary as each other: therefore, since W is constant,

$$-l\frac{d^2\theta}{dt^2} \text{ varies as } \sin\theta, = \frac{1}{2}c^2\sin\theta \text{ suppose,}$$

$$\therefore \left(l\frac{d\theta}{dt}\right)^2 = c^2 l(\cos\theta + \text{constant})$$

when $\theta = \alpha$, velocity $= 0$,

$$\therefore (\text{vel.})^2 = c^2 l(\cos\theta - \cos\alpha)$$

$$\therefore (\text{vel. at lowest point})^2 = c^2 l(1 - \cos\alpha),\ \theta = 0:$$

$= c^2$ ver-sine of the arc of descent: and this coincides with the experiment. Hence the law of connexion between pressure and motion holds for variable forces.

214. The product of the mass of a body and the accelerating force is called by Newton the *Moving Force* of the body: and the product of the velocity and mass of a body he calls its *Momentum*, or quantity of motion. These experiments therefore shew, that *the pressure communicating the motion varies as the moving force, or as the momentum generated in the body in a given time:* for moving force must be measured by the momentum generated in a given time, since accelerating force is measured by the velocity generated in a given time.

2. *Impulsive Forces.*

215. Impulsive forces are measured *dynamically* by the aggregate effect, which they produce upon the velocity. And therefore they should be measured *statically* by the sum of the pressures exerted during the action of the force.

The following experiments will shew, that this sum of pressures, or statical measure of the impulsive force, is a function of the momentum generated or destroyed.

Let A and B be two balls (fig. 77.) suspended by threads from two points C and D, so that they may just touch when at rest, and have their centres in the same horizontal line: FAE, fBe circular arcs with centres C, D: now the velocities of a ball in falling through different arcs of a circle to the lowest point are in the proportion of the chords of those arcs, as appears by Art. 213. Let therefore a scale be placed below A and B so graduated, as to mark the velocities of the balls

A and *B* when at the lowest positions, by knowing the arcs through which they move.

Now suppose a small steel point is fixed in *A*, so that when *A* and *B* come in contact separation is prevented. It is found that if *A* and *B* are drawn through arcs, of which the chords are inversely as the masses of the bodies, and then left to themselves, they will impinge, and exactly destroy each others velocity, a small allowance being made for the resistance of the air. If one of the balls be moved through a greater arc, then when the balls come in contact they will not be at rest, but move in the direction in which that ball was moving before impact. This shews, that when the bodies impinge on each other with equal momenta, their mutual pressures exactly balance the momenta; but, if the momentum of one ball be greater than the momentum of the other, the mutual pressure is not sufficient to overcome the momentum of the first; but not only overcomes the momentum of the second, but generates new momentum. This is found to be true for masses and velocities of all finite magnitudes.

Desaguliers mentions an experiment (*Experimental Philosophy*, Vol. II. Lecture VI. p. 62.) in which he replaced *A* and *B* by two cylinders closed at the outer extremities; one was introduced a short way into the other, the cavity being previously filled with gunpowder: it was found that after the explosion the cylinders rose through arcs the chords of which varied inversely as their masses. Consequently the momenta generated by the action of the impulsive force of the explosion were the same.

216. These experiments shew, that the same impulsive force acting upon different masses will generate or destroy equal momenta: and, on the other hand, that to generate or destroy equal momenta in different masses the impulsive force must be the same. Hence the aggregate of the pressures during the action of the force is some function of the momentum generated or destroyed. But it is evident, that this aggregate must vary *directly* as the mass; because, to produce the same dynamical effects in the change of velocity during the action of the force on different masses, *each* of the instan-

taneous pressures must be changed in the different cases in proportion to the change of mass: this appears from Art. 214. It follows, then, that this aggregate of pressures, or the *statical* measure of the impulsive force varies *directly* as the momentum generated or destroyed; *i.e.* as the product of the mass on which it acts, and the *dynamical* measure of the force.

217. Hence, then, the *Third Law of Motion* is this:

The statical *measure of force, whether finite or impulsive, varies directly as the product of the mass of the body upon which it acts, and the* dynamical *measure of the force.*

218. This Law may be stated also in the following forms.

When *finite* pressure generates or destroys motion in a body, the moving force is proportional to the pressure: and when *impulsive* pressure generates or destroys motion in a body, the momentum generated or destroyed is proportional to the pressure.

Newton has given this Law under the more general form, that *Action and Reaction are equal and opposite.* If action and reaction in dynamics be measured by the quantity of motion gained and lost, this is an immediate deduction from our Third Law of Motion.

Origin of the term VIS VIVA.

219. Leibnitz in the *Acta Eruditorum* 1695, p. 149, and after him Jean Bernoulli and others raised objections to Newton's measure of force, contending that it ought to be proportional to the product of the mass and the square of the velocity.

In their own words, "A force is said to be dead (*vis mortua*) which consists in nothing but the endeavour, or the tendency to motion. Such is gravity" it was said "as long as a heavy body hung by a thread endeavours to descend, but cannot actually descend. A force is said to be alive or quick (*vis viva*) which always accompanies actual motion, and

tends to produce a local motion. There is such a force in a body falling by gravity when it has already acquired some degrees of velocity." Professor Wolfius, quoted by Desaguliers; *Exp. Phil.* Vol. II. p. 72, 80.

Our object in making this quotation is to shew the origin of the term *vis viva*, which, as a term only, is still in use among us. The incorrectness of the above notion appears from the fact that it implies that matter has some inherent power of exerting force when in motion which it has not when at rest.

The reasoning by which these philosophers were led to the idea, that pressure should be measured by the product of mass and the square of the velocity generated, appears from the nature of the experiments from which they argued. It was found, that when balls of equal size and density impinged upon clay they penetrated the clay by spaces which are as the squares of the velocities of impact: as in the example of the note to Art. 202. It was reasoned (as in that note) that when balls are projected against different solid substances so as to penetrate to the same depth the forces will be as the squares of the velocities: and hence arises the mistake, for this supposes, that we measure force by the velocity generated or destroyed in moving through a given space irrespective of the time of motion: but we measure force by the velocity generated in a given time irrespective of the space described. If then we retain our definition of force, estimated dynamically, (the velocity generated in a given time), the force must vary as the product of the mass and the velocity generated in a given time: but if we were to adopt the second measure of force estimated dynamically, (the velocity generated in moving through a given space), we should find, that the force varies as the product of the mass and the square of the velocity generated: see Art. 198.

The term *vis viva* is still used to express the product of the mass and square of the velocity.

The Units of Pressure and Mass.

220. We shall now choose the *units* of pressure, or statical force, and mass.

Let P be a finite pressure, f the accelerating force and M the mass, then P varies as Mf. Let the unit of pressure be that of a body of which the mass is M' and the accelerating force f': then

$$P : 1 :: Mf : M'f';$$

$$\therefore P = \frac{Mf}{M'f'};$$

we shall choose M' and f' so as to simplify this formula as much as possible: let $M' = 1$, $f' = 1$; then

$$P = Mf \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the unit of pressure being the pressure of a body of a unit of mass and acted on by the unit of accelerating force.

When the pressure is impulsive *its unit is that of a body of mass unity moving with a unit of velocity:* if we, as above, suppose

$$P = Mv \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let W be the weight of a body of which the mass is M, and let the accelerating force of the Earth's attraction, or gravity, equal g: then

$$W = Mg \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also suppose that the body is homogeneous, of density ρ, and volume V: let ρ' and V' be the density and volume of a body of which the mass equals the unit of mass: then

$$M : 1 :: \rho V : \rho' V';$$

$$\therefore M = \frac{\rho V}{\rho' V'};$$

we shall choose ρ' and V' so as to simplify this formula as much as possible: let $\rho' = 1$, $V' = 1$: then

$$M = \rho V \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

the unit of mass being the mass of a body of a unit of volume and a unit of density.

By (3) (4) we have

$$W = \rho V g \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Now by experiments made by Atwood's Machine described in the note to Art. 216, it is found that the spaces described by a body falling freely from rest are 16.1, 3 × 16.1, 5 × 16.1,.... feet in the first, second, third,.... seconds of time. Hence gravity is a constant force and generates a velocity of 2 × 16.1 or 32.2 feet in a second of time. Wherefore if we take *a foot as the unit of length and a second as the unit of time we have*

$$g = 32.2 \dots\dots\dots\dots\dots\dots(6),$$

$$W = 32.2\ V\rho \dots\dots\dots\dots\dots(7);$$

and when $\rho = 1$ and $W = 1$, $V = \dfrac{1}{32.2}$; hence the relation among the units chosen gives this result, *that the unit of weight is the weight of a body of the unit of density and volume equal the* 32.2^{th} *part of the unit of volume.* The density of distilled water is generally taken as the unit of density; and a cubic foot as the unit of volume.

MEANS OF OBTAINING EQUATIONS TO CALCULATE MOTION.

221. The grand Problem of Dynamics is to find the relation, which exists between the motion of a system of bodies, and the forces which act upon them: so that when the forces are known the motion may be determined; and vice versâ.

We have seen, that, if no forces act upon a part of the system, that part will move uniformly in a straight line, when once put in motion. This will also happen if the forces acting upon that part of the system are in equilibrium with each other.

In the general case, however, when *finite* forces act on the system, each particle will move in a determinate curvilinear path, and the acceleration (or retardation) of its motion will take place whenever the forces acting on the particle are not in equilibrium, *i. e.* whenever they have a resultant. Let $x\ y\ z$ be the co-ordinates of position of any particle of the system at the expiration of the time t, then the resultant is measured dynamically by the accelerating forces $\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}, \dfrac{d^2z}{dt^2}$ acting

parallel to the fixed axes of co-ordinates. These are termed the *effective accelerating forces* of the particle at the time t parallel to the axes of co-ordinates. The forces which act upon the particle to produce the motion, not including the molecular actions of the particles on each other (if there be any), are termed the *impressed forces* by way of distinction.

Now it is immediately evident, that if, at any instant of the motion, we were to apply to each particle of the system forces equal in magnitude but opposite in direction to the effective forces of that particle, these would at that instant check the acceleration (or retardation) of the motion, or, in other words, would be in equilibrium with the impressed and molecular forces which act upon the system: and will therefore together with them satisfy the equations of condition we have deduced in the former part of this Work for the equilibrium of forces.

By this principle, the truth of which is self-evident, we shall obtain equations which connect together the forces that act upon the system and the analytical quantities $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ and all similar quantities for the other particles. If the question be to determine the motion when the forces are given in terms of xyz and t, the solution is effected by integrating these equations. If, on the other hand, the question be to determine the forces which will cause the system of particles to move in given curves, we must differentiate the equations to the curves with respect to t, and substitute for $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\cdots$ in the equations resulting from the application of the above principle: in this way the forces will become known.

222. In the case of *impulsive* forces suppose v_1, v_2, v_3, are the resolved parts, parallel to the axes, of the velocity of any particle of the system arising from the action of the impulsive forces. Then the effect of the impulsive forces upon this particle is the same as three impulsive accelerating forces acting parallel to the axes and equal to v_1, v_2, v_3; these are termed the *effective* impulsive accelerating forces: and the original forces are termed the *impressed* impulsive forces.

Wherefore it is immediately evident, that if, at the time of the action of the impulsive forces on the system, we were to apply to each particle impulsive forces equal but opposite to the effective impulsive forces of that particle, these would check the effect of the impulsive forces actually impressed on the system and would consequently with them satisfy the equations of condition for the equilibrium of forces.

As before, then, we obtain equations by means of which the motion of the system may be calculated.

223. Now in the calculations of the conditions of equilibrium of forces acting upon a single particle, a rigid body, or any material system given in Chapters I, II, and III of Statics we have considered the magnitudes of the forces to be estimated statically; in other words, we have supposed them to be pressures. Wherefore before we can make use of the results of those Chapters for determining the equations of motion of a system, in the manner explained above, we must refer all the forces to their statical measure or its equivalent. This the Third Law of Motion enables us to accomplish: and the foregoing observations lead us immediately to the two following Principles.

224. *Let* m *be the mass of that particle of the system, of which the co-ordinates are* x y z; *then*

I. *If the system be in motion under the action of* finite *forces, the forces*

$$-m\frac{d^2x}{dt^2}, \quad -m\frac{d^2y}{dt^2}, \quad -m\frac{d^2z}{dt^2},$$

acting on m *parallel to the axes, and similar forces acting on each of the other particles of the system, must, together with the impressed moving forces and the molecular forces satisfy the conditions of equilibrium.*

225. II. *If the system be acted on by* impulsive *forces, and* v_1, v_2, v_3 *be the velocities of* m *parallel to the axes, at the instant that the impulsive forces cease to act; then, the forces*

$$-mv_1, \quad -mv_2, \quad -mv_3$$

acting on m *parallel to the axes, and similar forces acting on each of the other particles of the system, must together with the impressed impulsive forces, or momenta, and the molecular forces, satisfy the conditions of equilibrium.*

226. In the case of a *rigid body* the molecular forces are of themselves in equilibrium: and therefore in that case mention of them may be neglected in stating these Principles. They will then coincide with the Principle which D'Alembert enunciated to the scientific world A.D. 1743, and which formed such an epoch in the history of Mechanics applied to the motion of rigid bodies: see Whewell's *History of the Inductive Sciences*, Vol. II. pp. 95, 96.

227. These Principles are the interpretation of the Three Laws of Motion into mathematical language. The Laws themselves are the results solely of observation and experiment. But these Principles are the results not only of the Laws, but also of certain conventional rules for measuring the quantities treated of; without which indeed we could not make the phenomena resulting from the Laws subjects of calculation. We must therefore be careful to interpret all results to which they lead us in conformity to these conventional rules.

CHAPTER II.

THE MOTION OF A MATERIAL PARTICLE.

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228. LET xyz be the co-ordinates to the particle at the end of the time t, and m its mass.

Suppose the accelerating forces acting on the particle are resolved parallel to the axes and compounded into three X, Y, Z in these directions.

Then by the Principle enunciated in Art. 224, the moving forces

$$mX, \quad mY, \quad mZ$$

$$-m\frac{d^2x}{dt^2}, \quad -m\frac{d^2y}{dt^2}, \quad -m\frac{d^2z}{dt^2}$$

will be in equilibrium with each other at the time t.

Hence by the conditions of equilibrium of a particle acted on by any forces given in Art. 23, we have the equations

$$mX - m\frac{d^2x}{dt^2} = 0, \quad mY - m\frac{d^2y}{dt^2} = 0, \quad mZ - m\frac{d^2z}{dt^2} = 0,$$

$$\text{or } \frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z.$$

These are called the equations of motion of the material particle; and by integration we shall have three equations involving x, y, z, t and constant quantities.

By eliminating t we have two equations involving x, y, z without t. These are the equations to the curve described by the particle.

229. In the course of the integration six arbitrary constants will be introduced: these are determined by the initial circumstances of the motion: by the term *initial* we mean at

the epoch from which t is measured*. The general integrals determine the *nature* only and not the *dimensions* of the curve described. The dimensions depend upon the initial conditions. These are, first, the three co-ordinates which give the position of the particle at the commencement of the motion. By substituting these in the three integrals and putting $t = 0$ we have three equations involving the six arbitrary constants and known quantities. The other initial quantities are the velocity and direction of projection, or, which amounts to the same, the initial velocities* parallel to the three axes.

By differentiating the three integrals with respect to t, we shall have three equations involving $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ and the arbitrary constants: and giving the variable quantities their initial values we have three more equations involving the arbitrary constants and known quantities.

From these six equations, then, we can determine the six arbitrary constants and the problem is completely solved.

230. Suppose, on the other hand, the problem to be solved be the converse of the one already considered, namely, to determine the forces which will make a body describe a given curve.

We shall in this case have given *two* equations involving xyz, from which we are to obtain the *three* quantities $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ or X, Y, Z: this shews that the problem is indeterminate. The following is the way to proceed.

* If any particle of the system commence its motion with a finite velocity, this is imparted to it by an impulsive force, which acts for so short a time as to produce its effect instantaneously: for this reason it is evidently indifferent whether we measure the time from the commencement or termination of the action of the impulsive force: and the term *initial velocity*, though there is no velocity, rigorously speaking, at the commencement of the motion, is perfectly allowable.

In short, when a system of material particles is projected into space and submitted to the action of surrounding bodies, two entirely different systems of forces act upon the particles. The first is a system of impulsive forces of the nature described in Art. 201: these produce their effect in an indefinitely short time, after which they cease to act. The second system consists of forces of the nature described in the same Articles: these require a length of time of sensible duration to produce their effect. This latter system differs from the former merely in *intensity*.

The two equations involving x, y and z must be differentiated twice with respect to t: by this means we have two equations involving the four quantities X, Y, Z, and velocity (v).

$$\text{But } v^2 = \frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2};$$

$$\therefore \tfrac{1}{2}\frac{d.v^2}{dt} = X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt},$$

$$\tfrac{1}{2}\frac{d.v^2}{dx} = \quad X + Y\frac{dy}{dx} + Z\frac{dz}{dx}.$$

This is a third equation involving X, Y, Z, v. By assuming a value of any one of these four quantities, or any condition connecting them, the other three may be determined, in terms of xyz.

RECTILINEAR MOTION.

231. Prop. *A body is acted on by a uniform force (that of gravity for instance) the motion being in the line of action of the force: required to determine the motion.*

Let x be the distance of the body at the time t from a fixed point in its course, measured in the direction of the force: and let g be the force.

Then the equation of motion is

$$\frac{d^2x}{dt^2} = g.$$

By integration we have

$$\frac{dx}{dt} = gt + C, \; C \text{ being an arbitrary constant.}$$

To determine C we must refer to the *initial* circumstances of the motion.

Suppose the body is projected with a velocity u in the direction in which the force acts.

$$\text{When } t = 0, \quad \frac{dx}{dt} = u; \quad \therefore u = C;$$

$$\therefore \frac{dx}{dt} = gt + u.$$

Integrating again

$$x = \tfrac{1}{2}gt^2 + ut + C'.$$

Let a be the distance of the body from the origin of x at the commencement of the motion: then the initial circumstances are that when $t = 0$, $x = a$; $\therefore\ C' = a$;

$$\therefore\ x = a + ut + \tfrac{1}{2}gt^2,$$

or the space described in the time t is $ut + \frac{1}{2}gt^2$.

This is a necessary consequence of the second law of motion.

If the body be not projected, then $u = 0$ and $x = a + \frac{1}{2}gt^2$.

If the body be projected with a velocity u in a direction *opposite* to that in which x is measured, then when $t = 0$, $-\dfrac{dx}{dt} = u$ since x is diminished as t increases:* and

$$x = a - ut + \tfrac{1}{2}gt^2.$$

PROP. *A body falls towards a centre of force the intensity of which varies directly as the distance of the body from the centre: required to determine the motion.*

232. Let μ be the magnitude of the force at a distance unity from the centre of force: this is called the *absolute* force of the centre: a the distance of the body from the centre at the commencement of the motion, x the distance at the time t.

Then μx is the magnitude of the force at the distance x: and the equation of motion is

$$-\frac{d^2x}{dt^2} = \mu x,$$

* In Art. 190, it was shewn that if s be the space described in the time t by a body, and v its velocity at the end of that time, then $\dfrac{ds}{dt} = v$.

But if the space be measured in a direction opposite to that in which the motion takes place, then b and s' being the distances of the point from which the space is measured at the commencement of the motion and at the end of the time t, then $s = b - s'$ and $-\dfrac{ds'}{dt} = v$.

Also in Art 200, it was shewn that if f be the magnitude of the force at the end of the time t, then $\dfrac{d^2s}{dt^2} = f$. If, as before, the space be measured in the direction opposite to that of the action of the *force*, then $-\dfrac{d^2s'}{dt^2} = f$.

the negative sign being taken because the tendency of the force is to diminish x;

$$\therefore 2\frac{dx}{dt}\frac{d^2x}{dt^2} = -2\mu x\frac{dx}{dt};$$

$$\text{integrating, } \frac{dx^2}{dt^2} = C - \mu x^2,$$

C being an arbitrary constant to be determined by the initial circumstances of the motion: these are that when $t = 0$, $x = a$, and the velocity, or $\frac{dx}{dt}$, $= 0$; $\therefore C = \mu a^2$;

$$\therefore \frac{dx^2}{dt^2} = \mu(a^2 - x^2);$$

$$\therefore -\frac{dt}{dx} = \frac{1}{\sqrt{\mu}}\frac{1}{\sqrt{a^2 - x^2}},$$

the negative sign being taken in extracting the square root because x diminishes as t increases.

$$\text{Integrating, } t = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a} + C'$$

$$\text{when } t = 0,\ x = a,\ \therefore C' = 0;$$

$$\therefore t = \frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}$$

when $x = 0$, the body arrives at the centre;

$$\therefore \text{time of falling into the centre} = \frac{\pi}{2\sqrt{\mu}}.$$

The velocity is zero when $\frac{dx}{dt} = 0$, or when $x = a$ and $-a$: hence the body passes through the centre and stops at a distance on the other side equal to the original distance. From this point it will return to its original position and continually oscillate over the same space: the time of oscillation from rest to rest is $\frac{\pi}{\sqrt{\mu}}$. It is remarkable that this is independent of the initial distance of the body from the centre of force.

The expression for the time shews, that the body will oscillate backwards and forwards: for suppose α is the least positive value of $\cos^{-1}\frac{x}{a}$ for any given value of x, then

$$t = \frac{\alpha}{\sqrt{\mu}} \text{ or } \frac{2\pi - \alpha}{\sqrt{\mu}} \text{ or } \frac{2\pi + \alpha}{\sqrt{\mu}} \ldots\ldots$$

or, generally, $\dfrac{2n\pi \mp \alpha}{\sqrt{\mu}}$, n being any integer.

This proves that the body will periodically arrive at any given point of its path: the intervals of time between the successive arrivals being $\dfrac{2\pi - 2\alpha}{\sqrt{\mu}}$ and $\dfrac{2\alpha}{\sqrt{\mu}}$ alternately.

Prop. *Suppose the body in the last Proposition is projected with a velocity* u *in the line in which the force acts.*

233. As before we have

$$\frac{dx^2}{dt^2} = C - \mu x^2$$

when $x = a$, $\dfrac{dx}{dt} = u$ or $-u$ according as the direction of projection is from or towards the centre: in both cases

$$u^2 = C - \mu a^2$$

$$\frac{dx^2}{dt^2} = u^2 + \mu (a^2 - x^2).$$

Considering the motion *towards* the centre

$$-\frac{dt}{dx} = \frac{1}{\sqrt{\mu}} \frac{1}{\sqrt{a^2 + \dfrac{u^2}{\mu} - x^2}}$$

$$t = \frac{1}{\sqrt{\mu}} \cos^{-1} \frac{x}{\sqrt{a^2 + \dfrac{u^2}{\mu}}} + C$$

when $t = 0$, $x = a$;

$$\therefore t = \frac{1}{\sqrt{\mu}}\left\{\cos^{-1}\frac{x}{\sqrt{a^2+\frac{u^2}{\mu}}} - \cos^{-1}\frac{a}{\sqrt{a^2+\frac{u^2}{\mu}}}\right\}.$$

The greatest distance to which the body goes from the centre is $\sqrt{a^2+\frac{u^2}{\mu}}$, and the time of a complete oscillation from rest to rest is as before $\frac{\pi}{\sqrt{\mu}}$.

PROP. *A body falls towards a centre of force the intensity of which varies inversely as the square of the distance of the body: required to determine the motion.*

234. Let μ be the *absolute* force of the centre as before: then the force at distance x is equal to $\frac{\mu}{x^2}$: and the equation of motion is

$$-\frac{d^2x}{dt^2} = \frac{\mu}{x^2};$$

$$\therefore 2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\frac{2\mu}{x^2}\frac{dx}{dt};$$

$$\text{integrating, } \frac{dx^2}{dt^2} = \frac{2\mu}{x} + C$$

$$\text{when } x = a, \frac{dx}{dt} = 0; \therefore 0 = \frac{2\mu}{a} + C$$

$$\frac{dx^2}{dt^2} = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right)$$

$$-\frac{dt}{dx} = \sqrt{\frac{a}{2\mu}}\sqrt{\frac{x}{a-x}} = \sqrt{\frac{a}{2\mu}}\cdot\frac{x}{\sqrt{ax-x^2}};$$

$$\therefore \frac{dt}{dx} = \sqrt{\frac{a}{2\mu}}\,\frac{\frac{1}{2}a - x - \frac{1}{2}a}{\sqrt{ax-x^2}}$$

$$t = \sqrt{\frac{a}{2\mu}}\left\{\sqrt{ax-x^2} - \frac{a}{2}\operatorname{vers}^{-1}\frac{2x}{a}\right\} + C$$

$$\text{when } t = 0,\ x = a;\ \therefore 0 = -\sqrt{\frac{a}{2\mu}}\frac{a\pi}{2} + C$$

$$t = \sqrt{\frac{a}{2\mu}}\left\{\frac{\pi}{2}a - \frac{a}{2}\text{vers}^{-1}\frac{2x}{a} + \sqrt{ax - x^2}\right\}$$

when the body arrives at the centre $x = 0$, therefore time of falling to the centre $= \frac{\pi}{\sqrt{\mu}}\left(\frac{a}{2}\right)^{\frac{1}{2}}$.

235. In a subsequent part of this work we shall see, that the attraction of the Earth on external bodies varies inversely as the square of the distance from its centre, supposing the Earth a sphere. And that the attraction on any bodies within the Earth varies directly as the distance from the centre.

It is for this reason that in the foregoing Propositions we have selected these particular laws of force. No other laws are known to exist in the universe.

Prop. *A body acted on by the constant force of gravity moves down an inclined plane: required to calculate the motion.*

236. Let the plane of the paper be the vertical plane in which the motion takes place: AB (fig. 78.) the intersection of this with the inclined plane: P the position of the body at the time t, A being its place when $t = 0$: α the angle the plane makes with the horizon. Now the forces which are acting upon the body at the time t are the force of gravity g, which acts vertically and the pressure of the plane on the body. If we resolve the forces in the direction of the motion we shall not introduce the pressure.

$$\text{Let } AP = x.$$

Now the part of g resolved along the line AP is $g \sin \alpha$, hence the equation of motion is

$$\frac{d^2 x}{dt^2} = g \sin \alpha,$$

and the results will be precisely the same as those in Art. 231, if we there substitute $g \sin \alpha$ instead of g.

If we wish to know the pressure P upon the plane, by resolving the forces perpendicularly to the line of motion we have, since no space is described by the body in that direction,

$$0 = mg \cos a - P.$$

Hence $P = mg \cos a$, and is constant and is in proportion to the weight of the body in the ratio $\cos a : 1$.

CURVILINEAR MOTION OF A PARTICLE.

PROP. *A body is acted on by the constant force of gravity, which acts in parallel lines: required to determine the motion of the body when it is projected in a direction not vertical.*

237. Let the axis of y be vertical and reckoned positive upwards and drawn through the point of projection. The motion will evidently take place wholly in a vertical plane. Let the axis of x be drawn in this plane the origin being the point of projection A, (fig. 79). Let also g be the accelerating force of gravity.

Then the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \quad -\frac{d^2y}{dt^2} = g,$$

$$\text{integrating, } \frac{dx}{dt} = c, \quad \frac{dy}{dt} = c' - gt,$$

c and c' being constants to be determined by the circumstances of projection.

Let u be the velocity of projection, a the angle its direction makes with the axis of x.

$$\text{Then when } t = 0, \quad \frac{dx}{dt} = u \cos a, \quad \frac{dy}{dt} = u \sin a;$$

$$\therefore u \cos a = c, \quad u \sin a = c';$$

$$\therefore \frac{dx}{dt} = u \cos a, \quad \frac{dy}{dt} = u \sin a - gt.$$

Integrating again

$$x = ut \cos a, \quad y = ut \sin a - \tfrac{1}{2}gt^2 \ldots\ldots (1),$$

no constants are added after integration because when $t=0$, $x=0$ and $y=0$ by the circumstances of the problem.

These two equations determine the position of the body at any time.

238. To find the curve described we eliminate t from equations (1); (Art. 228)

$$\therefore y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}.$$

This is the equation to a parabola.

For it may be written

$$\left(x - \frac{u^2}{g}\cos\alpha\sin\alpha\right)^2 = -\frac{2u^2}{g}\cos^2\alpha\left(y - \frac{u^2}{2g}\sin^2\alpha\right).$$

And by transferring the origin to a point of which the co-ordinates are

$$\frac{u^2}{g}\cos\alpha\sin\alpha \text{ and } \frac{u^2}{2g}\sin^2\alpha$$

the equation becomes

$$x^2 = -\frac{2u^2}{g}\cos^2\alpha y,$$

which is the equation to a parabola with its axis vertical and measured downwards, and latus rectum $= \frac{2u^2}{g}\cos^2\alpha$.

The *range* is the distance between the point of projection and the point where the body strikes the ground. The curve described is called the *projectile*.

PROP. *To find the range of the projectile, the time of flight, and the greatest height the body reaches.*

239. When $y=0$, $x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} = 0$;

$$\therefore x = 0 \text{ and } x = \frac{2u^2}{g}\cos^2\alpha\tan\alpha = \frac{u^2}{g}\sin 2\alpha,$$

this latter value of x is the *range on a horizontal plane.*

If the body be projected from an inclined plane perpendicular to the plane of the projectile, then, if i be the angle

of inclination of the plane to the horizon, $y = x \tan i$ is the equation to the intersection of this plane and the plane of motion: and the value of x when the body strikes the plane is found from

$$x \tan i = x \tan a - \frac{gx^2}{2u^2 \cos^2 a};$$

$$\therefore\ x = 0, \text{ and } x = \frac{2u^2}{g}\cos^2 a(\tan a - \tan i) = \frac{2u^2}{g}\frac{\cos a \sin(a-i)}{\cos i},$$

this latter value of x is the *range on the inclined plane.*

$$\text{By (1)}\quad x = ut\cos a;$$

therefore time of flight on the inclined plane

$$= \frac{x}{u\cos a} = \frac{2u}{g}\frac{\sin(a-i)}{\cos i}; \quad = \frac{2u}{g}\sin a, \text{ if } i = 0.$$

When the body reaches its greatest height

$$\frac{dy}{dx} = 0; \quad \therefore\ x = \frac{u^2}{g}\tan a\cos^2 a = \frac{u^2}{g}\sin a\cos a;$$

$$\therefore \text{ greatest height} = \frac{u^2}{g}\left\{\sin^2 a - \tfrac{1}{2}\sin^2 a\right\} = \frac{u^2}{2g}\sin^2 a.$$

CENTRAL FORCES.

240. Forces which continually tend towards a given point, and the intensity of which depends upon the distance from that point, whether fixed or in motion, are called *Central Forces.* All the forces with which we are acquainted in nature are of this description, as will appear in the sequel. For this reason we shall devote a large portion of these pages to the consideration of their action.

We shall, in the first place, investigate the most important general properties of orbits described by bodies moving under the influence of central forces, and in the next place determine the nature of the orbits when the law and intensity of the forces are given, and, conversely, determine the forces requisite to cause a body to describe given orbits.

PROP. *When a body is acted on by one central force the motion is wholly in one plane.*

241. Suppose xyz are the co-ordinates at the time t to a material particle moving about a centre of force, the origin of co-ordinates being at this centre: r the distance of the particle from the centre: and let P, some function of r, represent the intensity of the force at the distance r.

The resolved parts of this force parallel to the three axes of co-ordinates are

$$P\frac{x}{r},\quad P\frac{y}{r},\quad \text{and } P\frac{z}{r};$$

and since these tend to diminish the co-ordinates the equations of motion are

$$-\frac{d^2x}{dt^2} = P\frac{x}{r},\quad -\frac{d^2y}{dt^2} = P\frac{y}{r},\quad -\frac{d^2z}{dt^2} = P\frac{z}{r}\,\dots\dots\dots\,(1).$$

Multiplying the first by y and the second by x and subtracting the equations we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = h,\ \text{an arb}^{\text{y}}.\ \text{const.}$$

In like manner $z\dfrac{dx}{dt} - x\dfrac{dz}{dt} = h_1$, and $y\dfrac{dz}{dt} - z\dfrac{dy}{dt} = h_2$;

h_1 and h_2 being arbitrary constants, which, as well as h, are to be determined by the circumstances of the motion at any given time.

Now multiply these last three equations by z, y, x respectively and add them together;

$$\therefore\ 0 = hz + h_1 y + h_2 x.$$

This is the equation to an invariable plane passing through the origin of co-ordinates, its position depending on the values of $h\,h_1\,h_2$.

Hence the motion takes place wholly in a plane passing through the centre of force, the position depending upon the initial (or any other *given*) circumstances of the motion.

PROP. *The areas described by the body about the centre of force are proportional to the time.*

242. In consequence of the property proved in the last Proposition we shall refer the body's motion to two co-ordinates instead of three. Let the plane of motion be the plane xy.

Then the equations of motion are

$$\frac{d^2x}{dt^2} = -P\frac{x}{r} \ldots\ldots (1), \qquad \frac{d^2y}{dt^2} = -P\frac{y}{r} \ldots\ldots (2),$$

$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = h,$$

and let A be the sectorial area swept out during the time t by the radius-vector;

$$\therefore\ \frac{dA}{dt} = \tfrac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) \text{ by Diff. Calc.} = \tfrac{1}{2}h;$$

$$\therefore\ A = \tfrac{1}{2}ht,$$

if t and A be both measured from the commencement of the motion. This proves that the area swept out by the radius-vector is proportional to the time of describing it.

When polar co-ordinates are used let θ be the angle that the radius-vector r makes with the axis of x; then $x = r\cos\theta$ and $y = r\sin\theta$: and by substitution

$$\frac{dA}{dt} = \tfrac{1}{2}r^2\frac{d\theta}{dt}; \qquad \therefore\ r^2\frac{d\theta}{dt} = h.$$

The following is an immediate consequence of this property.

PROP. *To prove that the velocity of the body at different parts of its path is inversely proportional to the perpendicular on the tangent.*

243. Velocity $= v = \dfrac{ds}{dt} = \dfrac{ds}{d\theta}\dfrac{d\theta}{dt} = \dfrac{r^2}{p}\dfrac{d\theta}{dt}$, by Diff. Calc.

p is the perpendicular on the tangent at the distance r,

$$= \frac{r^2}{p}\frac{h}{r^2} \text{ by last Art.} = \frac{h}{p}.$$

PROP. *To prove that the velocity is independent of the path described.*

244. Multiply equations (1) (2) of Art. 242. by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ respectively and add them, then

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}+2\frac{dy}{dt}\frac{d^2y}{dt^2}=-\frac{2P}{r}\left(x\frac{dx}{dt}+y\frac{dy}{dt}\right)$$

$$=-2P\frac{dr}{dt},\quad \because\ x^2+y^2=r^2;$$

$$\text{but } v^2=\frac{ds^2}{dt^2}=\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2};$$

$$\therefore\ \frac{d\,.\,v^2}{dt}=-2P\frac{dr}{dt};$$

$$\therefore\ v^2=V^2-2\int_R^r P\,dr,\quad r=R \text{ when } v=V;$$

and since this, when integrated from one position of the body to another, will be a function only of the corresponding distances, it follows, that the velocity is independent of the orbit described, and at any given distance depends solely on the magnitude and law of the force and the velocity and distance of projection.

COR. This is true also when the body is acted on by any number of central forces tending to fixed centres.

There is one more property of central orbits which we shall demonstrate owing to its utility in determining the velocity whenever the force and orbit are known.

PROP. *To prove that the velocity at any point of a central orbit is that due to a body falling through one fourth of the chord of curvature at that point through the centre of force under the action of the force at that point supposed to remain constant.*

245. By last Article $v\frac{dv}{dr}=-P$: and by Art. 243. $v=\frac{h}{p}$: differentiate the logarithm of each side of the latter;

$$\therefore \frac{1}{v}\frac{dv}{dr} = -\frac{1}{p}\frac{dp}{dr},$$

divide the former equation by this;

$\therefore v^2 = Pp\dfrac{dr}{dp} = 2P\,\frac{1}{4}$ chord of curvature through the centre of force at dist. r. Hence the Proposition is true.

Having demonstrated these Properties of Central Orbits we shall proceed to the determination of the nature of the orbits themselves.

PROP. *A body being acted on by a central force: required to find the polar equation to its path.*

246. The equations of motion are

$$\frac{d^2x}{dt^2} = -P\frac{x}{r}\ \ldots\ (1), \quad \frac{d^2y}{dt^2} = -P\frac{y}{r}\ \ldots\ (2);$$

$$\therefore x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0,$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = \text{constant} = h,$$

putting $x = r\cos\theta$ and $y = r\sin\theta$; we have

$$r^2\frac{d\theta}{dt} = h.$$

Again, multiplying (1) and (2) by $2\dfrac{dx}{dt}$ and $2\dfrac{dy}{dt}$ and adding,

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -\frac{2P}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right);$$

$$\therefore \frac{d}{dt}\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\} = -2P\frac{dr}{dt}, \quad \because x^2 + y^2 = r^2,$$

and introducing polar co-ordinates

$$\frac{d}{dt}\left\{\left(\frac{dr^2}{d\theta^2} + r^2\right)\frac{d\theta^2}{dt^2}\right\} = -2P\frac{dr}{dt}.$$

$$\text{But } \frac{d\theta}{dt} = \frac{h}{r^2};$$

$$\therefore \frac{d}{d\theta}\left\{\frac{1}{r^4}\frac{dr^2}{d\theta^2} + \frac{1}{r^2}\right\} = -\frac{2P}{h^2}\frac{dr}{d\theta}.$$

$$\text{Put } \frac{1}{r} = u: \text{ and } \therefore -\frac{1}{r^2}\frac{dr}{d\theta} = \frac{du}{d\theta};$$

$$\therefore \frac{d}{d\theta}\left\{\frac{du^2}{d\theta^2} + u^2\right\} = \frac{2P}{h^2u^2}\frac{du}{d\theta}.$$

and then performing the differentiation on the left-hand side and dividing by $2\dfrac{du}{d\theta}$,

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}.$$

This is the differential equation to the orbit described. The force P being given in terms of r, we must integrate this equation: and the solution will be the equation to the orbit described.

The integral will contain three arbitrary constants, two introduced in the process of integration and the other, h, existing in the differential equation. These are determined by the initial (or any other given) circumstances of the motion: viz. the velocity, distance, and direction of projection.

The general integral determines only the *nature* of the orbit described: but the circumstances of the motion at any given time determine the *species* and *dimensions* of the orbit.

247. The differential equation $P = h^2u^2\left\{\dfrac{d^2u}{d\theta^2} + u\right\}$ may be used to ascertain the law of force which must act upon a body to cause it to describe a given curve. To effect this we must determine the relation between u and θ from the equation to the orbit: we must then differentiate u twice with respect to θ and substitute the result in the expression for P, eliminating θ, if it occur, by means of the relation between u and θ. In this way we shall obtain P in terms of u alone, and therefore of r alone.

248. When we know the relation between r and θ, we make use of the equation $r^2 \dfrac{d\theta}{dt} = h$ to determine the time of describing a given portion of the orbit: or, conversely, to find the position of the body in its orbit at any time. We proceed now to exemplify these principles by various applications.

PROP. *A body moves about a centre of force varying directly as the distance: required to determine the motion.*

249. Let μ be the absolute force: then $P = \mu r = \dfrac{\mu}{u}$.

In order to simplify the calculation we shall first suppose the body projected perpendicular to the radius vector.

Let V, R be the velocity and distance of projection;

$\therefore\ h = 2$ area described in $1'' = VR$ by Art. 243;

$$\therefore\ \frac{d^2u}{d\theta^2} + u = \frac{\mu}{V^2R^2u^3};$$

multiplying by $2\dfrac{du}{d\theta}$ and integrating

$$\frac{du^2}{d\theta^2} + u^2 = C - \frac{\mu}{V^2R^2u^2},$$

$$\text{when } \frac{1}{u} = R,\ \ \frac{dr}{d\theta} = 0 \text{ and } \therefore\ \frac{du}{d\theta} = 0;$$

$$\therefore\ C = \frac{1}{R^2} + \frac{\mu}{V^2},$$

$$\therefore\ \frac{du^2}{d\theta^2} = \frac{V^2 + R^2\mu}{R^2V^2} - \frac{\mu}{V^2R^2u^2} - u^2;$$

$$\tfrac{1}{4}\left(\frac{d\,.\,u^2}{d\theta}\right)^2 = \left(\frac{V^2 - R^2\mu}{2R^2V^2}\right)^2 - \left(u^2 - \frac{V^2 + R^2\mu}{2R^2V^2}\right)^2;$$

extracting the square root, inverting, and integrating

$$2\theta + C = \sin^{-1}\frac{2R^2V^2u^2 - (V^2 + R^2\mu)}{V^2 - R^2\mu},$$

$$\text{when } \theta = 0, \; u = \frac{1}{R}, \; \therefore \; C = \sin^{-1} 1 = \frac{\pi}{2};$$

$$\therefore \; \frac{1}{r^2} = u^2 = \frac{(V^2 + R^2\mu) + (V^2 - R^2\mu)\cos 2\theta}{2R^2V^2}$$

$$= \frac{V^2\cos^2\theta + R^2\mu\sin^2\theta}{R^2V^2}$$

$$1 = \left(\frac{r\cos\theta}{R}\right)^2 + \left(\frac{\sqrt{\mu}\,r\sin\theta}{V}\right)^2.$$

Hence the orbit is an ellipse, the force being in the centre. The semiaxes are R and $\dfrac{V}{\sqrt{\mu}}$.

250. The periodic time may be found by integrating the equation $\dfrac{dt}{d\theta} = \dfrac{r^2}{h}$, (Art. 242). But the following method is more simple.

$$\text{Periodic time} = \frac{2 \text{ area of ellipse}}{h}, \text{ (see Art. 242.)}$$

$$= \frac{2\pi R\dfrac{V}{\sqrt{\mu}}}{VR} = \frac{2\pi}{\sqrt{\mu}}.$$

This result is remarkable; for it shews that the period is independent of the dimensions of the ellipse and depends solely on the *intensity* of the force.

251. Cor. 1. If the angle of projection be β instead of $\frac{1}{2}\pi$ it will be found, that the orbit is still an ellipse, the force being in the centre; and if a, b be the semi-axes*,

* This may be demonstrated with greater facility by using the equations of motion, which are, in this case,

$$\frac{d^2x}{dt^2} = -\mu x, \quad \frac{d^2y}{dt^2} = -\mu y.$$

Multiplying

$$\frac{1}{a^2} \text{ and } \frac{1}{b^2} = \frac{V^2 + \mu R^2 \pm \sqrt{(V^2 + \mu R^2)^2 - 4\mu V^2 R^2 \sin^2\beta}}{2V^2 R^2 \sin^2\beta} \text{ respectively.}$$

$$\text{Hence periodic time} = \frac{2\pi ab}{(h =)\, VR \sin\beta} = \frac{2\pi}{\sqrt{\mu}},$$

the same result as before.

Multiplying them respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ and integrating we have

$$\frac{dx^2}{dt^2} = \mu(h^2 - x^2), \quad \frac{dy^2}{dt^2} = \mu(k^2 - y^2) \text{.................. (1)},$$

h and k being arbitrary constants, introduced in the above form for the sake of symmetry;

$$\therefore \frac{dy^2}{dx^2} = \frac{k^2 - y^2}{h^2 - x^2}, \quad \frac{1}{\sqrt{k^2 - y^2}} \frac{dy}{dx} = \frac{1}{\sqrt{h^2 - x^2}};$$

$$\therefore \sin^{-1}\frac{y}{k} = \sin^{-1}\frac{x}{h} + \sin^{-1} c \text{ .... (2)},$$

c being an arbitrary constant;

$$\therefore \frac{y}{k} = \frac{x}{h}\sqrt{1 - c^2} + c\sqrt{1 - \frac{x^2}{h^2}},$$

by transposing and squaring and transposing again

$$\frac{y^2}{k^2} + \frac{x^2}{h^2} - \frac{2\sqrt{1 - c^2}\, xy}{hk} = c^2.$$

This is the equation to an ellipse from the centre: since $B^2 - 4AC = \frac{4(1 - c^2)}{h^2k^2} - \frac{4}{h^2k^2}$ $= -\frac{4c^2}{h^2k^2}$ is essentially negative; A, B, C being the coefficients of y^2, xy, x^2 respectively.

In order to determine the constants h, k, c, let V be the velocity of projection, α the angle which the direction of projection makes with the axis of x, a, b, the co-ordinates to the point of projection: then equations (1) give

$$V^2 \cos^2\alpha = \mu(h^2 - a^2), \quad V^2 \sin^2\alpha = \mu(k^2 - b^2),$$

$$\text{and (2) gives } \sin^{-1}\frac{b}{k} - \sin^{-1}\frac{a}{h} = \sin^{-1} c,$$

by which h, k, c are known.

If, as in Art. 249, we suppose the body projected from the axis of x at right angles to that line, then $b = 0$, $\alpha = 90^\circ$;

$$\therefore h^2 = a^2, \quad \mu k^2 = V^2,$$

$$\sin^{-1} c = \sin^{-1}\frac{b}{k} - \sin^{-1}\frac{a}{h} = -\sin^{-1} 1, \text{ therefore } c^2 = 1,$$

and the equation to the orbit becomes

$$\frac{\mu}{V^2} y^2 + \frac{1}{a^2} x^2 = 1.$$

The equation to an ellipse of which the semi-axes are a and $\frac{V}{\sqrt{\mu}}$.

Cor. 2. The result of this Proposition is of great importance in Physical Optics. For the forces which act upon the disturbed molecules of the vibrating medium of light all vary as the distance so long as the displacements are not very great. Now the *colour* of the light is assumed to depend upon the *time of vibration* of the molecules: and the *intensity* of the light upon the extent and magnitude of the vibrations, that is, upon the quantity of motion. The preceding Proposition shews, then, that light may alter in intensity without changing in colour, since the time of vibration is independent of the magnitude of the motion, when the law of force is that of the direct distance.

Prop. *A body is acted on by a central force varying inversely as the square of the distance: required to determine the orbit described**.

* Many Propositions of this description may be solved in the following manner.

By Arts. 243, 244, $v^2 = V^2 - 2\int_R^r P\,dr$, $v = \frac{h}{p}$; $\therefore \frac{h^2}{p^2} = V^2 - 2\int_R^r P\,dr$.

$$\text{Ex. 1. Let } P = \mu r; \;\therefore\; \frac{h^2}{p^2} = V^2 + \mu R^2 - \mu r^2,$$

which is the equation to an ellipse about the centre, the axes being given by the equations

$$a^2 + b^2 = \frac{V^2}{\mu} + R^2, \quad a^2b^2 = \frac{h^2}{\mu}.$$

$$\text{Ex. 2. Let } P = \frac{\mu}{r^2}, \;\therefore\; \frac{h^2}{p^2} = \frac{2\mu}{r} - \frac{2\mu}{R} + V^2.$$

This is the equation to a conic section about the focus.

$$\text{The equation to the ellipse is } \frac{1}{p^2} = \frac{2a}{b^2r} - \frac{1}{b^2},$$

$$\text{........................ hyperbola is } \frac{1}{p^2} = \frac{2a}{b^2r} + \frac{1}{b^2},$$

$$\text{........................ parabola is } \frac{1}{p^2} = \frac{1}{mr}.$$

In the case of the ellipse

$$\frac{a}{b^2} = \frac{\mu}{h^2}, \quad \frac{1}{b^2} = \frac{2\mu}{Rh^2} - \frac{V^2}{h^2};$$

$$\therefore a = \frac{R\mu}{2\mu - V^2R}, \quad b = \sqrt{\frac{Rh^2}{2\mu - V^2R}}.$$

The path is an ellipse, hyperbola, or parabola according as V^2 is less than, greater than, or equal to $\frac{2\mu}{R}$.

252. Let μ be the absolute force: then $P = \frac{\mu}{r^2} = \mu u^2$, V, R, β the velocity, distance, and angle of projection: then $h = VR \sin\beta$ (Art. 243.)

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2},$$

multiplying by $2\frac{du}{d\theta}$ and integrating

$$\frac{du^2}{d\theta^2} + u^2 = \frac{2\mu}{h^2} u + C,$$

when $\frac{1}{u} = R$, $r\frac{d\theta}{dr}$, or the tangent of the angle between the radius-vector and the tangent line, $= \tan\beta$:

$$\therefore\ u\frac{d\theta}{du} = -\tan\beta;$$

$$\therefore\ C = \frac{1}{R^2\tan^2\beta} + \frac{1}{R^2} - \frac{2\mu}{h^2R} = \frac{1}{R^2\sin^2\beta} - \frac{2\mu}{h^2R} = \frac{V^2R - 2\mu}{h^2R};$$

$$\therefore\ \frac{du^2}{d\theta^2} = \frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4} - \left(u - \frac{\mu}{h^2}\right)^2;$$

$$\therefore\ \frac{d\theta}{du} = -\frac{1}{\sqrt{\frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4} - \left(u - \frac{\mu}{h^2}\right)^2}},$$

$$\theta + C' = \cos^{-1}\frac{u - \frac{\mu}{h^2}}{\sqrt{\frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4}}},$$

C' is found by the condition, that when $u = \frac{1}{R}$, $\theta = 0$.

$$\text{Then } \frac{1}{r} = \frac{\mu}{h^2} + \sqrt{\frac{V^2R - 2\mu}{h^2R} + \frac{\mu^2}{h^4}}\cos(\theta + C')$$

is the equation to the path: it is the equation to a conic section from the focus, and may be written

$$\frac{1}{r} = \frac{1 + e\cos(\theta + C')}{a(1 \sim e^2)};$$

the angle $\theta + C'$ being measured from the shorter length of the axis major, and $2a$ and $2a\sqrt{1 - e^2}$ being the axes:

$$\text{Then } e^2 = \frac{V^2R - 2\mu}{R\mu^2} h^2 + 1, \text{ subs. for } h$$

$$= \frac{V^2R - 2\mu}{\mu^2} RV^2 \sin^2\beta + 1 \ldots (1);$$

$$\text{and } a(1 \sim e^2) = \frac{h^2}{\mu} = \frac{V^2R^2 \sin^2\beta}{\mu} \ldots\ldots\ldots\ldots (2).$$

Now the conic section is an ellipse, parabola, or hyperbola according as e is less than, equal to, or greater than unity.

Hence, from equation (1), the orbit described is an ellipse, parabola, or hyperbola about the focus according as V^2 is less than, equal to, or greater than $\frac{2\mu}{R}$. This proves the remarkable property, that the species of the conic section described is independent of the direction of projection.

In the case of the ellipse and hyperbola the axis major $= 2a = \frac{2\mu R}{V^2R \sim 2\mu}$ and *this* is also independent of the direction of projection.

In the case of the parabola, the distance of the vertex from the focus, or $D = a(1 - e)$ $(e = 1) = \frac{V^2R^2 \sin^2\beta}{2\mu}$.

The position of the axis major with respect to the radius-vector R, is determined by C', which is the angle between these two lines.

Put $\theta = 0$ and $r = R$ in the value of $\frac{1}{r}$:

$$\therefore \cos C' = \frac{a(1 \sim e^2)}{Re} - \frac{1}{e} = \frac{V^2R\sin^2\beta - \mu}{\mu e}.$$

By referring to Art. 234, we see that the velocity of a body falling from an infinite distance to a distance R from a centre of force $\frac{\mu}{r^2}$ is equal to $\sqrt{\frac{2\mu}{R}}$. Hence the orbit described about this centre of force will be an ellipse, parabola, or hyperbola according as the velocity is less than, equal to, or greater than that from infinity.

253. We might make use of the equation

$$P = h^2 u^2 \left(\frac{d^2 u}{d\theta^2} + u\right)$$

to discover the law of force when the orbit is given.

Thus if the orbit be a conic section with the force in one of the foci, and m be the distance of the pole from the nearest vertex, then the equation to the orbit is

$$u = \frac{1 + e\cos\theta}{m(1+e)}; \quad \therefore P = \frac{h^2 u^2}{m(1+e)} = \frac{h^2}{m(1+e)}\frac{1}{r^2};$$

or the only law of force is that of the inverse square of the distance.

If the orbit be the ellipse the centre being the centre of force, then, a and b being the semi-axes,

$$u^2 = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}; \quad u\frac{du}{d\theta} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)\cos\theta\sin\theta,$$

$$u\frac{d^2u}{d\theta^2} + \frac{du^2}{d\theta^2} = \left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta);$$

$$\therefore P = \frac{h^2}{u}\left\{u^4 + u^3\frac{d^2u}{d\theta^2}\right\} = \frac{h^2}{u}\left\{u^4 - u^2\frac{du^2}{d\theta^2} + u^2\left(u\frac{d^2u}{d\theta^2} + \frac{du^2}{d\theta^2}\right)\right\}$$

$$= \frac{h^2}{u}\left\{\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)^2 - \left(\frac{1}{b^2} - \frac{1}{a^2}\right)^2\cos^2\theta\sin^2\theta\right.$$

$$\left. + \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)\left(\frac{1}{b^2} - \frac{1}{a^2}\right)(\cos^2\theta - \sin^2\theta)\right\}$$

$$= \frac{h^2}{u}\left\{\frac{\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta}{a^2b^2}\right\} = \frac{h^2}{a^2b^2}r,$$

and therefore the only law is that of the direct distance....It was said in Art. 230, that the general problem of finding the force when the orbit is known is indeterminate. In this Article we have assumed the condition, that the force shall be *central;* and so have made it here a determinate problem.

254. If the orbit be a circle the centre of force being in the centre of the circle: then a being the radius $r = a$ is its equation: and

$$P = h^2 u^3 = \frac{h^2}{a^3}.$$

Also by Art. 242, $\frac{dt}{d\theta} = \frac{r^2}{h} = \frac{a^2}{h}$ in this case, and therefore the velocity is constant;

$$\therefore ht = a^2\theta + \text{const.}$$

when $t = 0$, suppose $\theta = 0$; and when $t = T$, the time of revolution, $\theta = 2\pi$;

$$\therefore hT = 2\pi a^2.$$

Let V be the velocity, then $h = Va$, Art. 243;

$$\therefore P = \frac{V^2}{a} \text{ and } T = \frac{2\pi a}{V}.$$

Since the velocity is uniform it follows that the force produces no effect upon the velocity: in short, the only effect of the force is to deflect the body from the rectilinear path which it would describe with the uniform velocity V if no force acted. Consequently the central force is a measure of the tendency that the body has at every instant to preserve a rectilinear course. This tendency is sometimes called the *Centrifugal Force;* and the central force is then called in reference to this the *Centripetal Force.*

When a particle describes a curve in space the force which acts upon it is employed partly in changing the velocity and partly in deflecting the course of the body. A force equal and opposite to the part of the force which deflects the course of the body is called the centrifugal force in this general case as well as in that specified above.

Prop. *To prove that the centrifugal force of a particle moving in space at any point of its course equals the square of the velocity divided by the radius of absolute curvature at that point, and acts in the osculating plane.*

255. If X, Y, Z be the accelerating forces acting on the particle, the equations of motion are

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y, \quad \frac{d^2z}{dt^2} = Z.$$

Now if s, the arc described, be the independent variable, the absolute radius of curvature (ρ) is given by the equations

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \text{.........}(1).$$

Hence if we change the independent variable in the equations of motion from t to s, we have

$$X = \frac{\dfrac{dt}{ds}\dfrac{d^2x}{ds^2} - \dfrac{dx}{ds}\dfrac{d^2t}{ds^2}}{\dfrac{dt^3}{ds^3}} = v^2\frac{d^2x}{ds^2} + \frac{dx}{ds}\frac{d^2s}{dt^2}, \quad v = \frac{ds}{dt},$$

Similarly $Y = v^2\dfrac{d^2y}{ds^2} + \dfrac{dy}{ds}\dfrac{d^2s}{dt^2}$, $Z = v^2\dfrac{d^2z}{ds^2} + \dfrac{dz}{ds}\dfrac{d^2s}{dt^2}$.

If, then, P be the resultant of X, Y, Z we have

$$P^2 = X^2 + Y^2 + Z^2 = v^4\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}$$

$$+ 2v^2\frac{d^2s}{dt^2}\left\{\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2}\right\}$$

$$+ \left\{\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right\}\left(\frac{d^2s}{dt^2}\right)^2.$$

But $\dfrac{dx^2}{ds^2} + \dfrac{dy^2}{ds^2} + \dfrac{dz^2}{ds^2} = 1$, and therefore by equation (1)

$$P^2 = \left(\frac{v^2}{\rho}\right)^2 + \left(\frac{d^2s}{dt^2}\right)^2 \text{..................}(2).$$

Now $\frac{d^2s}{dt^2}$ is the part of the force P that produces the change in velocity (Art. 208): and the other part $\frac{v^2}{\rho}$ acts at right angles to the former, as the form of equation (2) shews, and consequently is what we have termed the centrifugal force: this expression proves the first part of the Proposition.

The force P acts through the point (xyz), let

$$x_1 - x = A(z_1 - z), \quad y_1 - y = B(z_1 - z)$$

be the equations to its direction: then the cosines of the angles which this line makes with the axes are

$$\frac{A}{\sqrt{1 + A^2 + B^2}}, \quad \frac{B}{\sqrt{1 + A^2 + B^2}}, \quad \frac{1}{\sqrt{1 + A^2 + B^2}},$$

but these cosines are also

$$\frac{X}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \frac{Y}{\sqrt{X^2 + Y^2 + Z^2}}, \quad \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}}.$$

$$\text{Hence } A = \frac{X}{Z}, \quad B = \frac{Y}{Z}:$$

and the equations to the direction of the resultant are

$$Z(x_1 - x) = X(z_1 - z), \quad Z(y_1 - y) = Y(z_1 - z),$$

the equations to the tangent line, or the line in which the force $\frac{d^2s}{dt^2}$ acts, are

$$(x_1 - x)\frac{dz}{ds} = (z_1 - z)\frac{dx}{ds}, \quad (y_1 - y)\frac{dz}{ds} = (z_1 - z)\frac{dy}{ds}:$$

Hence the equation to the plane passing through these lines, or the plane in which the centrifugal force acts, is

$$(z_1 - z)\left(X\frac{dy}{ds} - Y\frac{dx}{ds}\right) + (y_1 - y)\left(Z\frac{dx}{ds} - X\frac{dz}{ds}\right)$$
$$+ (x_1 - x)\left(Y\frac{dz}{ds} - Z\frac{dy}{ds}\right) = 0,$$

and substituting in this equation the values of X, Y, Z it becomes

$$(x_1 - x)\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right) + (y_1 - y)\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right)$$

$$+ (z_1 - z)\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right) = 0,$$

which is the equation to the osculating plane at the point (xyz), the arc s being the independent variable.

Hence the second part of the Proposition is true.

After this digression respecting centrifugal force we shall return to the subject of central forces.

256. Kepler discovered by calculation depending on observations, that the planet Mars moves in an ellipse having the Sun in the focus. He also discovered that the areas described by the planet when near its perihelion and aphelion distances (that is, the nearest and farthest distances from the Sun) were proportional to the times of describing them. These two empiric laws have since been proved to hold for the other planets and also for every part of their course. Kepler likewise discovered that the squares of the periodic times of the planets about the Sun were in the same proportion as the cubes of their axes major. These three laws are known by the name of *Kepler's Laws* and may be thus enunciated.

I. *The planets move in ellipses, each having one of its foci in the Sun's centre.*

II. *The areas swept out by each planet about the Sun are, in the same orbit, proportional to the time of describing them.*

III. *The squares of the periodic times of the planets about the Sun are proportional to the cubes of the axes major.*

We shall shew how we are led by these empiric laws to conjecture respecting the nature of the force which acts upon the planetary system.

Prop. *To determine the nature of the force which acts upon the planetary system.*

257. Let XY be the forces which act on a planet parallel to two co-ordinate axes drawn through the Sun in the plane of motion of the planet: then the equations of motion are

$$\frac{d^2x}{dt^2} = X, \quad \frac{d^2y}{dt^2} = Y;$$

$$\therefore\ x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = xY - yX.$$

But by Kepler's Second Law the area is proportional to the time: therefore area $= c \,.\, t$, c being the area described in a unit of time:

$$\therefore\ \frac{d\,.\,\text{area}}{dt} \text{ or } \tfrac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = c;$$

$$\therefore\ x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore\ xY - yX = 0; \quad \therefore\ \frac{X}{Y} = \frac{x}{y}.$$

This shews that the resolved parts of the force acting upon the planet are proportional to the co-ordinates from the Sun's centre: and therefore, by the composition of forces, the force itself must pass through the Sun's centre.

Hence the forces acting on the planets all pass through the Sun's centre.

Let $\frac{1}{r}$ or $u = \frac{1 + e\cos(\theta - \alpha)}{a(1 - e^2)}$ be the equation to the elliptic orbit: Kepler's First Law.

Then the force P, since we have shewn it to be central,

$$= h^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\}, \text{ Art. 246.}$$

$$= h^2u^2\left\{\frac{-e\cos(\theta - \alpha)}{a(1 - e^2)} + \frac{1 + e\cos(\theta - \alpha)}{a(1 - e^2)}\right\}$$

$$= \frac{h^2}{a(1 - e^2)}\frac{1}{r^2}.$$

Hence the *law* of the force acting upon the planets is that of the inverse square of the distance.

Let T be the periodic time of a planet and a the semi-axis major of its orbit.

$$\text{Then } T = \frac{2 \text{ area of ellipse}}{h} = \frac{2\pi a^2\sqrt{1-e^2}}{h};$$

$$\therefore \frac{h^2}{a(1-e^2)} = \frac{4\pi^2 a^3}{T^2},$$

$$\text{and } P = \frac{4\pi^2 a^3}{T^2}\frac{1}{r^2}.$$

But by Kepler's Third Law $\frac{a^3}{T^2}$ is the same for all the planets. Hence not only is the *law* of force the same for all the planets; but the *absolute* force is the same: and consequently the same cause seems to act on all the planets.

From this calculation, then, we conclude that the Sun attracts the planets, and that with a force which varies as the inverse square of their distances from his centre.

258. The elliptical orbits of the planets are nearly *circular:* since, then, in a circle there is no *variation* of distance, it may at first sight be a matter of doubt whether the calculations which prove Kepler's Laws are sufficiently accurate to allow us to believe the *law of variation* of the Sun's attraction to be correctly determined.

This doubt is, however, easily removed. For in the case here contemplated the Third of Kepler's Laws determines both the *law* and *intensity* of the Sun's attraction.

In this case u is constant and $= \frac{1}{a}$;

$$\therefore P = \frac{h^2}{a^3} = \frac{4\pi^2 a}{T^2} = \frac{4\pi^2 a^3}{T^2}\frac{1}{a^2}.$$

But T^2 varies as a^3, for different planets;

$$\therefore P \text{ varies as } \frac{1}{a^2} \text{ for different planets,}$$

and therefore the *law* of attraction is that of the inverse square as before: and the *magnitude* is the same.

259. Now the greatest diameters of the planets are proved by observation to be exceedingly small when compared with their distances from the Sun. But in Art. 165, Cor. we have shewn that the constituent particles of bodies of this description, if they attract, will attract according to the same law as that according to which the bodies themselves attract. And we have just shewn (Art. 257.) that the Sun attracts the planets with a force varying inversely as the square of the distance from his centre. It is therefore highly probable that the particles of the Sun attract the particles of the planets, and vice versâ, with a force varying directly as the mass of the attracting particle and inversely as the square of the distance.

260. These consequences to which we have been led by Kepler's Laws are equally satisfied whether we suppose the centre alone of each body to have an inherent property of attraction, or each particle of the system to attract. But this ambiguity is removed by Dr. Maskelyne's observations on the stars from stations near the mountain Shehallien in Scotland. By these it was proved that the mountain produced a sensible effect in drawing the plumb line out of the vertical: see the *Philosophical Transactions*, 1775. Also some beautiful experiments by Cavendish on the attraction of leaden balls, recorded in the *Philosophical Transactions*, 1798, shew the same thing; that the property of attraction does not reside only in the centres of the heavenly bodies but in every portion of their mass. We are therefore led to conjecture that matter is endowed with a general gravitating principle by which every particle attracts every other particle according to the law before mentioned.

261. Were, however, this principle universally true, not only would the Sun attract the planets, but the planets would attract the Sun (which we have imagined *immoveable**) and likewise one another: and our calculations are erroneous, but these depend on Kepler's Laws. Wherefore it follows, that

* See Arts. 240......246. In these the centre of force is *fixed*.

either Kepler's Laws are not true, or that Universal Gravitation is not a Principle of Nature.

Now in point of fact observations of greater nicety than those made by Kepler prove that his laws are not accurately true, though they differ but slightly from the reality.

Here then is an additional argument (as far as it goes) in favour of Universal Gravitation. For since the magnitudes of the planets are very small in comparison with that of the Sun, we should anticipate that the perturbations of their elliptic motion about the Sun and of the position of the Sun in space by the action of the heavenly bodies would be small; and, consequently, that the deviation from Kepler's Laws would not be considerable.

262. Our investigations thus far are only a *first approximation* to the truth: it yet remains to be determined whether the perturbations actually experienced agree, both in their *nature* and *magnitude*, with those which are calculated on this hypothesis of Universal Gravitation. These are the real tests of the existence of such a principle. Probably many imaginary laws would explain the ordinary phenomena of the motion of the heavenly bodies; but that alone is the law of nature which will stand the test of the more refined calculations of the perturbations.

It is by the complete harmony which is found to subsist between the numerical results deduced from theory and observation that we become convinced of the truth of the Law of Universal Gravitation. To prove this complete accordance is the object of Physical Astronomy.

263. Having stated the main arguments which lead us to conjecture that the motions of the heavenly bodies are regulated by a universal principle of attraction with which all matter is endued, we proceed to a more strict investigation of the consequences of this principle, and shall now enter upon the consideration of the motion of a given number of material particles attracting each other with forces varying directly as the mass of the attracting body and inversely as the square of the distance. This Problem is one of insuperable difficulty when considered in a general point of view, and has baffled the com-

bined exertions of mathematicians from the days of Newton to the present time. In our Solar System, however, the masses of the planets are so small in comparison with that of the Sun, and the inclinations of the planes of their orbits to one another is also so small that the Problem is rendered capable of solution by methods of approximation. But it must not be imagined, that the results are for this reason not to be relied upon. For by the process of *successive approximation* in which we begin by obtaining a *first* approximation, thence proceeding to a *second*, and so on, we can by extending our calculations approach as near the truth as we please: and although the number of calculations must, strictly speaking, be infinite in order to arrive, by this method, at an *exact* result, yet the error in stopping at the third or fourth approximation is so slight as in fact to be inappreciable to our senses. Suppose, for example, that the longitude of the Sun's centre is *calculated* to be $134^0. 0'. 1''$ at some given time and that the *real* longitude is 134^0: what difference does this make in a practical point of view? But even if we were able to obtain an exact solution of the Problem, yet in calculating numerical results we are obliged to reduce the whole to decimals; and though the labour in this case would be perhaps diminished, yet the result would still be only approximate.

We shall first calculate the motion of two bodies, considered as particles, attracting each other, and then proceed to the more general question.

CHAPTER III.

MOTION OF TWO MATERIAL PARTICLES ATTRACTING EACH OTHER.

PROP. *Two material particles attract each other with forces varying inversely as the square of their distance and directly as the mass of the attracting body: required to determine the motion of their centre of gravity.*

264. Let M and m be the masses of the two particles: r their distance at the time t: then, if the unit of attraction be the attraction of a unit of mass at a unit of distance, the accelerating force produced in M by the attraction of $m = \dfrac{m}{r^2}$;

and that produced in m by M's attraction $= \dfrac{M}{r^2}$.

Let xyz be co-ordinates to M at time t,

$x'y'z'$ .... m ....

Then resolving the attractions parallel to the axes, and attending to the directions in which the resolved parts act, the equations of motion of M are

$$\frac{d^2x}{dt^2} = -\frac{m(x-x')}{r^3}, \quad \frac{d^2y}{dt^2} = -\frac{m(y-y')}{r^3}, \quad \frac{d^2z}{dt^2} = -\frac{m(z-z')}{r^3};$$

and those of m are

$$\frac{d^2x'}{dt^2} = \frac{M(x-x')}{r^3}, \quad \frac{d^2y'}{dt^2} = \frac{M(y-y')}{r^3}, \quad \frac{d^2z'}{dt^2} = \frac{M(z-z')}{r^3}.$$

Multiply the first three equations by M and the last three by m, and add the first, second, and third of the first set to the first, second, and third of the second set respectively;

$$\left.\begin{aligned} \therefore\ M\frac{d^2x}{dt^2}+m\frac{d^2x'}{dt^2}=0,\quad M\frac{d^2y}{dt^2}+m\frac{d^2y'}{dt^2}=0,\\ M\frac{d^2z}{dt^2}+m\frac{d^2z'}{dt^2}=0.\end{aligned}\right\}\ldots\ldots(1).$$

Let $\bar{x}\bar{y}\bar{z}$ be the co-ordinates to the centre of gravity of the two bodies at the time t: then

$$(M+m)\,\bar{x}=Mx+mx',\quad (M+m)\,\bar{y}=My+my',$$

$$(M+m)\,\bar{z}=Mz+mz'.$$

Differentiating these twice with respect to t and making use of equations (1), we obtain

$$\frac{d^2\bar{x}}{dt^2}=0,\quad \frac{d^2\bar{y}}{dt^2}=0,\quad \frac{d^2\bar{z}}{dt^2}=0\ldots\ldots\ldots(2);$$

$$\therefore\ \frac{d\bar{x}}{dt}=a,\quad \frac{d\bar{y}}{dt}=b,\quad \frac{d\bar{z}}{dt}=c,$$

a, b, c being constants to be determined by the initial circumstances of the motion of the bodies.

Hence the velocity of the centre of gravity $=\sqrt{a^2+b^2+c^2}$, (Art. 207) and is therefore uniform.

$$\text{Also } \frac{d\bar{x}}{d\bar{z}}=\frac{a}{c},\quad \frac{d\bar{y}}{d\bar{z}}=\frac{b}{c};$$

$$\therefore\ \bar{x}=\frac{a}{c}\bar{z}+a',\quad \bar{y}=\frac{b}{c}\bar{z}+b',$$

a', b' being constants to be determined as before.

These are the equations to the path of the centre of gravity; and, since they are the equations to a straight line in space, they prove that that point will move in a straight line.

If a, b, c each $=0$, then the expression for the velocity of the centre of gravity vanishes: and the general conclusion is, That *the centre of gravity of the two bodies will either remain at rest during the motion of the bodies, or move*

uniformly in a straight line. Which of these will be the case is determined by the initial circumstances of the motion of the bodies.

Prop. *To determine the orbits the bodies describe about each other, and about their centre of gravity.*

265. Let us subtract the equations of motion for m from those of M respectively, and we obtain

$$\frac{d^2(x-x')}{dt^2} = -\frac{(M+m)(x-x')}{r^3}, \quad \frac{d^2(y-y')}{dt^2} = -\frac{(M+m)(y-y')}{r^3},$$

$$\frac{d^2(z-z')}{dt^2} = -\frac{(M+m)(z-z')}{r^3}.$$

These are the equations we should obtain by supposing either of the bodies at rest, and the force acting on the other to be the *sum* of the masses divided by the square of the distance.

Hence (Art. 252) each will describe relatively to the other a conic section, the nature of the path being determined by the circumstances of projection of the bodies.

266. To determine their paths about their centre of gravity, let $r_{,}$ and r' be the distances of M and m from that point at the time t: then

$$r_{,} = \frac{m}{M+m} r, \quad r' = \frac{M}{M+m} r.$$

Also, if P and Q be the two particles (fig. 80), G their centre of gravity,

$$\frac{PN}{PQ} = \frac{PN'}{PG}, \quad \therefore \frac{x-x'}{r} = \frac{x-\bar{x}}{r_{,}}:$$

and in the same way $\dfrac{y-y'}{r} = \dfrac{y-\bar{y}}{r_{,}}$ and $\dfrac{z-z'}{r} = \dfrac{z-\bar{z}}{r_{,}}$:

Now subtract equations (2) of Art. 264. from the equations of motion of M in that Article respectively:

$$\therefore \frac{d^2(x-\bar{x})}{dt^2} = -\frac{m(x-x')}{r^3} = -\frac{m^3}{(M+m)^2}\frac{x-\bar{x}}{r_{,}^{3}}$$

$$\frac{d^2(y-\bar{y})}{dt^2} = -\frac{m^3}{(M+m)^2}\frac{y-\bar{y}}{r_{,}^{3}} \text{ and } \frac{d^2(z-\bar{z})}{dt^2} = -\frac{m^3}{(M+m)^2}\frac{z-\bar{z}}{r_{,}^{3}}.$$

These are the equations of motion of M relatively to the centre of gravity of M and m, which as we have seen is at rest, or is moving uniformly in a straight line. They prove that the path about the centre of gravity is such as would be described about a force $\frac{m^3}{(M+m)^2} \cdot \frac{1}{r_{,}^{2}}$ residing in that point.

Hence the orbits of M and m relatively to the centre of gravity are conic sections, their nature and magnitude being determined by the circumstances of projection *relatively* to the centre of gravity of M and m.

PROP. *To compare the relative orbits of* M *and* m *about their centre of gravity.*

267. Let v, v' be the absolute velocities of projection of M and m: $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ the angles the directions of these velocities make with the axes. V and V' the relative vels. of project. about centre of gravity, R and R' the initial distances from the centre of gravity, δ and δ' the relative angles of projection, a and a' the semi-axes major of the orbits, e and e' the eccentricities of the orbits, μ and μ' the absolute forces.

Then by equations (1) (2) of Art. 252,

$$\frac{1-e^2}{1-e'^2} = \frac{2\mu - V^2R}{2\mu' - V'^2R'}\frac{RV^2\sin^2\delta}{R'V'^2\sin^2\delta'}\frac{\mu'^2}{\mu^2},$$

$$\text{and } \frac{a(1-e^2)}{a'(1-e'^2)} = \frac{V^2R^2\sin^2\delta}{V'^2R'^2\sin^2\delta'}\frac{\mu'}{\mu}.$$

Also $\frac{R}{R'} = \frac{m}{M}$, and $\frac{\mu'}{\mu} = \frac{M^3}{m^3}$ by Art. 266.

To find V, V', δ, δ', we proceed as follows.

The velocities of the centre of gravity parallel to the axes are at first, and therefore during the motion, respectively (Art. 264.)

$$\frac{Mv\cos\alpha+mv'\cos\alpha'}{M+m}, \quad \frac{Mv\cos\beta+mv'\cos\beta'}{M+m}, \quad \frac{Mv\cos\gamma+mv'\cos\gamma'}{M+m}.$$

Also the *absolute* velocities of projection of M parallel to the axes are $v\cos\alpha$, $v\cos\beta$, $v\cos\gamma$: and therefore the *relative* velocities of projection of M about the centre of gravity parallel to the axes are

$$\frac{m(v\cos\alpha-v'\cos\alpha')}{M+m}, \quad \frac{m(v\cos\beta-v'\cos\beta')}{M+m}, \quad \frac{m(v\cos\gamma-v'\cos\gamma')}{M+m}.$$

Adding the squares of these, (Art. 207.) the square of the relative velocity of M about the centre of gravity (V^2) =

$$\frac{m^2}{(M+m)^2}\{(v\cos\alpha-v'\cos\alpha')^2+(v\cos\beta-v'\cos\beta')^2+(v\cos\gamma-v'\cos\gamma')^2\}$$

$$=\frac{m^2}{(M+m)^2}(v^2+v'^2-2vv'\cos A),$$

where A is the angle between the directions of projection of M and m: and therefore determined by the equation

$$\cos A=\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'.$$

Similarly $V'^2=\dfrac{M^2}{(M+m)^2}(v^2+v'^2-2vv'\cos A)=\dfrac{M^2}{m^2}V^2.$

Let the line joining M and m at the commencement of the motion be the axis of x: then the cosine of the angle which the direction of V makes with the distance of projection (which coincides with the axis of x), or $\cos\delta$, equals the relative velocity parallel to the axis of x divided by the whole relative velocity (V) =

$$\frac{m}{M+m}\,\frac{v\cos\alpha-v'\cos\alpha'}{V}.$$

$$\text{Similarly } \cos\delta' = \frac{M}{M+m}\,\frac{v'\cos\alpha' - v\cos\alpha}{V'} = -\cos\delta.$$

Substituting these in the expression given above for $\dfrac{1-e^2}{1-e'^2}$ we find

$$\frac{1-e^2}{1-e'^2} = 1\,;\ \therefore e = e',$$

or the orbits are similar to each other.

$$\text{Also } \frac{a}{a'} = \frac{m^4}{M^4}\cdot\frac{M^3}{m^3} = \frac{m}{M},$$

or the linear dimensions of the orbits of M and m are in the ratio of m to M.

268. Cor. 1. It follows from this that the perturbation of the Sun by any planet is very small, because his mass is so much the greater of the two masses.

In the same way it will be shewn that the combined effect of the heavenly bodies in moving the Sun is very slight; and therefore the error in Kepler's Laws, anticipated in Art. 261, owing to the supposed immobility of the Sun, is not very great. Thus far, then, we are confirmed in our hypothesis of Universal Gravitation.

269. Cor. 2. We have seen (Art. 257) that if μ be the absolute force of a centre of which the law is that of the inverse square, and a the semi-axis major of the orbit described, the periodic time

$$(T) = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}} \quad \text{(Art. 265)},$$

M and m being the masses of the Sun and a planet.

Let m' be the mass of another planet: and a' the semi-axis major of its orbit, T' its period;

$$\therefore T' = \frac{2\pi a'^{\frac{3}{2}}}{\sqrt{M+m'}}, \text{ and } \therefore \frac{T^2}{T'^2} = \frac{a^3}{a'^3}\,\frac{M+m'}{M+m}.$$

This shews that Kepler's Third Law would not be true even if we suppose that the planets do not attract each other,

unless their masses were equal to each other. The deviation, however, from the truth is extremely small.

270. The investigations in Arts. 252, 265, shew us that if our law of gravitation be true, the only orbits which a heavenly body will describe, supposed to be acted on only by the Sun, are an ellipse, a parabola, or a hyperbola with the Sun's centre in the focus.

The manner in which the magnitude and position of the orbit of a heavenly body is determined by actual observation will be found in Works on Plane Astronomy. We shall here briefly explain the process. There are six quantities which determine the position and magnitude of an elliptic or hyperbolic orbit, and the place of the body in its orbit: these are called the *elements* of the body's orbit, and are (1) the inclination of the orbit to the ecliptic, and (2) the longitude of the ascending node, these determine the *position of the plane of the orbit* in space: next (3) the longitude of the perihelion, (or point of the orbit nearest the Sun), which determines the *position of the orbit itself:* then (4) the mean distance, and (5) eccentricity, which determine the *magnitude* of the orbit, and lastly (6) the epoch, or the time of the planet's being in the perihelion, this determines the *position of the body itself* in its orbit.

The elements of a parabolic orbit are five in number, being the same as the above, if we replace the mean distance and eccentricity by the perihelion distance.

The elements of a circular orbit are only four in number, the eccentricity and longitude of the perihelion not being required.

In order to determine the numerical values of the elements of any heavenly body (supposed to move in a conic section with the Sun in the focus) two Trigonometrical equations* are deduced connecting the elements with the right ascension and

* For a parabolic and circular orbit see Maddy's Plane Astronomy, Chap. XIV. Woodhouse's Plane Astronomy, Chap. XXIV.

But for other orbits the reader may consult the Work of Lalande; Gauss's Theoria Motus Corporum Cœlestium: the Mécanique Céleste, Vol. I.; Lagrange's Mec. Analytique; Pontécoulant's Théorie Anal. du Système du Monde, and Mr Lubbock's Mathematical Tracts and various Papers in the Transactions of the Philosophical and Astronomical Societies.

declination of the body and the distance of the Earth from the Sun.

Since there are five or six quantities to be determined three independent observations must be made on the declination and right ascension of the body: when these are substituted successively in the two equations mentioned above we shall have six equations involving the elements: by means of which we shall be able to calculate the magnitude and position of the orbit.

271. By methods of this nature Kepler discovered his three planetary Laws.

Also Astronomers have in this way proved, that comets move in orbits most of which are parabolic, some elliptic, and others probably hyperbolic. In consequence of the vast distances to which comets penetrate into space, they are invisible except when near the Sun. During their appearance numerous observations are made, in order that the elements may be determined with the greatest possible accuracy. The calculations for parabolic motion are less laborious than for elliptic or hyperbolic motion. The elements are therefore first calculated on the supposition that the orbit is a parabola. If the elements thus calculated shew that the comet has passed so near any of the planets as to have experienced a sensible perturbation the elements must be corrected in a manner to be explained hereafter.

If a parabola will not coincide with the orbit calculations must be made for an ellipse or hyperbola. It is thus found that "three or four comets describe very long ellipses: and nearly all the others that have been observed are found to move in curves which cannot be distinguished from parabolas. There is reason to think that two or three comets move in hyperbolas." (Airy's *Gravitation*, page 15.)

272. Our calculations have been hitherto respecting the nature of the orbits described. We now proceed to deduce formulæ for determining the time that the body occupies in moving through a given angle; and conversely the angle described in a given time: by the former we know the time of the body being at a given place, and by the latter we know the place of the body at a given time.

PROP. *To find the time of motion of a planet or comet through any portion of an elliptic orbit, the Sun's centre being in the focus.*

273. Let θ and ϖ be the longitudes of the body and the perihelion, that is, the point of the orbit nearest the Sun: a the semi-axis major of the orbit: e the eccentricity: μ the sum of the masses of the Sun and the body (Art. 265): then the equation to the orbit is

$$\frac{1}{r} = \frac{1 + e\cos(\theta - \varpi)}{a(1 - e^2)}. \quad \text{Also } \frac{dt}{d\theta} = \frac{r^2}{h}, \text{ Art. 242.}$$

Now h must be determined in terms of the quantities above given, since the orbit to be described is known and not the original circumstances of projection. The following method, which we here apply to the ellipse, will answer our purpose in every case. By Art. 243, $h = vp$ at every point of a central orbit; v being the velocity and p the perpendicular from the centre of force on the tangent at that point: also by Art. 245, the velocity is that due to one-fourth the chord of curvature through the centre of force;

$$\therefore\ v^2 = \frac{2\mu}{r^2}\cdot\tfrac{1}{2}p\frac{dr}{dp}: \text{ but } p^2 = \frac{b^2 r}{2a - r} \text{ from the focus;}$$

$$\therefore\ h = vp = \sqrt{\frac{\mu}{r^2}p^3\frac{dr}{dp}} = \sqrt{\frac{\mu b^2}{a}} = \sqrt{\mu a(1 - e^2)}.$$

Then the time of moving from the perihelion through the angle $\theta - \varpi =$

$$t = \int_\varpi^\theta \frac{r^2 d\theta}{h} = \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_\varpi^\theta \frac{d\theta}{\{1 + e\cos(\theta - \varpi)\}^2}$$

$$= \frac{a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_\varpi^\theta \frac{d\theta}{\{(1 + e)\cos^2\frac{1}{2}(\theta - \varpi) + (1 - e)\sin^2\frac{1}{2}(\theta - \varpi)\}^2}$$

$$= \frac{2a^{\frac{3}{2}}(1 - e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_\varpi^\theta \frac{\sec^2\frac{1}{2}(\theta - \varpi)\ \dfrac{d\tan\frac{1}{2}(\theta - \varpi)}{d\theta}\, d\theta}{\{(1 + e) + (1 - e)\tan^2\frac{1}{2}(\theta - \varpi)\}^2}.$$

To simplify this let

$$\tan \tfrac{1}{2}(\theta - \varpi) = \sqrt{\frac{1+e}{1-e}} \tan \frac{u}{2} \dots\dots\dots (1);$$

$$\therefore t = \frac{2a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u \frac{\left(1 + \frac{1+e}{1-e}\tan^2\frac{u}{2}\right)}{(1+e)^2 \sec^4 \frac{u}{2}} \frac{d}{du}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}\right) du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u \left\{(1-e)\cos^2\frac{u}{2} + (1+e)\sin^2\frac{u}{2}\right\} du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \int_0^u (1 - e\cos u)\, du$$

$$= \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} (u - e\sin u), \text{ let } \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{1}{n};$$

$$\therefore nt = u - e\sin u \dots\dots\dots (2).$$

When θ is given we calculate u by (1), and substituting in (2) we know t.

The angle $\theta - \varpi$, or the excess of the longitude of the body over the longitude of the perihelion, is called the *true anomaly;* and nt is called the *mean anomaly*, since it varies uniformly with the time and coincides with the true anomaly at the end of each revolution, as the formulæ (1) (2) shew. Also the angle u is called the *eccentric anomaly*, since it equals the angle QCA (fig. 81), as may easily be proved: P is the body, APa the ellipse, S the focus, AQa a circle on Aa.

Cor. 1. If t be not measured from the epoch of passing the perihelion, but from the time when $u = u_{,}$, then

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \left\{(u - u_{,}) - e(\sin u - \sin u_{,})\right\}.$$

Cor. 2. Whenever u increases by 2π, θ increases by 2π, and t by $\dfrac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$. This, then, is the periodic time of the

planet: it is remarkable that it is independent of the eccentricity of the orbit.

To solve the converse of this Proposition, that is, to find the position of a heavenly body in its elliptic orbit at any time in terms of the time and the elements of the orbit, we must effect several expansions.

PROP. *To expand the true anomaly in terms of the eccentric anomaly.*

274. By last Article $\tan\dfrac{\theta-\varpi}{2}=\sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{u}{2}$.

Substituting the exponential expressions for the tangents,

$$\frac{\varepsilon^{(\theta-\varpi)\sqrt{-1}}-1}{\varepsilon^{(\theta-\varpi)\sqrt{-1}}+1}=m\,\frac{\varepsilon^{u\sqrt{-1}}-1}{\varepsilon^{u\sqrt{-1}}+1},\quad \sqrt{\frac{1+e}{1-e}}=m,$$

in which ε is the base of Napierian logarithms.

$$\therefore\ \varepsilon^{(\theta-\varpi)\sqrt{-1}}=\frac{(m+1)\varepsilon^{u\sqrt{-1}}-(m-1)}{(m+1)-(m-1)\varepsilon^{u\sqrt{-1}}}=\varepsilon^{u\sqrt{-1}}\frac{1-\lambda\varepsilon^{-u\sqrt{-1}}}{1-\lambda\varepsilon^{u\sqrt{-1}}},\ \lambda=\frac{m-1}{m+1};$$

$$\therefore\ (\theta-\varpi)\sqrt{-1}=u\sqrt{-1}+\log_\varepsilon(1-\lambda\varepsilon^{-u\sqrt{-1}})-\log_\varepsilon(1-\lambda\varepsilon^{u\sqrt{-1}})$$

$$=u\sqrt{-1}+\lambda\,(\varepsilon^{u\sqrt{-1}}-\varepsilon^{-u\sqrt{-1}})+\frac{\lambda^2}{2}(\varepsilon^{2u\sqrt{-1}}-\varepsilon^{-2u\sqrt{-1}})+\ldots$$

$$\therefore\ \theta-\varpi=u+2\,(\lambda\sin u+\tfrac{1}{2}\lambda^2\sin 2u+\tfrac{1}{3}\lambda^3\sin 3u+\ldots)$$

$$\text{in which } \lambda=\frac{\sqrt{1+e}-\sqrt{1-e}}{\sqrt{1+e}+\sqrt{1-e}}=\frac{1-\sqrt{1-e^2}}{e}.$$

PROP. *To expand the eccentric anomaly in terms of the mean anomaly.*

275. By Art. 273, $u=nt+e\sin u$.

Hence by Lagrange's Theorem, putting $nt=z$,

$$u = x + e \sin x + \frac{e^2}{1.2} \frac{d \sin^2 x}{dx} + \frac{e^3}{1.2.3} \frac{d^2 \sin^3 x}{dx^2} + \dots$$

$$= x + e \sin x + \tfrac{1}{2} e^2 \sin 2x + \tfrac{1}{2} e^3 (2 \sin x - 3 \sin^3 x) + \dots$$

$$= nt + e \sin nt + \tfrac{1}{2} e^2 \sin 2nt + \tfrac{1}{8} e^3 (3 \sin 3nt - \sin nt) + \dots$$

Prop. *To expand* sin u, *sin* 2 u, ... *in terms of the mean anomaly.*

276. By Lagrange's Theorem,

$$\sin u = \sin x + e \sin x \frac{d \sin x}{dx} + \frac{e^2}{1.2} \frac{d}{dx} \left\{ \sin^2 x \frac{d \sin x}{dx} \right\} + \dots$$

$$= \sin x + e \sin x \cos x + \tfrac{1}{2} e^2 (2 \cos^2 x \sin x - \sin^3 x) + \dots$$

$$= \sin x + \tfrac{1}{2} e \sin 2x + \tfrac{1}{8} e^2 (2 \sin 3x - \sin x) + \dots$$

$$= \sin nt + \tfrac{1}{2} e \sin 2nt + \tfrac{1}{8} e^2 (3 \sin 3nt - \sin nt) + \dots$$

Again, $\sin 2u = \sin 2x + e \sin x \dfrac{d \sin 2x}{dx} + \dots$

$$= \sin 2nt + 2e \sin nt \cos 2nt + \dots$$

$$= \sin 2nt + e (\sin 3nt - \sin nt) + \dots$$

$$\sin 3u = \sin 3nt + \dots$$

and so on.

Prop. *To expand the true anomaly in terms of the mean anomaly.*

277. By Art. 274 we have

$$\theta - \varpi = u + 2 (\lambda \sin u + \tfrac{1}{2} \lambda^2 \sin 2u + \tfrac{1}{3} \lambda^3 \sin 3u + \dots)$$

$$\text{where } \lambda = \frac{1 - \sqrt{1 - e^2}}{e} = \frac{e}{2} + \frac{e^3}{8} + \dots$$

Then substituting for u, $\sin u$, $\sin 2u$ ... the values obtained in the last two Articles, and retaining powers of e as far as the cube,

$$\theta - \varpi = nt + e \sin nt + \tfrac{1}{2}e^2 \sin 2nt + \tfrac{1}{8}e^3 (3 \sin 3nt - \sin nt) + \dots$$
$$+ 2\lambda \left\{ \sin nt + \tfrac{1}{2} e \sin 2nt + \tfrac{1}{8}e^2 (3 \sin 3nt - \sin nt) + \dots \right\}$$
$$+ \lambda^2 \left\{ \sin 2nt + e (\sin 3nt - \sin nt) + \dots\dots\dots\dots\dots \right\}$$
$$+ \tfrac{2}{3} \lambda^3 \sin 3nt + \dots\dots\dots\dots\dots\dots\dots\dots$$
$$= nt + (2e - \tfrac{1}{4}e^3) \sin nt + \tfrac{5}{4}e^2 \sin 2nt + \tfrac{13}{12} e^3 \sin 3nt + \dots$$

which is true as far as terms involving e^3.

278. Cor. If the time t be not measured from the time of perihelion passage, suppose ϵ is the mean longitude of the body when $t = 0$; then the mean longitude at the time t is $nt + \epsilon$; and the mean anomaly is $nt + \epsilon - \varpi$: in this case, then,

$$\theta - \varpi = nt + \epsilon - \varpi + (2e - \tfrac{1}{4}e^3) \sin (nt + \epsilon - \varpi)$$
$$+ \tfrac{5}{4} e^2 \sin 2 (nt + \epsilon - \varpi) + \dots$$

ϵ is called the *epoch*.

Prop. *To expand the radius-vector* r *in terms of the mean anomaly.*

279. The radius-vector

$$r = \frac{a(1-e^2)}{1 + e \cos (\theta - \varpi)} = \frac{a(1-e^2)}{(1+e) \cos^2 \frac{1}{2}(\theta - \varpi) + (1-e) \sin^2 \frac{1}{2}(\theta - \varpi)}$$

$$= \frac{a(1-e^2) \sec^2 \frac{1}{2}(\theta - \varpi)}{1 + e + (1-e) \tan^2 \frac{1}{2}(\theta - \varpi)} = \frac{a(1-e^2)\left\{1 + \dfrac{1+e}{1-e} \tan^2 \frac{1}{2} u\right\}}{(1+e) \sec^2 \frac{1}{2} u}$$

$$= a \left\{(1-e) \cos^2 \tfrac{1}{2} u + (1+e) \sin^2 \tfrac{1}{2} u\right\} = a (1 - e \cos u).$$

But $u = nt + e \sin u$; putting $nt = z$,

$$\cos u = \cos z + e \sin z \frac{d \cos z}{dz} + \frac{e^2}{1 . 2} \frac{d}{d} \left\{ \sin^2 z \frac{d \cos z}{dz} \right\} + \dots$$

$$= \cos z - \tfrac{1}{2} e (1 - \cos 2z) - \tfrac{1}{8} e^2 (3 \cos z - 3 \cos 3z) + \dots$$

$$\therefore r = a \left\{ 1 + \tfrac{1}{2}e^2 - e \cos nt - \tfrac{1}{2} e^2 \cos 2nt - \tfrac{3}{8} e^3 (\cos 3nt - \cos nt) + \dots \right\}$$

280. Cor. If t be measured as in Art. 278, then

$$r = a\left\{1 + \tfrac{1}{2}e^2 - e\cos(nt + \epsilon - \varpi) - \tfrac{1}{2}e^2\cos 2(nt + \epsilon - \varpi) - \ldots\right\}.$$

The time of describing a given portion of an elliptic hyperbolic or parabolic orbit may be found in terms of the radius-vectors at the extremities of the arc and the chord of the arc. These expressions are useful in determining the elements of a heavenly body. They will be found in Maddy's *Plane Astronomy*, Chapter XIII. New Edition: and in the *Système du Monde* of M. Pontécoulant, Tom. I. Liv. II. Chap. V.

Prop. *To find the time of describing a given portion of a parabolic orbit about the Sun in the focus.*

281. We have $r^2 \dfrac{d\theta}{dt} = h$: $h = \sqrt{2\mu D}$, and $r = \dfrac{D}{\cos^2 \frac{1}{2}(\theta - \varpi)}$ is the equation to the parabola, θ and ϖ being the longitude of the comet and of its perihelion measured from the Sun, and D the perihelion distance;

$$\therefore t = \frac{D^{\frac{3}{2}}}{\sqrt{2\mu}} \int_{\varpi}^{\theta} \frac{d\theta}{\cos^4 \frac{1}{2}(\theta - \varpi)}$$

$$= \frac{2D^{\frac{3}{2}}}{\sqrt{2\mu}} \int_{\varpi}^{\theta} \frac{d\tan\frac{1}{2}(\theta - \varpi)}{d\theta} \left\{1 + \tan^2 \tfrac{1}{2}(\theta - \varpi)\right\} d\theta$$

$$= \sqrt{\frac{2}{\mu}} D^{\frac{3}{2}} \left\{\tan\tfrac{1}{2}(\theta - \varpi) + \tfrac{1}{3}\tan^3 \tfrac{1}{2}(\theta - \varpi)\right\},$$

t being measured from the time of the perihelion passage.

By this equation it is easy to calculate the time of describing a given angle.

Prop. *To find the position of the comet in a parabolic orbit at a given time.*

282. This would require the solution of the cubic equation in the last Article. This is, however, obviated in the following manner.

$$\text{Let } \sqrt{\frac{\mu}{2D^3}} = n;$$

$$\therefore nt = \tan\tfrac{1}{2}(\theta - \varpi) + \tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta - \varpi).$$

A Table is formed consisting of two columns: one with values of t and the other with the corresponding values of $\theta - \varpi$ calculated from this formula for an orbit in which $n = 1$. Suppose, then, that we wish to find the position of a comet in a given parabolic orbit (the mean motion in which is n) at a given time t. We must multiply t by n and look for the value of $\theta - \varpi$ opposite the value of nt in the first column. This gives the position of the comet.

PROP. *To find the place of a comet at a given time in a very eccentric elliptic orbit.*

283. By Art. 273. $\dfrac{dt}{d\theta} = \dfrac{a^{\frac{3}{2}}(1-e^2)^{\frac{3}{2}}}{\sqrt{\mu}} \dfrac{1}{\{1 + e\cos(\theta - \varpi)\}^2}$.

Let D be the perihelion distance; $\therefore D = a(1-e)$;

$$\therefore \frac{dt}{d\theta} = \frac{D^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\sqrt{\mu}} \frac{\sec^4 \frac{1}{2}(\theta-\varpi)}{\{(1+e) + (1-e)\tan^2\frac{1}{2}(\theta-\varpi)\}^2}$$

$$= \frac{D^{\frac{3}{2}}}{\sqrt{\mu(1+e)}} \sec^4\tfrac{1}{2}(\theta-\varpi)\left\{1 + \frac{1-e}{1+e}\tan^2\tfrac{1}{2}(\theta-\varpi)\right\}^{-2}.$$

Expanding in powers of $1-e$, and neglecting powers of $1-e$ higher than the first, because $e = 1$ nearly;

$$\therefore nt = \tfrac{1}{2}\left(1 - \frac{1-e}{2}\right)^{-\frac{1}{2}} \int_\varpi^\theta \sec^4\tfrac{1}{2}(\theta-\varpi)\{1-(1-e)\tan^2\tfrac{1}{2}(\theta-\varpi)\}\,d\theta$$

$$= \int_\varpi^\theta \frac{d\tan\frac{1}{2}(\theta-\varpi)}{d\theta}\Big\{1 + \tan^2\tfrac{1}{2}(\theta-\varpi)$$

$$+ (1-e)\left[\tfrac{1}{4} - \tfrac{3}{4}\tan^2\tfrac{1}{2}(\theta-\varpi) - \tan^4\tfrac{1}{2}(\theta-\varpi)\right]\Big\}\,d\theta;$$

$$\therefore nt = \tan\tfrac{1}{2}(\theta-\varpi) + \tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta-\varpi)$$

$$+ (1-e)\left\{\tfrac{1}{4}\tan\tfrac{1}{2}(\theta-\varpi) - \tfrac{1}{4}\tan^3\tfrac{1}{2}(\theta-\varpi) - \tfrac{1}{5}\tan^5\tfrac{1}{2}(\theta-\varpi)\right\}.$$

The following is a convenient method for calculating the value of $\theta - \varpi$ for a given value of t.

Suppose $\theta' - \varpi$ is the true anomaly of a comet at the time t moving in a parabolic orbit of which D is the perihelion distance; then by Art. 282,

$$nt = \tan\tfrac{1}{2}(\theta' - \varpi) + \tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta' - \varpi).$$

Let $\theta - \varpi = \theta' - \varpi + x$: then putting this for $\theta - \varpi$ in the first expression for nt, and neglecting the squares and products of x and e, we have by Taylor's Theorem

$$nt = \tan\tfrac{1}{2}(\theta' - \varpi) + \tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta' - \varpi)$$
$$+ \tfrac{1}{2}x\sec^4\tfrac{1}{2}(\theta' - \varpi) + \tfrac{1}{4}(1-e)\tan\tfrac{1}{2}(\theta' - \varpi)\left\{1 - \tan^2\tfrac{1}{2}(\theta' - \varpi)\right.$$
$$\left. - \tfrac{4}{5}\tan^4\tfrac{1}{2}(\theta' - \varpi)\right\},$$

and eliminating nt from these last two equations

$$x = \tfrac{1}{10}(1-e)\tan\tfrac{1}{2}(\theta' - \varpi)\left\{4 - 3\cos^2\tfrac{1}{2}(\theta' - \varpi) - 6\cos^4\tfrac{1}{2}(\theta' - \varpi)\right\}.$$

A third column must now be added to the table mentioned in Art. 282. consisting of values of $\dfrac{x}{1-e}$ for the corresponding values of t and $\theta - \varpi$. When this is constructed the manner of using it is as follows. Suppose $\sqrt{\dfrac{\mu}{2D^3}} = n$ in our orbit: then in the first column look for the time nt; and take the corresponding values of $\theta - \varpi$ and $\dfrac{x}{1-e}$: multiply the latter by $1-e$, which will depend upon the form of the orbit, and then the true anomaly at the time t will be this quantity added to the value of $\theta - \varpi$ thus found.

CHAPTER IV.

CALCULATION OF THE LUNAR PERTURBATIONS, ACCORDING TO NEWTON'S METHOD.

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284. In the last Chapter we have calculated completely the motion of two bodies, considered as particles, attracting each other according to the assumed law of gravitation. When the various formulæ there obtained are applied to calculate the motion of the planets about the Sun, and for that purpose are reduced to tables, they manifest an agreement with observation so far complete as to leave no doubt of the correctness of the principles, which form the basis of the calculation; provided that observations be made at times separated by moderately long intervals. If, however, we proceed to a more rigorous nicety, and especially if we compare together observations which embrace a very long series of years, it is found that the agreement is not so perfect. Minute irregularities are detected, and the planets are found sometimes a little in advance, sometimes a little falling short, sometimes a little above or below, to the right or left, of their places, calculated on the theory of elliptic motion.

Now this is exactly what was to be anticipated. For if the principle of gravitation be universal the heavenly bodies disturb each other in their motion about the Sun, and so derange the elliptic form of the orbits and the equable description of areas by the radius-vector.

285. It is our object in Chapters V and VI to deduce formulæ by which the mutual perturbations of the heavenly bodies may be calculated. The equations of motion for three or more particles attracting each other according to the law of gravitation have never yet been integrated. In fact, their integrals depend upon the integration of a function analogous to the function V in Art. 169. See note to Art. 319. We must therefore have recourse to methods of approximation.

286. The peculiar configuration of the Solar System renders this approximation practicable, though under most other arrangements it would not be so; the bodies of our system are arranged either singly as the planets Mercury, Venus and Mars, or in groups as the Earth and the Moon, Jupiter, Saturn, and Herschel each with their Satellites, the central masses of the groups being much greater than that of the attending bodies: likewise the single bodies and the groups are always at considerable distances from each other, and describe orbits about the Sun very nearly circular and in planes nearly coinciding. There is, however, an exception in the case of the four asteroids Ceres, Vesta, Pallas, and Juno, the orbits of these being not very far different from each other in magnitude; but their masses are so small that in this way a compensation takes place. The mass of the Sun is of enormous magnitude in comparison with that of the other bodies; since, as we have remarked in Art. 284, the calculations made on the supposition that the Sun is the only attracting body nearly coincide with observation. Again the mass of the Earth is large when compared with that of the Moon; because the Earth moves about the Sun and the Moon about the Earth, nearly as if the Moon did not disturb the Earth's elliptic motion about the Sun, and as if the Sun did not disturb the Moon's elliptic motion about the Earth. In the same way we argue that the mass of Jupiter is much larger than that of his satellites by observing that Kepler's Laws are nearly verified, and so of the other bodies.

It is in consequence of this peculiar configuration of the Solar System that we are able to approximate to the solutions of our equations of motion by converging series.

287. In the present Chapter we intend to explain the nature of the perturbations of the Moon's motion about the Earth by the attraction of the Sun. We shall introduce a few calculations as interpretations of Newton's geometry into analytical language (*Principia*, Book I. Prop. 66. and Book III); but shall reserve for the next Chapter the solution of the problem by systematic approximation.

We shall first explain a principle of great importance in calculating the combined effect of several small perturbing causes, which we shall find important throughout this and the two following Chapters.

PROP. *To explain the principle of the superposition of small motions.*

288. Let xyz be the co-ordinates of a body at the time t when undisturbed by any other body; a a very small numerical quantity which depends upon the disturbing force, of which the square and higher powers may therefore be always neglected.

$x + ax'$, $y + ay'$, $z + az'$ the co-ordinates of the body at time t when disturbed by the body (m') only.

$x + ax''$, $y + ay''$, $z + az''$ the co-ordinates of the body at time t when disturbed by the body (m'') only, and so on.

Now suppose the planets m', m'', ... all to disturb together.

In this case the alterations in x, y, z arising from the several planets will not be the same as before; but they will themselves suffer perturbations, since the action of each planet is now modified by that of all the others.

Thus the value of x will not become

$$x + a(x' + x'' + \ldots):$$

for each of the terms after the first will be modified, but since this modification arises from the disturbing forces, it follows, that the quantities to be added will be multiplied by a^2, a^3, ... and may therefore be neglected and, under this restriction, the co-ordinates of the body subjected to the combined perturbations of all the others will be

$$x + a(x' + x'' + \ldots\ldots\ldots),$$

$$y + a(y' + y'' + \ldots\ldots\ldots),$$

$$z + a(z' + z'' + \ldots\ldots\ldots), \text{ at the time } t.$$

Hence the perturbation in any quantities $\phi(x, y, z) = (u)$ which depends upon the co-ordinates of the planet will

$$= \frac{du}{dx} a (x' + x'' + \ldots\ldots) + \frac{du}{dy} a (y' + y'' + \ldots\ldots)$$

$$+ \frac{du}{dz} a (z' + z'' + \ldots\ldots).$$

$$\text{And this} = \frac{du}{dx} a x' + \frac{du}{dy} a y' + \frac{du}{dz} a z'$$

$$+ \frac{du}{dx} a x'' + \frac{du}{dy} a y'' + \frac{du}{dz} a z''$$

$$+ \ldots\ldots\ldots\ldots$$

But this latter form shews that the perturbation in $\phi(x, y, z)$ is equal to the sum of the separate perturbations of each planet supposing the others not to exist. Hence the Principle we enunciated is true. Its great use in our calculations is this, that it reduces the problem from one of several bodies to that of only three bodies. Hence the famous *Problem of Three Bodies.*

289. At an early period it was observed that the apparent motion of the Sun and Moon round the Earth was not uniform. This had been remarked by the Greek Astronomers. By observing the motion of the shadow of the gnomon they discovered a considerable difference in the intervals of time between the equinoxes and the solstices.

Hipparchus was the first who endeavoured to explain this: he supposed the orbits of the Sun and Moon described about the Earth to be eccentric circles, or circles of which the centres do not coincide with that of the Earth.

290. After a lapse of three centuries Ptolemy discovered that there was an error in the Moon's place in the heavens, which could not be accounted for on the hypothesis of Hipparchus: and he shewed that the magnitude of the error depended upon the position of the line of apsides, or axis major of the lunar orbit. This inequality was called the *Evection* of the Moon: we shall hereafter explain from what cause it arises.

291. The next remarkable inequality of the Moon's motion was discovered by Tycho Brahe in the sixteenth century. This was proved to depend upon the angular distance of the Sun and Moon: and was greatest when the

Moon was about 45^0 or 135^0 from the Sun. In this respect this inequality, which was called the *Variation*, differed from the Evection, which Ptolemy found to be greatest when the Moon was ninety degrees from the Sun.

292. Tycho Brahe was the discoverer of one more inequality, which was called the *Annual Equation*, since it depends on the distance of the Sun from the Earth, and therefore goes through its changes in a year.

Not many years after these discoveries of Tycho Brahe, Kepler published to the world his Three Laws, which he had calculated with almost incredible labour and perseverance. Theory has led to the discovery of many other inequalities in the Moon's motion, but the above have been specified for their historical interest and because they are more sensible than the others.

293. All these, however, were merely bare facts, the results of continued and indefatigable observations and calculations. No common law appeared to connect them, no one cause was known of which they were necessary consequences. It was the glory of Newton, that he unravelled the mystery and demonstrated that these were all results of a universal principle with which matter is endowed by the Creator of the World.

Kepler and other Astronomers had conceived the notion of a universal gravitating principle: but it needed the master genius of Newton to demonstrate its existence.

We now proceed to explain the causes of these perturbations.

PROP. *To calculate the disturbing forces of the Sun on the Moon.*

294. The *disturbing forces* are the differences of the forces of the Sun on the Moon and Earth. Let E, M, S represent the masses of the Earth, Moon, and Sun. The law of force we assume to be that of the inverse square of the distance between the centres of the bodies. We shall consider the orbits of the Sun and Moon about the Earth nearly circular, since this is proved to be the case by observation*.

* The slight variations in the apparent magnitudes of the Sun and Moon convince us of this.

Let r be the distance between M and E (fig. 82),

r' .... S and E,

y .... S and M,

ω the angle SEM,

measured in the direction in which the hands of a watch move.

We must consider the motion of M about E as if E were fixed in order that we may discover the apparent perturbations: but the accelerating forces acting on E are $\frac{M}{r^2}$ and $\frac{S}{r'^2}$ in the directions EM and ES respectively: and in order that the relative motion may not be affected by supposing E fixed we must apply forces equal to these upon each body of the system in an opposite direction: the second law of motion shews the legitimacy of this process.

Hence the forces acting on M, E being considered fixed, are

$\frac{M+E}{r^2}$ in the direction ME,

$\frac{S}{y^2}$ .... MS,

$\frac{S}{r'^2}$ .... MK, MK being parallel to SE.

Then, resolving the second of these in the directions ME and ML (Art. 18), and combining the resolved parts of this with the other forces, the *disturbing* forces of S on M are

$\frac{Sr}{y^3}$ in the direction ME, and $\frac{Sr'}{y^3} - \frac{S}{r'^2}$ in the direction ML.

Again resolve these in the directions ME and perpendicular to this line: then the whole forces which act upon M about E at rest, are

$$\frac{M+E}{r^2} + \frac{Sr}{y^3} - \frac{S}{r'^2}\left(\frac{r'^3}{y^3} - 1\right)\cos\omega \text{ in the direction } ME,$$

$$\frac{S}{r'^2}\left(\frac{r'^3}{y^3} - 1\right)\sin\omega \text{ in the direction perpendicular to } ME,$$

and acting towards the nearest syzygy.

The Moon is said to be in *syzygy* when it is new or full; and in *quadrature* when ninety degrees from syzygy.

The first of the above forces deprived of its first term is called the *central disturbing force*: and the second is called the *tangential disturbing force*.

295. We proceed to obtain approximate expressions for these. Since the orbit of the Moon is nearly circular; and since $\frac{r}{r'}$ is a very small fraction, being about equal to $\frac{1}{400}$, we shall neglect its square and higher powers.

By the figure, $y^2 = r'^2 + r^2 - 2r'r\cos\omega$;

$$\therefore \quad \frac{r'^3}{y^3} = \left(1 - \frac{2r}{r'}\cos\omega\right)^{-\frac{3}{2}} = 1 + \frac{3r}{r'}\cos\omega;$$

$$\therefore \text{ central dist}^g\text{. force} = \frac{Sr}{r'^3} - \frac{3Sr}{r'^3}\cos^2\omega = -\frac{Sr}{2r'^3}(1 + 3\cos 2\omega)$$

$$\text{tangential dist}^g\text{. force} = \frac{3Sr}{r'^3}\sin\omega\cos\omega = \frac{3Sr}{2r'^3}\sin 2\omega.$$

In order still further to simplify these expressions; let f be the mean force of E upon M, E being supposed fixed: m the ratio of the periodic times of the Moon and Sun about the Earth ($m = \frac{1}{13}$ nearly): a and a' the mean distances of the Moon and Sun from the Earth: therefore by Art. 273. Cor. 2,

$$m^2 = \frac{4\pi^2 a^3}{M+E} \div \frac{4\pi^2 a'^3}{S+E} = \frac{S}{M+E}\frac{a^3}{a'^3} \text{ nearly};$$

$$\frac{M+E}{a^2} = f: \text{ and } \therefore \frac{Sr}{r'^3} = m^2 f \frac{a'^3}{r'^3}\frac{r}{a};$$

$$\therefore \text{ central disturbing force} = -\tfrac{1}{2}m^2 f \frac{a'^3}{r'^3}\frac{r}{a}(1 + 3\cos 2\omega),$$

$$\text{tangential disturbing force} = \tfrac{3}{2}m^2 f \frac{a'^3}{r'^3}\frac{r}{a}\sin 2\omega.$$

296. If we suppose the orbit of the Earth about the Sun and the undisturbed orbit of the Moon about the Earth to be circular, then $r' = a'$ and $r = a$; and

$$\text{central disturbing force} = -\tfrac{1}{2} m^2 f (1 + 3\cos 2\omega),$$

$$\text{tangential disturbing force} = \tfrac{3}{2} m^2 f \sin 2\omega.$$

297. The central disturbing force vanishes when

$$\omega = \tfrac{1}{2}\cos^{-1}(-\tfrac{1}{3}) = 55^0,\ 125^0,\ 235^0,\ 305^0 \text{ nearly}.$$

The points in the Moon's orbit determined by these angles are called *Octants*.

The central disturbing force is said to be ***addititious*** at points between those octants between which the quadratures lie, because at those points the above expression for the force is positive and consequently adds to the force of M to E. For a like reason the central disturbing force is said to be ***ablatitious*** between those octants between which the syzygies lie.

298. We proceed to examine the effects which the disturbing forces have upon the form and position of the Moon's orbit. We shall neglect quantities which depend upon the square and higher powers of the disturbing force. Whenever the undisturbed orbit is supposed to be nearly circular we may neglect all such terms as $\dfrac{dr^2}{d\theta^2}$. Thus

$$\text{radius of curvature} = \frac{\left\{r^2 + \dfrac{dr^2}{d\theta^2}\right\}^{\frac{3}{2}}}{r^2 + 2\dfrac{dr^2}{d\theta^2} - r\dfrac{d^2 r}{d\theta^2}} = \frac{r^2}{r - \dfrac{d^2 r}{d\theta^2}}.$$

$$\text{Also (velocity)}^2 = \left(\frac{ds}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2 = r^2\left(\frac{d\theta}{dt}\right)^2.$$

$$\text{Again central force} = h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2}\right) \text{ by Art. 246.}$$

$$= \frac{h^2}{r^2}\left\{\frac{1}{r} + \frac{2}{r^3}\frac{dr^2}{d\theta^2} - \frac{1}{r^2}\frac{d^2 r}{d\theta^2}\right\} = \frac{h^2}{r^2}\left\{\frac{1}{r} - \frac{1}{r^2}\frac{d^2 r}{d\theta^2}\right\}$$

$$= r^2 \left(\frac{d\theta}{dt}\right)^2 \left\{\frac{1}{r} - \frac{1}{r^2}\frac{d^2 r}{d\theta^2}\right\} \text{ by Art. 242.} = \frac{(\text{vel.})^2}{\text{rad. of curv.}}.$$

We have introduced these expressions since we shall find them useful hereafter.

PROP. *To find the effect of the Sun's disturbing forces on the periodic time of the Moon.*

299. Since the tangential force passes through all its degrees of magnitude positive and negative during half a revolution of the Moon, it will compensate during one quarter of the revolution for any loss or gain that the angular velocity of the Moon may have experienced in the preceding quarter. In fact the *mean** tangential force equals zero. For this reason the tangential force has no effect on the periodic time.

For the same reason we neglect the periodic term of the central force. The *mean* central force $= f(1 - \frac{1}{2}m^2)$. Since this is less than f, it follows that the mean distance is increased by the disturbing forces.

The absolute force $= f a^2 (1 - \frac{1}{2}m^2)$;

$\therefore$ the periodic time $= \dfrac{2\pi a^{\frac{3}{2}}}{\sqrt{\text{abs. force}}}$ nearly, (Art. 273. Cor. 2.)

$$= 2\pi\sqrt{\frac{a}{f}}\,(1 + \tfrac{1}{4}m^2) \text{ nearly.}$$

This is greater than if there were no disturbing force (or $m = 0$); especially when we remember that a is greater than in the undisturbed orbit.

Hence the periodic time is increased by the disturbing force. (*Principia*, Lib. I. Prop. LXVI. Cor. 6.)

* By the *mean* value of a function of a variable angle θ we mean, the part of the function which is independent of periodical terms which pass through all their changes positive and negative as θ is increased by certain equal increments. Thus a is the mean value of $a + b\sin n\theta$.

300. COR. 1. If we suppose the Earth's orbit about the Sun not circular, then by Art. 295, the mean central force

$$= f\left\{1 - \frac{m^2}{2}\frac{a'^3}{r'^3}\right\},$$

and the length of the month

$$= 2\pi\sqrt{\frac{a}{f}}\left\{1 + \frac{m^2}{4}\frac{a'^3}{r'^3}\right\}.$$

Hence the months are longest when the Earth is in perihelion, and shortest when in aphelion. This accounts for the Winter Months at this epoch being longer than the Summer Months.

301. COR. 2. The mean velocity (V) of the Moon =

$$\frac{2\pi a}{\text{per. time}} = a\sqrt{\frac{f}{a}}(1 - \tfrac{1}{4}m^2) = \sqrt{af}\,(1 - \tfrac{1}{4}m^2).$$

PROP. *To find the effect of the Sun's disturbing force on the velocity of the Moon, supposing the undisturbed orbit of the Moon to be circular.*

302. Since the angle ω is referred to a line moveable in space we must adopt another means of measuring the position of the Moon. Let θ be the longitude of the Moon at the time t: then $m\theta$ is the longitude of the Sun, supposing that θ is measured from the time when the Sun and Moon were in Aries together; and supposing the orbits nearly circular: and therefore $\omega = (1-m)\theta$:

$$\therefore \text{ tangential disturbing force } = \tfrac{3}{2}m^2 f \sin 2(1-m)\theta,$$

and this is the only disturbing force which *directly* affects the velocity. We shall see in Art. 305, that the velocity is affected *indirectly* by the central disturbing force.

Now the space described by the Moon in the time t is $a\theta$; and the tangential force is the only force which acts in the line of the Moon's motion:

$$\therefore\ a\frac{d^2\theta}{dt^2} = -\tfrac{3}{2}m^2 f \sin 2(1-m)\theta,$$

the *negative* sign being taken because the tangential force acts always towards the nearest syzygy, Art. 294, and consequently tends to diminish θ when the Moon is in the first and third quadrants, through which angles $\sin 2(1-m)\theta$ is positive, and to increase θ in the second and fourth quadrants, through which angles $\sin 2(1-m)\theta$ is negative.

Multiplying by $2a\dfrac{d\theta}{dt}$ and integrating

$$v^2 \text{ or } a^2\frac{d\theta^2}{dt^2} = \text{const.} + \frac{3m^2}{2(1-m)} fa \cos 2(1-m)\theta.$$

Let V be the mean velocity; then

$$v^2 = V^2 + \frac{3m^2}{2(1-m)} fa\cos 2(1-m)\theta$$

$$= V^2\left\{1 + \frac{3m^2}{2(1-m)}\cos 2(1-m)\theta\right\} \text{ neg}^g.\ m^4,$$

since $V^2 = fa(1-\frac{1}{2}m^2)$ by Art. 301.

The effect, then, of the disturbing force is to increase the velocity above what it would be in the circular orbit, when the Moon is not more than 45° from syzygy; and in other positions to diminish it. This follows directly from the fact proved in Art. 294, that the tangential force always acts towards the nearest syzygy.

The velocity is greatest in syzygies and $= V\left\{1 + \dfrac{3m^2}{4(1-m)}\right\}$

.... least in quadratures and $= V\left\{1 - \dfrac{3m^2}{4(1-m)}\right\}$.

(*Principia*, Lib. I. Prop. LXVI. Cors. 2, 3.)

PROP. *To find the effect of the Sun's disturbing force on the form of the Moon's orbit; supposing the undisturbed orbit to be circular.*

303. The curvature at any point of the orbit is measured by the reciprocal of the radius of curvature: hence, by Art. 298, the curvature equals $\dfrac{\text{central force}}{(\text{velocity})^2}$.

Now the central force $= f - \frac{1}{2} m^2 f(1 + 3 \cos 2\omega)$, Art. 296,

$$\text{also (velocity)}^2 = V^2 \left\{1 + \frac{3m^2}{2(1-m)} \cos 2\omega\right\}, \text{ Art } 302.$$

$$V = \text{mean vel.} = \sqrt{af}\,(1 - \tfrac{1}{4} m^2) \text{ by Art. } 301\,;$$

$$\therefore \text{ curvature} = \frac{f}{V^2}\left\{1 - \tfrac{1}{2} m^2 - \tfrac{3}{2} m^2 \left(1 + \frac{1}{1-m}\right) \cos 2\omega\right\}$$

$$= \frac{1}{a}\left\{1 - \tfrac{3}{2} m^2 \left(1 + \frac{1}{1-m}\right) \cos 2\omega\right\}.$$

This is greatest in quadrature, when $\omega = 90^0$ and 270^0; and is least in syzygy, when $\omega = 0$ and 180^0.

This shews that the orbit will assume an oval form *with respect to the Earth*, having its minor axis in syzygy, (*Principia*, Lib. I. Prop. LXVI. Cor. 4). Its form in space will be an irregular curve, nearly circular, but not *re-entering*. Also the expression for the curvature shews that the equation to the orbit is $r = a\,(1 - x \cos 2\omega)$, the major and minor axes being $2a\,(1 + x)$ and $2a\,(1 - x)$.

PROP. *To find the ratio of the axes of the oval orbit.*

304. The equation to the orbit is $r = a\,(1 - x \cos 2\omega)$; but since ω is measured from a moveable line we must put $\omega = (1 - m)\,\theta$ as in Art. 302;

$$\therefore\ r = a\,\{1 - x \cos 2\,(1 - m)\,\theta\}\,;$$

$$\therefore \text{ curvature} = \frac{1}{r} - \frac{1}{r^2}\frac{d^2 r}{d\theta^2} \text{ by Art. } 298.$$

$$= \frac{1}{a}\left\{1 + x\left[1 - 4(1-m)^2\right] \cos 2\,(1 - m)\,\theta\right\}.$$

Equating this to the expression found in the last Article,

$$\frac{1}{a}\left\{1 + x\left[1 - 4(1-m)^2\right] \cos 2\,(1 - m)\,\theta\right\}$$

$$= \frac{1}{a}\left\{1 - \frac{3m^2}{2}\left(1 + \frac{1}{1-m}\right) \cos 2\,(1 - m)\,\theta\right\};$$

$$\therefore\ x = \frac{3m^2}{2}\,\frac{1 + \dfrac{1}{1-m}}{4(1-m)^2 - 1}.$$

If we put $m = \dfrac{1}{13.3}$, $x = \dfrac{1}{139}$ nearly;

$$\therefore\ \frac{1+x}{1-x} = \frac{70}{69},$$

the ratio in the *Principia*, Lib. III. Prop. XXVIII.

This is the ratio of the axes of the oval orbit which moves round with the Sun while the Moon moves in it.

In Art. 302, we found the effect of the tangential disturbing force on the velocity of the Moon; and we have just shewn that the tangential and central disturbing forces draw the orbit of the Moon into an oval figure with respect to the Earth. We proceed, then, to calculate the velocity of the Moon when both disturbing forces are considered.

PROP. *To find the velocity of the Moon in the oval orbit: and the Variation.*

305. In Art. 302, we found the effect of the tangential force on the velocity: but the velocity will be affected by the change in form of the orbit: and thus we see the indirect effect of the central disturbing force upon the velocity.

Let v be the velocity of the Moon at any distance,

v_1 .... the mean distance,

then $v^2 = r^2\dfrac{d\theta^2}{dt^2}$, neglecting $\dfrac{dr^2}{dt^2}$ which depends on the square of the disturbing force: Art. 298.

$$v^2 = \frac{1}{r^2}\left(r^2\frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^2};\quad v_1^2 = \frac{h^2}{a^2};$$

$$\therefore\ v^2 = \frac{a^2}{r^2}v_1^2.$$

By substituting for $\frac{a^2}{r^2}$ we shall correct v^2 for the oval form of the orbit: and by substituting for v_1^2 we shall correct for the change in velocity in the circular orbit: and in this way the complete velocity in the oval orbit is found.

$$\frac{a^2}{r^2} = 1 + 2x \cos 2(1-m)\theta,$$

$$v_1^2 = V^2 \left\{1 + \frac{3m^2}{2(1-m)} \cos 2(1-m)\theta\right\}. \quad \text{See Art. 302;}$$

$$\therefore v^2 = V^2 \left\{1 + \left(2x + \frac{3m^2}{2(1-m)}\right) \cos 2(1-m)\theta\right\}.$$

306. Let $\delta\theta$ be the error in longitude owing to this change in the velocity: then

$$\frac{d\theta}{dt} = \frac{V}{a}, \quad \frac{d(\theta + \delta\theta)}{dt} = \frac{v}{r} \text{ nearly;}$$

$$\therefore \frac{d.\delta\theta}{d\theta} = \frac{v}{V}\frac{a}{r} - 1$$

$$= \left\{1 + \left(x + \frac{3m^2}{4(1-m)}\right) \cos 2(1-m)\theta\right\} \left\{1 + x\cos 2(1-m)\theta\right\} - 1$$

$$= \left(2x + \frac{3m^2}{4(1-m)}\right) \cos 2(1-m)\theta;$$

$$\therefore \delta\theta = \left(\frac{x}{1-m} + \frac{3m^2}{8(1-m)^2}\right) \sin 2(1-m)\theta.$$

This error in longitude is greatest (disregarding its sign) when the Moon is 45° or 135° from the Sun on either side of syzygy: and therefore explains the cause of the error in the Moon's place discovered by Tycho Brahe, and called the ***Variation*** (Art. 291).

If we put $x = \frac{1}{139}$ and $m^2 = \frac{1}{179}$.

Variation $= m^2 (1 + \frac{3}{8}) \sin 2(1-m)\theta$ nearly, $= \frac{11}{8} m^2 \sin 2(1-m)\theta$, which accords with the rigorous approximation of the second order in the next Chapter, Art. 337.

307. For a general and popular explanation of the nature of the lunar perturbations we refer the reader to Airy's *Gravitation*.

308. We shall now proceed to explain the effect of the disturbing forces on the inclination and the motion of the line of nodes.

Hitherto we have supposed the Moon to move in the plane of the ecliptic: let us now suppose that the planes of the ecliptic and the Moon's orbit are slightly inclined to each other, as is the case in nature. Let the disturbing force of the Sun on the Moon be resolved into two parts, one in the plane of the Moon's orbit, and the other perpendicular to this plane: this latter is the part which affects the inclination and the position of the line of nodes.

If we bear in mind that the Sun's disturbing force always acts *towards* the Sun when the Moon is nearer the Sun than the Earth, and *from* the Sun in the contrary case, we shall easily see, by referring to fig. 88, that the part of the disturbing force which is perpendicular to the plane of the Moon's orbit always acts *towards* the ecliptic, except when the Moon is between quadrature and the nearest node, in which case it acts *from* the ecliptic.

By the plane of the Moon's orbit we mean the plane drawn through the centres of the Moon and Earth, and through the direction of the Moon's motion at any instant. And since in consequence of the Sun's disturbing force the Moon is continually drawn out of the plane in which it is moving, the plane of the orbit is continually shifting its position by revolving about the Moon's radius-vector as an instantaneous axis.

309. Before we begin to explain the effect on the inclination and line of nodes we shall enunciate the following Lemma, the truth of which is self-evident.

LEMMA. When a body is moving from or towards a plane and a force acts upon it in a direction from or towards the plane, then the inclination of the body's resulting motion will be increased or diminished according as the original motion and the force act in the same or opposite directions with respect to the plane.

PROP. *To explain the effect of the Sun's disturbing force upon the inclination of the Moon's orbit to the ecliptic and on the position of the line of nodes.*

310. I. *Suppose the line of nodes is in syzygies.*

It is clear that in this case the inclination and line of nodes will not be affected; since no part of the force acts perpendicularly to the plane of the orbit. The line of nodes would remain in this position were it not for the Sun's motion.

311. II. *Suppose the line of nodes is in advance of the Sun:* fig. 89.

Let Nn be the line of nodes: take $Nm = 90^0$ on the orbit: let Qq be the quadratures. Then as the Moon moves from N to Q she moves from the ecliptic, and the disturbing force acts from the ecliptic. Hence the inclination of the Moon's path (Art. 309), and therefore of the plane of her orbit, is increasing, and therefore in revolving about the radius-vector EM the node N must move towards quadratures, or the line of nodes Nn must progress.

As the Moon moves from Q to m, her motion and the disturbing force act in opposite directions, and therefore the inclination is decreasing (Art. 309), and the line of nodes regressing.

As the Moon moves from m to n her motion and the disturbing force both tend towards the ecliptic and therefore the inclination of the plane of her orbit is increasing, and therefore in revolving about the radius-vector EM' causes the point n to move back, or the line of nodes to regress.

In the other half of the orbit the effect will be exactly the same.

Hence, if ϕ be the angular distance of the line of nodes from syzygies (ϕ being less than 90^0), the inclination is increasing as the Moon is moving through an angle

$$= 2\,(NEQ + mEn) = 360^0 - 2\,\phi:$$

and is decreasing as the Moon is moving through the remaining angle $2\,\phi$. And the line of nodes regresses while the Moon is

moving through an angle $180^0 + 2\phi$; and progresses while the Moon is moving through the remaining angle $180^0 - 2\phi$.

312. III. *Suppose the line of nodes is in quadratures.*

Then as the Moon moves from quadrature to syzygy the disturbing force and motion tend in different directions, and therefore the inclination is decreasing and the line of nodes is regressing. And as the Moon moves from syzygy to quadrature the inclination is increasing and therefore the line of nodes still regresses.

Wherefore the increase and decrease of inclination counteract each other; but the motion of the nodes is wholly *regressive*.

313. IV. *Suppose the line of nodes is behind the Sun:* fig. 90.

Then as the Moon moves from N to m the inclination is decreasing and the line of nodes regressing: and as she moves from m to q the inclination is increasing and the nodes regressing. As the Moon moves from q to n the inclination is decreasing and the nodes progressing.

Hence, if as before ϕ be the angular distance of the line of nodes from syzygy, the inclination is increasing as the Moon moves through an angle 2ϕ, and decreasing as she moves through the angle $360^0 - 2\phi$. But the nodes regress and progress respectively while the Moon is moving through the angles $180^0 + 2\phi$ and $180^0 - 2\phi$.

It appears, then, that on the whole the nodes *regress* pretty steadily: but the inclination is much more fluctuating and on the whole is not affected during a revolution of the line of nodes.

We introduce the two following Propositions as examples of the method used by Newton in the Third Book of the Principia: they will be found in Props. 30 and 31. Newton's geometry is translated into analysis.

Prop. *To calculate the motion of the nodes of the Moon's orbit considered circular.*

314. Let MM' be the arc described by the Moon in a unit of time, fig. 91: $M'L = 2$ space through which the dis-

turbing force would draw the Moon in the same time: Nn the line of nodes, Qq the line of quadratures, AB of syzygies: Mm is a tangent to the Moon's orbit at M meeting the ecliptic in m: join LM and produce it to meet the ecliptic in l: this gives the position of the tangent at M after the small time of describing MM' and therefore $\angle mEl$ represents the motion of the node.

Now LM' is parallel to the ecliptic and therefore can meet no line in the ecliptic: but it is in the same plane with lm, therefore LM' is parallel to lm.

$$\text{Hence } \frac{\text{motion of Node}}{\text{motion of Moon}} = \frac{\angle lEm}{\angle MEM'} = \frac{\sin mlE}{\angle MEM'}\frac{lm}{Em} \text{ nearly}$$

$$= \frac{\sin AEn}{\angle MEM'}\frac{LM'}{MM'}\frac{Mm}{Em} = \frac{\sin AEn}{\angle MEM'}\frac{LM'}{MM'}\sin MEm.$$

Now the disturbing force in the direction $M'L$

$$= \frac{Sr'}{y^3} - \frac{S}{r'^2} \text{ (see Art. 294.)} = \frac{S}{r'^2}\left(\frac{r'^3}{y^3} - 1\right)$$

$$= \frac{3Sr}{r'^3}\cos MEA, \because y = r' - r\cos MEA;$$

$$\therefore LM' = \frac{3Sr}{r'^3}\cos MEA.$$

$$MM' \,.\, \angle MEM' = \frac{MM'^2}{r} = \frac{(\text{vel.})^2}{\text{rad.}} = \text{force of Moon to } E$$

$$= \frac{E+M}{r^2};$$

$$\therefore \frac{LM'}{MM' \,.\, \angle MEM'} = \frac{3S}{M+E}\frac{r^3}{r'^3}\cos MEA = 3m^2 \cos MEA,$$

Art. 295;

$\therefore$ motion of Node

$$= 3m^2 \cos MEA \sin MEN \sin AEn \,.\, \text{motion of Moon.}$$

Let N = longitude of the Node,

θ = ... Moon,

$m\theta$ = ... Sun,

supposing that the Sun, Moon, and Node were all in the first point of Aries when $\theta = 0$.

Hence the above equation gives

$$\frac{dN}{d\theta} = -3m^2 \cos(\theta - m\theta) \sin(\theta - N) \sin(m\theta - N),$$

the negative sign being taken because the mean motion of the node is regressive.

315. We shall now transform this by the ordinary trigonometrical formula

$$2 \sin a \cos b = \sin(a + b) + \sin(a - b),$$

we have

$$\frac{dN}{d\theta} = -\tfrac{3}{4}m^2\{1 + \cos 2(\theta - m\theta) - \cos 2(\theta - N) - \cos 2(m\theta - N)\}.$$

For a first approximation we neglect the periodical terms and take the mean values:

$$\frac{dN}{d\theta} = -\tfrac{3}{4}m^2 = -i \text{ suppose};$$

$$\therefore N = -i\theta, \text{ constant} = 0.$$

For a second approximation we shall put this value of N in the periodical terms;

$$\therefore \frac{dN}{d\theta} = -i\{1 + \cos 2(1 - m)\theta - \cos 2(1 + i)\theta - \cos 2(m + i)\theta\};$$

$$\therefore N = -i\theta - \frac{i}{2(1 - m)} \sin 2(1 - m)\theta + \frac{i}{2(1 + i)} \sin 2(1 + i)\theta$$

$$+ \frac{i}{2(m + i)} \sin 2(m + i)\theta.$$

For a third approximation we shall put this value of N in $\frac{dN}{d\theta}$ after neglecting the terms divided by $1 - m$ and $1 + i$, because they are smaller than the term divided by $m + i$; then

$$N = -i\theta + \frac{i}{2(m + i)} \sin 2(m + i)\theta;$$

$$\therefore \frac{dN}{d\theta} = -i - i\cos 2(1-m)\theta + i\cos\left\{2(1+i)\theta - \frac{i}{m+i}\sin 2(m+i)\theta\right\}$$

$$+ i\cos\left\{2(m+i)\theta - \frac{i}{m+i}\sin 2(m+i)\theta\right\}.$$

If we expand these the last term gives

$$\frac{i^2}{m+i}\sin^2 2(m+i)\theta \text{ or } \frac{i^2}{2(m+i)} - \frac{i^2}{2(m+i)}\cos 4(m+i)\theta.$$

Hence we obtain

$$\frac{dN}{d\theta} = -i + \frac{i^2}{2(m+i)} + \text{periodic functions of } \theta,$$

and therefore the *mean* value of N is

$$N = -\left\{i - \frac{i^2}{2(m+i)}\right\}\theta = -\left\{1 - \frac{3m}{8(1+\frac{3}{4}m)}\right\}\frac{3m^2}{4}\theta$$

$$= -\left(\tfrac{3}{4}m^2 - \tfrac{9}{32}m^3 + \tfrac{27}{128}m^4 + \ldots\ldots\right)\theta.$$

316. In this calculation we have supposed the Moon's angular velocity to be uniform. To correct for the oval orbit let N_1 be the corrected value of N.

Now the motion of the node varies as the magnitude of the disturbing force, which varies as the square of the time of the Moon's describing MM', and therefore as the square of the velocity at M inversely;

$$\therefore \frac{dN_1}{dN} = \frac{(\text{vel.})^2 \text{ in octants}}{(\text{vel.})^2 \text{ at } M} = 1 - \frac{3m^2}{2(1-m)}\cos 2(1-m)\theta;$$

$$\therefore \frac{dN_1}{d\theta} = -i\left\{1 - \frac{i}{2(m+i)} + \cos 2(1-m)\theta \ldots\right\}$$

$$\times\left\{1 - \frac{2i}{1-m}\cos 2(1-m)\theta + \ldots\right\}$$

$$= -i\left\{1 - \frac{i}{2(m+i)} - \frac{2i}{1-m}\cos^2 2(1-m)\theta + \ldots\right\}$$

$$= -i\left\{1 - \frac{i}{2(m+i)} - \frac{i}{1-m}\right\} + \text{periodical terms,}$$

and the mean value of N_1 is

$$N_1 = -i\left(1 - \frac{i}{2(m+i)} - \frac{i}{1-m}\right)\theta$$

$$= -\frac{3m^2}{4}\left\{1 - \frac{3m}{8(1+\frac{3}{4}m)} - \frac{3m^2}{4(1-m)}\right\}\theta$$

$$= -\left(\tfrac{3}{4}m^2 - \tfrac{9}{32}m^3 - \tfrac{45}{128}m^4 + \ldots\right)\theta,$$

this correction does not affect the first and second terms.

PROP. *To calculate the inclination of the Moon's orbit to the ecliptic at any time.*

317. Let ENm be the line of nodes (fig. 92), El its position after a unit of time: Mp perpendicular to the plane of the ecliptic, pG perpendicular to the line of nodes EN; produce pG to cut El in g; join MG, Mg, and draw Gr perpendicular to Mg: I the inclination of the plane of the Moon's orbit to the ecliptic.

$$\text{Now } \delta I = \angle MGp - \angle Mgp = \angle GMg = \frac{Gr}{GM}.$$

$$\text{Also } \delta N = \angle GEg = \frac{Gg}{GE};$$

$$\therefore \delta I = \frac{Gr}{Gg}\frac{GE}{GM}\delta N = \sin I \,.\, \cot MEN \,.\, \delta N;$$

$$\therefore \frac{dI}{d\theta} = 3m^2 \sin I \cos MEA \cos MEN \sin AEn$$

$$= -3m^2 \sin I \cos(\theta - m\theta)\cos(\theta - N)\sin(m\theta - N)$$

$$= -\tfrac{3}{4}m^2 \sin I\{\sin 2(\theta - N) - \sin 2(\theta - m\theta) + \sin 2(m\theta - N)\}.$$

Now I is always small: and therefore $\sin I = I$ nearly; let γ be the mean value of I: also

$$N = -\tfrac{3}{4}m^2\theta = -i\theta, \text{ (see Art. 315.)}$$

$$\therefore \frac{dI}{d\theta} = -\tfrac{3}{4}m^2\gamma\{\sin 2(1+i)\theta - \sin 2(1-m)\theta + \sin 2(m+i)\theta\};$$

$$\therefore I = \tfrac{3}{4} m^2 \gamma \left\{ \frac{1}{2(1+i)} \cos 2(1+i)\theta - \frac{1}{2(1-m)} \cos 2(1-m)\theta \right.$$

$$\left. + \frac{1}{2(m+i)} \cos 2(m+i)\theta \right\} + \text{const.}$$

The constant $= \gamma$, the mean value of I. Therefore, neglecting the first and second terms because they are of an order higher than the third,

$$I = \gamma \left\{ 1 + \frac{3m}{8(1+\frac{3}{4}m)} \cos 2(m+i)\theta \right\}$$

$$= \gamma \left\{ 1 + \tfrac{3}{8} m \cos 2 (\text{Sun's longitude} - \text{Node's longitude}) \right\},$$

neglecting γm^2, &c.

This accords with Chapter V. Art. 338.

CHAPTER V.

LUNAR THEORY.

318. We now enter upon the calculation of the perturbations of the Moon by a process of systematic approximation; and shall proceed in the next Chapter to calculate those of the planets. In the Lunar and Planetary Theories we use different methods of calculation for this reason. The perturbations of the Moon are much greater than those of any planet, because the Sun, the mass of which is so enormous (Art. 286), is one of the disturbing bodies. Likewise the ratio of the distances of the disturbed and disturbing bodies from the central body about which they move is very different in the two theories; being about $\frac{1}{400}$ in that of the Moon, and sometimes so large as $\frac{3}{4}$ in that of the planets. The difference of the methods of approximation will be seen in the calculations of this and the following Chapter.

Before entering upon the immediate subject of the present Chapter we must investigate the following Proposition.

Prop. *A number of bodies considered as material particles attract each other with forces which vary inversely as the square of their distances, and directly as the mass of the attracting body: required the equations of motion of any one of the bodies relatively to a second.*

319. Let M, m, m', m'' .... be the masses of the bodies; M being that of the body about which the motion is to be calculated: and m the mass of that body of which the equations of motion are to be determined.

Let X, Y, Z be the co-ordinates of M,

$X+x$, $Y+y$, $Z+z$ .... m,

$X+x'$, $Y+y'$, $Z+z'$ .... m'

....

Then the distance between m and m' is

$$\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}},$$

and the attraction of m' on m is

$$\frac{m'}{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

Let this be resolved into three parts parallel to the axes: that parallel to the axis of x is

$$\frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}},$$

$$\text{or } \frac{1}{m}\frac{d}{dx}\left\{\frac{mm'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}}\right\},$$

and so of the other bodies m''.........

$$\text{Now assume } \lambda = \Sigma . \frac{mm'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}},$$

which expression is the sum of the quantities found by dividing the product of every two of the masses m, m', m'' ...... by their respective distances.

Then the sum of the attractions of m', m'' ...... on m parallel to the axis of $x = \frac{1}{m}\frac{d\lambda}{dx}$. And $\frac{1}{m}\frac{d\lambda}{dy}$, $\frac{1}{m}\frac{d\lambda}{dz}$ are, in like manner, the attractions of m', m''.... on m parallel to the axes of y and z.

Let $r\,r'\,r''$ .... be the distances of $m\,m'\,m''$ .... from M. Then the attraction of M on m parallel to x is $\frac{Mx}{r^3}$, and con-

sequently the equation of motion of m in space parallel to x is

$$\frac{d^2(X+x)}{dt^2} = \frac{1}{m}\frac{d\lambda}{dx} - \frac{Mx}{r^3}.$$

But $\frac{mx}{r^3}$, $\frac{m'x'}{r'^3}$, ... are the attractions of m, m'... on M parallel to x: and therefore the equation of motion of M parallel to x is

$$\frac{d^2X}{dt^2} = \Sigma . \frac{mx}{r^3},$$

and by subtracting this from the equation above we have the equation of motion of m relatively to M

$$\frac{d^2x}{dt^2} + \frac{Mx}{r^3} + \Sigma . \frac{mx}{r^3} - \frac{1}{m}\frac{d\lambda}{dx} = 0.$$

And in like manner

$$\frac{d^2y}{dt^2} + \frac{My}{r^3} + \Sigma . \frac{my}{r^3} - \frac{1}{m}\frac{d\lambda}{dy} = 0,$$

$$\frac{d^2z}{dt^2} + \frac{Mz}{r^3} + \Sigma . \frac{mz}{r^3} - \frac{1}{m}\frac{d\lambda}{dz} = 0.$$

Now assume

$$R^* = \frac{m'(xx'+yy'+zz')}{r'^3} + \frac{m''(xx''+yy''+zz'')}{r''^3} + \ldots - \frac{\lambda}{m},$$

* The function R satisfies Laplace's Equation (Art. 168).

$$\text{For } \frac{dR}{dx} = \frac{m'x}{r'^3} + \ldots\ldots - \frac{1}{m}\frac{d\lambda}{dx}$$

$$= \frac{m'x'}{r'^3} + \ldots\ldots - \frac{1}{m}\Sigma . \frac{mm'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}};$$

$$\therefore \frac{d^2R}{dx^2} = \frac{1}{m}\Sigma . \frac{mm'\{(y'-y)^2+(z'-z)^2-2(x'-x)^2\}}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{5}{2}}},$$

$$\text{and so } \frac{d^2R}{dy^2} = \frac{1}{m}\Sigma . \frac{mm'\{(x'-x)^2+(z'-z)^2-2(y'-y)^2\}}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{5}{2}}},$$

$$\frac{d^2R}{dz^2} = \frac{1}{m}\Sigma . \frac{mm'\{(x'-x)^2+(y'-y)^2-2(z'-z)^2\}}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{5}{2}}};$$

$$\therefore \frac{d^2R}{dx^2} + \frac{d^2R}{dy^2} + \frac{d^2R}{dz^2} = 0.$$

$$\text{then } \frac{dR}{dx} = \frac{m'x'}{r'^3} + \frac{m''x''}{r''^3} + \ldots - \frac{1}{m}\frac{d\lambda}{dx}$$

$$= \Sigma . \frac{mx}{r^3} - \frac{mx}{r^3} - \frac{1}{m}\frac{d\lambda}{dx},$$

and the first equation of motion becomes

$$\left.\begin{aligned} &\frac{d^2x}{dt^2} + \frac{(M+m)\,x}{r^3} + \frac{dR}{dx} = 0 \\ &\text{and similarly} \\ &\frac{d^2y}{dt^2} + \frac{(M+m)\,y}{r^3} + \frac{dR}{dy} = 0 \\ &\frac{d^2z}{dt^2} + \frac{(M+m)\,z}{r^3} + \frac{dR}{dz} = 0 \end{aligned}\right\} \ldots\ldots (1).$$

These are the equations by which the motion of the Moon about the Earth, or of a planet about the Sun, is determined when under the action of all the other bodies of the Solar System. They have never yet been completely integrated. For this reason we must resort to approximation. To effect this R, which is called the *disturbing function*, must be developed in a converging series. The difference of the methods adopted in the Lunar and Planetary Theories depends upon the different modes of expanding the function R. In the Lunar Theory R is expanded in powers of the ratio of the distances of the Sun and Moon, a very small fraction nearly equal $\frac{1}{400}$; but in the Planetary Theory R is expanded in powers of the eccentricities and inclinations of the planetary orbits, all of which are very small, with the exception of those of Juno and Pallas; the eccentricities of these being about $\frac{1}{4}$ and the inclination of the orbit of Pallas to the ecliptic being about 35^0: but the masses of these planets are very small.

320. We intend throughout our calculations in this and the following Chapter to neglect quantities which depend upon the square and higher powers of the disturbing forces. In consequence of this we may calculate separately the perturbations caused by the Sun or a planet on the supposition, that the

other heavenly bodies do not attract, and then add together the separate perturbations: this follows from the Principle explained in Art. 288.

PROP. *To obtain equations for calculating the radius-vector of the Moon; and the inclination of the lunar orbit to the ecliptic.*

321. The equations of motion are, by the last Article,

$$\left.\begin{array}{r} \dfrac{d^2x}{dt^2}+\dfrac{\mu x}{r^3}+\dfrac{dR}{dx}=0, \quad \dfrac{d^2y}{dt^2}+\dfrac{\mu y}{r^3}+\dfrac{dR}{dy}=0, \\ \text{and} \quad \dfrac{d^2z}{dt^2}+\dfrac{\mu z}{r^3}+\dfrac{dR}{dz}=0, \end{array}\right\}\ldots(1)$$

where μ = mass of the Earth + mass of the Moon.

Let the plane of the ecliptic be the plane of xy: also let ρ be the projection of r on the ecliptic: s the tangent of inclination of r to the same plane: θ the longitude, the axis of x passing through Aries: then

$$x^2+y^2=\rho^2; \quad z^2+\rho^2=r^2;$$

$$x=\rho\cos\theta, \quad y=\rho\sin\theta, \quad z=\rho s.$$

Multiply the first equation by y, and the second by x, and subtract;

$$\therefore\ x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=y\frac{dR}{dx}-x\frac{dR}{dy};$$

$$\therefore\ \frac{d}{dt}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)=\rho\left\{\sin\theta\frac{dR}{dx}-\cos\theta\frac{dR}{dy}\right\},$$

$$\text{or } \frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right)=\rho T,$$

$$\text{if we put } \sin\theta\frac{dR}{dx}-\cos\theta\frac{dR}{dy}=T.$$

Multiply each side by $\rho^2 \frac{d\theta}{dt}$;

$$\therefore \rho^2 \frac{d\theta}{dt} \frac{d}{dt} \left\{\rho^2 \frac{d\theta}{dt}\right\} = \rho^3 T \frac{d\theta}{dt};$$

$$\therefore \left(\rho^2 \frac{d\theta}{dt}\right)^2 = h^2 + 2 \int \rho^3 T d\theta;$$

h^2 being a constant introduced by integration ;

$$\therefore \frac{d\theta^2}{dt^2} = \frac{h^2}{\rho^4} + \frac{2}{\rho^4} \int \rho^3 T d\theta; \quad \rho = \frac{1}{u}$$

$$= h^2 u^4 + 2u^4 \int \frac{T d\theta}{u^3} \ldots\ldots\ldots (2).$$

Again, multiply the first and second of equations (1) respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, and add,

$$\therefore \frac{d}{dt}\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\} + \frac{2\mu}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) + 2\frac{dx}{dt}\frac{dR}{dx} + 2\frac{dy}{dt}\frac{dR}{dy} = 0,$$

putting $x = \rho \cos\theta$, $y = \rho \sin\theta$, $x^2 + y^2 = \rho^2$;

$$\frac{d}{dt}\left\{\frac{d\rho^2}{dt^2} + \rho^2 \frac{d\theta^2}{dt^2}\right\} + \frac{2\mu\rho}{r^3}\frac{d\rho}{dt} - 2\rho\frac{d\theta}{dt}\left(\sin\theta \frac{dR}{dx} - \cos\theta \frac{dR}{dy}\right)$$

$$+ 2\frac{d\rho}{dt}\left(\cos\theta \frac{dR}{dx} + \sin\theta \frac{dR}{dy}\right) = 0.$$

$$\text{Let } \frac{\mu\rho}{r^3} + \cos\theta \frac{dR}{dx} + \sin\theta \frac{dR}{dy} = P;$$

$$\therefore \frac{d}{dt}\left\{\frac{d\rho^2}{dt^2} + \rho^2 \frac{d\theta^2}{dt^2}\right\} + 2P\frac{d\rho}{dt} - 2\rho\frac{d\theta}{dt} T = 0;$$

$$\therefore \frac{d}{d\theta}\left\{\frac{d\rho^2}{dt^2} + \rho^2 \frac{d\theta^2}{dt^2}\right\} + 2P\frac{d\rho}{d\theta} - 2\rho T = 0, \quad \rho = \frac{1}{u};$$

$$\therefore \quad \frac{d}{d\theta}\left\{\frac{1}{u^4}\frac{du^2}{dt^2}+\frac{1}{u^2}\frac{d\theta^2}{dt^2}\right\}-\frac{2P}{u^2}\frac{du}{d\theta}-\frac{2T}{u}=0.$$

Now $\dfrac{d\theta^2}{dt^2}=h^2u^4+2u^4\displaystyle\int\frac{Td\theta}{u^3}$ by (2),

by this equation we can eliminate t, and we have

$$\frac{d}{d\theta}\left\{\left(\frac{du^2}{d\theta^2}+u^2\right)\left(h^2+2\int\frac{Td\theta}{u^3}\right)\right\}-\frac{2P}{u^2}\frac{du}{d\theta}-\frac{2T}{u}=0,$$

performing the differentiation

$$2\frac{du}{d\theta}\left(\frac{d^2u}{d\theta^2}+u\right)\left(h^2+2\int\frac{Td\theta}{u^3}\right)$$

$$+\frac{2T}{u^3}\left(\frac{du^2}{d\theta^2}+u^2\right)-\frac{2P}{u^2}\frac{du}{d\theta}-\frac{2T}{u}=0;$$

$$\therefore \quad \frac{d^2u}{d\theta^2}+u-\frac{\dfrac{P}{u^2}-\dfrac{T}{u^3}\dfrac{du}{d\theta}}{h^2+2\displaystyle\int\frac{Td\theta}{u^3}}=0\ldots\ldots\ldots(3).$$

322. To obtain an equation for calculating the inclination of the radius-vector to the ecliptic, we have by the last of equations (1),

$$\frac{d^2z}{dt^2}=-\frac{\mu z}{r^3}-\frac{dR}{dz}=-S \text{ suppose,}$$

but $z=\dfrac{s}{u}$, $\therefore \quad \dfrac{dz}{dt}=\dfrac{dz}{d\theta}\dfrac{d\theta}{dt}=\dfrac{1}{u^2}\left(u\dfrac{ds}{d\theta}-s\dfrac{du}{d\theta}\right)\dfrac{d\theta}{dt}$

$$=\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\sqrt{h^2+2\int\frac{Td\theta}{u^3}} \text{ by (2);}$$

$$\therefore \quad \frac{d^2z}{dt^2}=\frac{d}{d\theta}\left\{\left(u\frac{ds}{d\theta}-s\frac{du}{d\theta}\right)\sqrt{h^2+2\int\frac{Td\theta}{u^3}}\right\}\frac{d\theta}{dt}$$

$$= \left\{\left(u\frac{d^2s}{d\theta^2} - s\frac{d^2u}{d\theta^2}\right)\sqrt{h^2 + 2\int\frac{Td\theta}{u^3}}\right.$$

$$\left. + \left(u\frac{ds}{d\theta} - s\frac{du}{d\theta}\right)\frac{T}{u^3\sqrt{h^2 + 2\int\frac{Td\theta}{u^3}}}\right\}\frac{d\theta}{dt}$$

$$= \left(u\frac{d^2s}{d\theta^2} - s\frac{d^2u}{d\theta^2}\right)u^2\left(h^2 + 2\int\frac{Td\theta}{u^3}\right) + \left(u\frac{ds}{d\theta} - s\frac{du}{d\theta}\right)\frac{T}{u} = -S;$$

$$\therefore \frac{d^2s}{d\theta^2} - \frac{s}{u}\frac{d^2u}{d\theta^2} + \frac{\frac{S}{u^3} + \frac{T}{u^3}\left(\frac{ds}{d\theta} - \frac{s}{u}\frac{du}{d\theta}\right)}{h^2 + 2\int\frac{Td\theta}{u^3}} = 0;$$

but by the last Article

$$\frac{s}{u}\frac{d^2u}{d\theta^2} + s - \frac{\frac{Ps}{u^3} - \frac{Ts}{u^4}\frac{du}{d\theta}}{h^2 + 2\int\frac{Td\theta}{u^3}} = 0,$$

adding these last equations we have

$$\frac{d^2s}{d\theta^2} + s + \frac{\frac{S - Ps}{u^3} + \frac{T}{u^3}\frac{ds}{d\theta}}{h^2 + 2\int\frac{Td\theta}{u^3}} = 0 \quad \text{.........(4).}$$

323. It is necessary to estimate the comparative magnitude of the various small quantities involved in our calculations.

Let e, e' be the eccentricities of the lunar and solar orbits; k the tangent of the mean inclination of the lunar orbit to the ecliptic; m the ratio of the Sun's mean motion to the Moon's mean motion, a and a' the mean distances of the Moon and Sun from the Earth: the values of these quantities are nearly

$$e = \tfrac{1}{20}, \quad e' = \tfrac{1}{60}, \quad k = \tfrac{1}{12}, \quad m = \tfrac{1}{13},$$

these we shall reckon of the first order of small quantities.

But $\frac{a}{a'} = \frac{1}{400}$ nearly is a quantity of the second order of magnitude, since it $= e^2$ nearly. The Sun's disturbing force is greatest when the Moon is between the Sun and Earth: in which case it $= \frac{m'}{(a'-a)^2} - \frac{m'}{a'^2}$; the ratio which this bears to the action of the Earth on the Moon

$$= \left\{\frac{m'}{(a'-a)^2} - \frac{m'}{a'^2}\right\} \div \frac{\mu}{a^2} = \frac{m'}{\mu}\frac{2a^3}{a'^3} \text{ nearly } = 2m^2$$

by Kepler's third law. Hence the disturbing force is of the second order.

We proceed to expand the values of T, P, S.

Prop. *To expand the values of* T, P, S *neglecting small quantities of the fourth order.*

324. By Arts. 321, 322,

$$T = \sin\theta \frac{dR}{dx} - \cos\theta \frac{dR}{dy};$$

$$P = \frac{\mu\rho}{r^3} + \cos\theta \frac{dR}{dx} + \sin\theta \frac{dR}{dy};$$

$$S = \frac{\mu z}{r^3} + \frac{dR}{dz}, \text{ and by Arts. 319, 320,}$$

$$R = \frac{m'(xx' + yy' + zz')}{r'^3} - \frac{m'}{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}},$$

m' = mass of Sun; $x'y'z'$ co-ordinates of Sun: r' = dist. of Sun.

Let $x = \rho\cos\theta$, $y = \rho\sin\theta$, $z = \rho s$ as before;

$$x' = r'\cos\theta', \quad y' = r'\sin\theta', \quad z' = 0,$$

since the plane of xy is the plane of the ecliptic.

$$\therefore\ R = \frac{m'\rho}{r'^2}\cos(\theta-\theta') - \frac{m'}{\{\rho^2 + z^2 + r'^2 - 2\rho r'\cos(\theta-\theta')\}^{\frac{1}{2}}}$$

$$R = \frac{m'\rho}{r'^2}\cos(\theta-\theta') - \frac{m'}{r'}\left\{1 - \frac{2\rho}{r'}\cos(\theta-\theta') + (1+s^2)\frac{\rho^2}{r'^2}\right\}^{-\frac{1}{2}}$$

$$= -\frac{m'}{r'} + \frac{m'}{2r'}\{1 + s^2 - 3\cos^2(\theta-\theta')\}\frac{\rho^2}{r'^2}$$

$$= -\frac{m'}{r'} - \frac{m'}{4r'}\{1 - 2s^2 + 3\cos 2(\theta-\theta')\}\frac{\rho^2}{r'^2}.$$

Also $\theta = \tan^{-1}\dfrac{y}{x}$, $\rho = \sqrt{x^2+y^2}$, $s = \dfrac{z}{\sqrt{x^2+y^2}}$.

Hence $$\frac{dR}{dx} = \frac{dR}{d\theta}\frac{d\theta}{dx} + \frac{dR}{d\rho}\frac{d\rho}{dx} + \frac{dR}{ds}\frac{ds}{dx}$$

$$= -\frac{3m'\rho}{2r'^3}\sin 2(\theta-\theta')\sin\theta - \frac{m'\rho}{2r'^3}\{1 + 3\cos 2(\theta-\theta')\}\cos\theta$$

$$= -\frac{m'\rho}{2r'^3}\cos\theta - \frac{3m'\rho}{2r'^3}\cos(\theta-2\theta');$$

$$\frac{dR}{dy} = \frac{dR}{d\theta}\frac{d\theta}{dy} + \frac{dR}{d\rho}\frac{d\rho}{dy} + \frac{dR}{ds}\frac{ds}{dy}$$

$$= \frac{3m'\rho}{2r'^3}\sin 2(\theta-\theta')\cos\theta - \frac{m'\rho}{2r'^3}\{1 + 3\cos 2(\theta-\theta')\}\sin\theta$$

$$= -\frac{m'\rho}{2r'^3}\sin\theta + \frac{3m'\rho}{2r'^3}\sin(\theta-2\theta');$$

$$\frac{dR}{dz} = \frac{dR}{d\theta}\frac{d\theta}{dz} + \frac{dR}{d\rho}\frac{d\rho}{dz} + \frac{dR}{ds}\frac{ds}{dz} = \frac{m's\rho}{r'^3}.$$

Hence $$T = \sin\theta\frac{dR}{dx} - \cos\theta\frac{dR}{dy} = -\frac{3m'\rho}{2r'^3}\sin 2(\theta-\theta');$$

$$P = \frac{\mu\rho}{r^3} + \cos\theta\frac{dR}{dx} + \sin\theta\frac{dR}{dy}$$

$$= \frac{\mu\rho}{\{\rho^2 + \rho^2 s^2\}^{\frac{3}{2}}} - \frac{m'\rho}{2r'^3} - \frac{3m'\rho}{2r'^3}\cos 2(\theta - \theta')$$

$$= \frac{\mu}{\rho^2}\left(1 - \frac{3}{2}s^2\right) - \frac{m'\rho}{2r'^3} - \frac{3m'\rho}{2r'^3}\cos 2(\theta - \theta');$$

$$S = \frac{\mu z}{r^3} + \frac{dR}{dz} = \frac{\mu\rho s}{\{\rho^2 + \rho^2 s^2\}^{\frac{3}{2}}} + \frac{m' s\rho}{r'^3}$$

$$= \frac{\mu}{\rho^2}\left(s - \frac{3}{2}s^3\right) + \frac{m' s\rho}{r'^3}.$$

Prop. ***To integrate the differential equations, first approximation.***

325. We here neglect all small quantities of an order higher than the first, and therefore the disturbing force (Art. 323): hence by last Article

$$T = 0, \quad P = \frac{\mu}{\rho^2} = \mu u^2, \quad S = \frac{\mu s}{\rho^2} = \mu s u^2,$$

and the differential equations (3) (4) of Arts. 321, 322 become

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{h^2} = 0;$$

$$\text{and } \frac{d^2s}{d\theta^2} + s \qquad = 0.$$

The solutions of these equations are

$$u = \frac{\mu}{h^2}\{1 + e\cos(\theta - \alpha)\} = b\{1 + e\cos(\theta - \alpha)\}, \quad b = \frac{\mu}{h^2},$$

and $s = k\sin(\theta - \gamma)$: e, α, k, γ are constants.

The first of these proves, that the orbit of the Moon is an ellipse: and the second proves that the tangent of the latitude bears a constant ratio to the sine of the longitude reckoned from the node; and therefore the Moon moves in a constant plane, as Napier's Rules in Spherical Trigonometry will immediately shew.

PROP. *To shew, that to integrate the differential equations to a second approximation we must introduce all terms of the* third *order, in which the coefficient of* θ *is either nearly equal to unity, or is small.*

326. By approximating to the values of the small quantities we shall arrive at a differential equation in u of the form

$$\frac{d^2u}{d\theta^2} + u + a + a' \cos(n\theta + n') + \ldots\ldots = 0,$$

the integral of which is of the form

$$u = -a + A\cos(\theta + B) + C\cos(n\theta + n') + \ldots$$

A, B being arbitrary constants, and C...constants to be determined by putting this value of u in the differential equation. This gives

$$C(1 - n^2) = -a'; \quad \therefore\ C = \frac{-a'}{1 - n^2},$$

from which we learn, that if $1 - n^2$ be a small quantity of the first order, then C will be of an order lower than a': and therefore, that our integral may contain all quantities of the *second* order, a' must be calculated to the *third* order. In the following pages $1 - n^2$ is never of an order higher than the first, otherwise it would be necessary to calculate a' to a higher order than even the third.

Wherefore when the coefficient of the argument of a cosine or sine is nearly unity we must retain coefficients of the *third* order, since these terms rise into importance by the process of integration.

Again, the function R and therefore the differential equation in u contains terms depending on r': and the reciprocal of r'

$$= b'\{1 + e'\cos(\theta' - \alpha')\} = b'\{1 + e'\cos(m\theta + \beta - \alpha') + \ldots\}:$$

the accented letters refer to the Sun: m = ratio of the Moon's period to the Sun's period: β = longitude of the Sun when the Moon is in Aries. Hence

$$\frac{dt}{d\theta} \text{ calculated from } \frac{dt}{d\theta} = \frac{1}{hu^2}\left\{1 + 2\int\frac{Td\theta}{h^2u^3}\right\}^{-\frac{1}{2}}$$

(Art. 321, equation (2)) will contain a term $D\cos(m\theta + \beta - \alpha')$, and therefore t contains a term

$$\frac{D}{m}\sin(m\theta + \beta - \alpha'):$$

hence D must be calculated to the *third* order. Wherefore all terms in which the coefficient of θ is small must be calculated to the third order; as well as those in which the coefficient of θ is nearly equal to unity.

PROP. *To calculate* $\sin 2(\theta - \theta')$ *and* $\cos 2(\theta - \theta')$ *to the first order.*

327. We need calculate these only to the first order because they occur only in terms multiplied by quantities of the second order.

$$\frac{dt}{d\theta} = \frac{1}{hu^2}\left\{1 + 2\int\frac{Td\theta}{h^2u^3}\right\}^{-\frac{1}{2}} = \frac{1}{hu^2}, \text{ first order}:$$

$$= \frac{1}{b^2h}\{1 - 2e\cos(\theta - \alpha)\},\ b^2h = n = \text{Moon's mean motion};$$

$$\therefore\ nt = \theta - 2e\sin(\theta - \alpha),$$

$t = 0$ when the Moon's mean longitude $= 0$; also let the Sun's mean longitude then $= \beta$:

$$\therefore\ n't + \beta = \theta' - 2e'\sin(\theta' - \alpha'),\ n' = \text{Sun's mean motion}.$$

Now $\frac{n'}{n} = m$: hence multiplying the first equation by m and subtracting, we have

$$\theta' - 2e'\sin(\theta' - \alpha') = m\theta + \beta;\ \text{neg}^{\text{g}}.\ me,\ \text{of second order};$$

$$\therefore\ \theta' = m\theta + \beta + 2e'\sin(m\theta + \beta - \alpha');$$

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$$\therefore \sin 2(\theta-\theta') = \sin\{[(2-2m)\theta-2\beta]-4e'\sin(m\theta+\beta-\alpha')\}$$

$$= \sin\{(2-2m)\theta-2\beta\} - 4e'\cos\{(2-2m)\theta-2\beta\}\sin(m\theta+\beta-\alpha')$$

$$= \sin\{(2-2m)\theta-2\beta\} - 2e'\sin\{(2-m)\theta-\beta-\alpha'\}$$
$$+ 2e'\sin\{(2-3m)\theta-3\beta+\alpha'\}$$

$$\cos 2(\theta-\theta') = \cos\{(2-2m)\theta-2\beta\}$$
$$+ 4e'\sin\{(2-2m)\theta-2\beta\}\sin(m\theta+\beta-\alpha')$$

$$= \cos\{(2-2m)\theta-2\beta\} + 2e'\cos\{(2-3m)\theta-3\beta+\alpha'\}$$
$$- 2e'\cos\{(2-m)\theta-\beta-\alpha'\}.$$

PROP. *To calculate* $\frac{T}{h^2u^3}, \int\frac{T\,d\theta}{h^2u^3}, \frac{T}{h^2u^3}\frac{du}{d\theta}$, *retaining the necessary terms of the third order.*

328. By Art. 324, $\frac{T}{h^2u^3} = -\frac{3m'}{2u^4h^2r'^3}\sin 2(\theta-\theta')$

$$= -\frac{3m'b'^3}{2h^2b^4}\{1+e\cos(\theta-\alpha)\}^{-4}\{1+e'\cos(\theta'-\alpha')\}^3\sin 2(\theta-\theta').$$

But $b = \frac{\mu}{h^2}$ (Art. 325); $\therefore \frac{m'b'^3}{h^2b^4} = \frac{m'b'^3}{\mu b^3} = m^2$ (Art. 323);

$$\therefore \frac{T}{h^2u^3} = -\tfrac{3}{2}m^2\{1-4e\cos(\theta-\alpha)+3e'\cos(m\theta+\beta-\alpha')\}$$
$$\times\{\sin[(2-2m)\theta-2\beta]-2e'\sin[(2-m)\theta-\beta-\alpha']$$
$$+2e'\sin[(2-3m)\theta-3\beta+\alpha']\}$$

$$= -\tfrac{3}{2}m^2\{\sin[(2-2m)\theta-2\beta]-2e\sin[(1-2m)\theta-2\beta+\alpha]\}.$$

Again $u = b\{1+e\cos(\theta-\alpha)\}$; $\therefore \frac{du}{d\theta} = -be\sin(\theta-\alpha)$;

$$\therefore \frac{T}{h^2u^3}\frac{du}{d\theta} = \tfrac{3}{4}bm^2e\cos\{(1-2m)\theta-2\beta+\alpha\}.$$

Again $\displaystyle\int \frac{T\,d\theta}{h^2 u^3} = \frac{3}{2} m^2 \left\{ \frac{1}{2-2m} \cos\left[(2-2m)\theta - 2\beta\right] \right.$

$$\left. - \frac{2e}{1-2m} \cos\left[(1-2m)\theta - 2\beta + a\right] \right\}$$

$$= \tfrac{3}{4} m^2 \cos\left\{(2-2m)\theta - 2\beta\right\} - 3m^2 e \cos\left\{(1-2m)\theta - 2\beta + a\right\}.$$

PROP. *To calculate* $\dfrac{\mathrm{P}}{\mathrm{h}^2\mathrm{u}^2}$, *retaining the necessary terms of the third order.*

329. By Art. 324.

$$\frac{P}{h^2 u^2} = b\left(1 - \tfrac{3}{2} s^2\right) - \frac{m'}{2u^3 h^2 r'^3}\left\{1 + 3\cos 2(\theta - \theta')\right\}.$$

First. $\quad b\left(1 - \tfrac{3}{2} s^2\right) = b\left\{1 - \tfrac{3}{4} k^2 + \tfrac{3}{4} k^2 \cos 2(\theta - \gamma)\right\}.$

Secondly.

$$-\frac{m'}{2u^3 h^2 r'^3} = -\frac{m' b'^3}{2h^2 b^3}\left\{1 + e\cos(\theta - a)\right\}^{-3}\left\{1 + e'\cos(\theta' - a')\right\}^3$$

$$= -\tfrac{1}{2} b m^2 \left\{1 - 3e\cos(\theta - a) + 3e'\cos(m\theta + \beta - a')\right\};$$

both terms must be retained, since in the first the coefficient of $\theta = 1$, and in the second it is small.

Thirdly. $\quad -\dfrac{3m'}{2u^3 h^4 r'^3} \cos 2(\theta - \theta') =$

$$-\tfrac{3}{2} b m^2 \left\{1 - 3e\cos(\theta - a) + 3e'\cos(m\theta + \beta - a')\right\}$$

$$\times \left\{\cos\left[(2-2m)\theta - 2\beta\right] + 2e'\cos\left[(2-3m)\theta - 3\beta + a'\right]\right.$$

$$\left. - 2e'\cos\left[(2-m)\theta - \beta - a'\right]\right\}.$$

Multiplying these together by the formula $2\cos a \cos b = \cos(a-b) + \cos(a+b)$, neglecting quantities of the third order, except those in which the coefficient of θ is small or nearly unity, we have this third part of $\dfrac{P}{h^2 u^2} =$

$$-\tfrac{3}{2} b m^2 \left\{\cos\left[(2-2m)\theta - 2\beta\right] - \tfrac{3}{2} e \cos\left[(1-2m)\theta - 2\beta + a\right]\right\}.$$

Hence the value of $\frac{P}{h^2u^2}$ is

$$b\left\{1-\tfrac{3}{4}k^2+\tfrac{3}{4}k^2\cos 2(\theta-\gamma)\right\}-\tfrac{1}{2}bm^2\left\{1-3e\cos(\theta-a)\right.$$
$$+3e'\cos(m\theta+\beta-a')+3\cos\left[(2-2m)\theta-2\beta\right]$$
$$\left.-\tfrac{9}{2}e\cos\left[(1-2m)\theta-2\beta+a\right]\right\}.$$

PROP. *To form the differential equation for* u.

330. By Art. 321, equation (3),

$$\frac{d^2u}{d\theta^2}+u-\frac{\dfrac{P}{u^2}-\dfrac{T}{u^3}\dfrac{du}{d\theta}}{h^2+2\displaystyle\int\frac{T\,d\theta}{u^3}}=0.$$

By expanding the reciprocal of the denominator of the fractional part and neglecting the square of the disturbing force, which is of the fourth order, and neglecting all other quantities of the fourth order, and observing that P contains a term μu^2, or by Art. 325 $b\,h^2u^2$, which is not small, we have

$$\frac{d^2u}{d\theta^2}+u-\frac{P}{h^2u^2}+\frac{T}{h^2u^3}\frac{du}{d\theta}+\frac{2b}{h^2}\int\frac{T\,d\theta}{u^3}=0.$$

By the last two Articles this becomes

$$\frac{d^2u}{d\theta^2}+u-b\left(1-\tfrac{3}{4}k^2-\tfrac{1}{2}m^2\right)-\tfrac{3}{2}bm^2e\cos(\theta-a)$$
$$-\tfrac{3}{4}bk^2\cos 2(\theta-\gamma)+3\,bm^2\cos\left\{(2-2m)\theta-2\beta\right\}$$
$$-\tfrac{15}{8}bm^2e\cos\left\{(1-2m)\theta-2\beta+a\right\}+\tfrac{3}{2}bm^2e'\cos(m\theta+\beta-a')=0.$$

Now this equation cannot be integrated, as it stands, according to the method mentioned in Art. 326; because the term $\frac{3}{2}bm^2e\cos(\theta-a)$ would introduce an infinite coefficient into the expression of u, since the coefficient of $\theta=$ unity. But this may be remedied by putting for $be\cos(\theta-a)$ in the term $\frac{3}{2}bm^2e\cos(\theta-a)$, which is of the third order, its first approximate value $u-b$: then the equation becomes

$$\frac{d^2u}{d\theta^2} + (1 - \tfrac{3}{2}m^2)(u - b) + \tfrac{1}{4}b(3k^2 + 2m^2) - \tfrac{3}{4}bk^2\cos 2(\theta - \gamma)$$
$$+ 3bm^2\cos\{(2-2m)\theta - 2\beta\} - \tfrac{15}{2}bm^2e\cos\{(1-2m)\theta - 2\beta + \alpha\}$$
$$+ \tfrac{3}{2}bm^2e'\cos(m\theta + \beta - \alpha') = 0.$$

Let $1 - \frac{3}{2}m^2 = c^2$; then if we integrate this equation, we have

$$u = b\{1 + e\cos(c\theta - \alpha) + \dots\dots\dots\dots\}.$$

Now although c differs from unity only by a quantity of the second order, yet $\cos(c\theta - \alpha)$ will differ very sensibly from $\cos(\theta - \alpha)$ after several revolutions of the Moon. Wherefore the value of u in Art. 325 will cease to be even a *first* approximate value after several revolutions of the Moon; the true first approximate value being $b\{1 + e\cos(c\theta - \alpha)\}$. We must therefore carefully retrace our steps, and replace θ by $c\theta$ in every place where θ is introduced in consequence of its depending immediately on the first approximate value of u. This may very easily be accomplished by putting $\alpha + (1 - c)\theta$ instead of α in every place where it occurs: for α enters the equations solely in consequence of its dependence on θ and u by the equation $u = b\{1 + e\cos(\theta - \alpha)\}$.

The same will be the case with the value of s, as we shall shew in the next Proposition. We shall write down the corrected equations of u and s in Art 332.

PROP. *To form the differential equation for* s.

331. By Art. 322, equation (4),

$$\frac{d^2s}{d\theta^2} + s + \frac{\dfrac{S - Ps}{u^3} + \dfrac{T}{u^3}\dfrac{ds}{d\theta}}{h^2 + 2\displaystyle\int\frac{Td\theta}{u^3}} = 0.$$

Now $\dfrac{S - Ps}{h^2u^3} = \dfrac{3m's}{2u^4h^2r'^3}\{1 + \cos 2(\theta - \theta')\}$

$$= \frac{3m'kb'^3}{2h^2b^4}\sin(\theta - \gamma)\{1 + \cos[2(1 - m)\theta - 2\beta]\}$$

$$= \tfrac{3}{2}m^2k\{\sin(\theta - \gamma) - \tfrac{1}{2}\sin[(1 - 2m)\theta - 2\beta + \gamma]\},$$

retaining those terms of the third order which have the multiplier of θ nearly $= 1$.

Again, $\dfrac{ds}{d\theta} = k \cos(\theta - \gamma)$;

$$\therefore \frac{T}{h^2u^3}\frac{ds}{d\theta} = -\frac{3m'k}{2u^4h^2r'^3}\cos(\theta-\gamma)\sin 2(\theta-\theta')$$

$$= -\tfrac{3}{2}m^2k\cos(\theta-\gamma)\sin\{2(1-m)\theta - 2\beta\}$$

$$= -\tfrac{3}{4}m^2k\sin\{(1-2m)\theta - 2\beta + \gamma\}.$$

Then the equation in s becomes

$$\frac{d^2s}{d\theta^2} + s + \tfrac{3}{8}m^2k\{\sin(\theta-\gamma) - \sin[(1-2m)\theta - 2\beta + \gamma]\} = 0.$$

This (as in the case of the equation in u) cannot be integrated by the method explained in Art. 326, because the term $\frac{3}{8}m^2k\sin(\theta-\gamma)$ would introduce an infinite coefficient into the value of s. But by putting for $k\sin(\theta-\gamma)$ its first approximate value s in the term $\frac{3}{8}m^2k\sin(\theta-\gamma)$, which is of the third order, this difficulty is overcome; the differential equation then becomes

$$\frac{d^2s}{d\theta^2} + (1 + \tfrac{3}{8}m^2)s - \tfrac{3}{8}m^2k\sin\{(1-2m)\theta - 2\beta + \gamma\} = 0.$$

Let $1 + \frac{3}{8}m^2 = g^2$; then if we integrate this equation, we have

$$s = k\sin(g\theta - \gamma) + \ldots\ldots$$

Hence (as in the last Article) although g differs from unity only by a quantity of the second order, yet $\sin(g\theta-\gamma)$ will differ sensibly from $\sin(\theta-\gamma)$ after several revolutions of the Moon. Therefore $k\sin(\theta-\gamma)$ ceases to be a first approximation of s after several revolutions of the Moon: and we must retrace our steps and put $g\theta$ for θ in every place where θ enters in consequence of its immediately depending on s. This may be done by putting $\gamma + (1-g)\theta$ for γ in every place where γ occurs.

Prop. *To integrate the differential equations in* u *and* s.

332. After replacing θ by $c\theta$ and $g\theta$ in all such places as θ enters the equations in consequence of its immediate dependence on u and s respectively, the equations of Arts. 330, 331, become

$$\frac{d^2u}{d\theta^2} + c^2(u-b) + \tfrac{1}{4}b\,(3k^2+2m^2) - \tfrac{3}{4}\,b\,k^2\cos 2\,(g\theta-\gamma)$$

$$+\,3bm^2\cos\{(2-2m)\theta-2\beta\} - \tfrac{15}{2}bm^2e\cos\{(2-2m-c)\theta-2\beta+a\}$$
$$+\,\tfrac{3}{2}bm^2e'\cos(m\theta+\beta-a') = 0\,;$$

$$\text{and}\ \frac{d^2s}{d\theta^2} + g^2s - \tfrac{3}{2}m^2k\sin\{(2-2m-g)\,\theta-2\beta+\gamma\} = 0.$$

To integrate the first assume $u =$

$$b\,\{A + e\cos(c\theta-a) + B\cos 2\,(g\theta-\gamma) + C\cos[(2-2m)\theta-2\beta]$$
$$+D\cos[(2-2m-c)\,\theta-2\beta+a] + E\cos(m\theta+\beta-a')\},$$

A, B, C, D, E being indeterminate coefficients: to find these substitute this value of u in the differential equation, and equate the coefficients to zero: then, remembering that $c^2 = 1-\frac{3}{2}m^2$ and $g^2 = 1+\frac{3}{2}m^2$, and neglecting small quantities of orders higher than the second, we have

$$c^2A = c^2 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2, \qquad \therefore\ A = 1 - \tfrac{3}{4}k^2 - \tfrac{1}{2}m^2$$
$$(c^2-4g^2)\,B = \tfrac{3}{4}k^2, \qquad \therefore\ B = -\tfrac{1}{4}k^2$$
$$\{c^2-(2-2m)^2\}\,C = -3m^2, \qquad \therefore\ C = m^2$$
$$\{c^2-(2-2m-c)^2\}\,D = \tfrac{15}{2}m^2e, \qquad \therefore\ D = \tfrac{15}{8}me$$
$$(c^2-m^2)\,E = -\tfrac{3}{2}m^2e', \qquad \therefore\ E = -\tfrac{3}{2}m^2e'.$$

Hence $u = b\,\{1 - \frac{3}{4}k^2 - \frac{1}{2}m^2 + e\cos(c\theta-a) - \frac{1}{4}k^2\cos 2\,(g\theta-\gamma)$

$$+\ m^2\cos[(2-2m)\,\theta-2\beta] + \tfrac{15}{8}me\cos[(2-2m-c)\,\theta-2\beta+a]$$
$$-\ \tfrac{3}{2}m^2e'\cos(m\,\theta+\beta-a)\}.$$

333. Again, to integrate the equation in s assume

$$s = k\sin(g\theta-\gamma) + F\sin\{(2-2m-g)\,\theta-2\beta+\gamma\},$$

F being an indeterminate coefficient. Substitute this in the differential equation, and we have

$$\{g^2 - (2 - 2m - g)^2\} F = \tfrac{3}{4} m^2 k, \quad \therefore \; F = \tfrac{3}{8} mk\,;$$

$$\therefore \; s = k \sin (g\theta - \gamma) + \tfrac{3}{8} mk \sin \{(2 - 2m - g)\,\theta - 2\beta + \gamma\}.$$

We shall now make use of these values of u and s to calculate the distance and longitude of the Moon.

PROP. *To find the distance of the Moon from the Earth.*

334. Let r be this distance; $\therefore \; r = \rho \sqrt{1 + s^2}$;

$$\therefore \; \frac{1}{r} = u\,(1 - \tfrac{1}{2} s^2), \text{ neglecting quantities of the fourth order,}$$

$$= u \left\{1 - \tfrac{1}{4}k^2 + \tfrac{1}{4}k^2 \cos 2\,(g\theta - \gamma)\right\}, \text{ (Art. 333.)}$$

$$= b \left\{1 - k^2 - \tfrac{1}{2} m^2 + e \cos (c\theta - a) + m^2 \cos \left[(2 - 2m)\,\theta - 2\beta\right]\right.$$
$$+ \tfrac{15}{8} me \cos \left[(2 - 2m - c)\,\theta - 2\beta + a\right]$$
$$\left. - \tfrac{3}{2} m^2 e' \cos (m\theta + \beta - a')\right\}, \text{ (Art. 332.)}$$

$$= \mathrm{b} \left\{1 + e \cos (c\theta - a) + m^2 \cos \left[(2 - 2m)\,\theta - 2\beta\right]\right.$$
$$\left. + \tfrac{15}{8} me \cos \left[(2 - 2m - c)\,\theta - 2\beta + a\right] - \tfrac{3}{2} m^2 e' \cos (m\theta + \beta - a')\right\},$$
$$\text{where } \mathrm{b} = b\,(1 - k^2 - \tfrac{1}{2} m^2).$$

PROP. *To find the longitude of the Moon in terms of the time.*

335. By Art. 321 equation (2) we have

$$\frac{dt}{d\theta} = \frac{1}{hu^2} \left\{1 + 2 \int \frac{T\,d\theta}{h^2 u^3}\right\}^{-\frac{1}{2}} = \frac{1}{hu^2} \left\{1 - \int \frac{T\,d\theta}{h^2 u^3}\right\}.$$

Then substituting for $\int \frac{T\,d\theta}{h^2 u^3}$ and u by Arts. 328, 332, and retaining those terms of the third order in which the coefficient of θ is small (Art. 326),

$$\frac{dt}{d\theta} = \frac{1}{hb^2} \{1 + \tfrac{1}{2}k^2 + m^2 + \tfrac{3}{2}e^2 - 2e \cos (c\theta - a)$$

$$+\tfrac{3}{4}e^2\cos 2(c\theta-a)+\tfrac{1}{4}k^2\cos 2(g\theta-\gamma)$$
$$-\tfrac{11}{4}m^2\cos\left[(2-2m)\theta-2\beta\right]-\tfrac{15}{4}me\cos\left[(2-2m-c)\theta-2\beta+a\right]$$
$$+3m^2e'\cos(m\theta+\beta-a')\big\}.$$

Then putting $hb^2(1-\frac{1}{4}k^2-m^2-\frac{1}{2}e^2)=n$, multiplying by n and integrating,

$$nt=\theta-2e\sin(c\theta-a)+\tfrac{3}{4}e^2\sin 2(c\theta-a)+\tfrac{1}{4}k^2\sin 2(g\theta-\gamma)$$
$$-\tfrac{11}{8}m^2\sin\{(2-2m)\theta-2\beta\}-\tfrac{15}{4}me\sin\{(2-2m-c)\theta-2\beta+a\}$$
$$+3me'\sin(m\theta+\beta-a').$$

To obtain θ in terms of t proceed as follows. Transpose all the terms but θ to one side of the equation. If all small quantities are neglected $\theta=nt$; then for a *first approximation* neglect small quantities of the second order, and put $\theta=nt$ in the small terms;

$$\therefore\ \theta=nt+2e\sin(cnt-a).$$

For a *second approximation* put this value of θ in small terms and neglect small quantities of the third order;

$$\therefore\theta=nt+2e\sin(cnt-a)+\tfrac{5}{4}e^2\sin 2(cnt-a)-\tfrac{1}{4}k^2\sin 2(gnt-\gamma)$$
$$+\tfrac{11}{8}m^2\sin\{(2-2m)nt-2\beta\}+\tfrac{15}{4}me\sin\{(2-2m-c)nt-2\beta+a\}$$
$$-3me'\sin(mnt+\beta-a').$$

336. These expressions for the radius-vector and the longitude of the Moon shew, that her distance from the Earth preserves nearly a constant value, fluctuating between very small limits: and that her longitude varies nearly as the time of motion, departing from this law only by small quantities.

It will be an interesting enquiry to examine these formulæ for the radius-vector and the longitude, and see whether they will enable us to explain the various inequalities that observations have pointed out in the motion of the Moon. The principle of the superposition of small motions (Art. 288.) allows us to examine the cause of each small term upon the supposition, that all the other small terms do not exist.

PROP. *To interpret the physical meaning of the various terms in the analytical expressions for the radius-vector and the longitude of the Moon.*

337. To examine the effect of the first small term we neglect all the other small terms, and suppose the equation to the Moon's orbit to be

$$\frac{1}{r} = \mathrm{b}\left\{1 + e\cos(c\theta - \alpha)\right\}.$$

If $c = 1$ this would be an ellipse from the focus: but c is not $= 1$, though it differs but slightly from unity. We shall now shew how the motion may be found: fig. 83*.

Let E be the focus and aEA' the major-axis of the ellipse, which the above equation represents when $c = 1$: then $\theta - \frac{\alpha}{c}$ is measured about E as pole and from the line EA': the point A' is both in the ellipse and also in the Moon's orbit; because, when $\theta - \frac{\alpha}{c} = 0$, r has the same value in both equations. Let $A'EM = \theta - \frac{\alpha}{c}$, and $EM =$ the Moon's radius-vector: $A'M$ her orbit: also let $\angle A'EM' = c \angle A'EM$, and let EM' cut the ellipse above mentioned in M': then

$$\frac{1}{EM} = \mathrm{b}\left\{1 + e\cos c\angle A'EM\right\} = \mathrm{b}\left\{1 + e\cos A'EM'\right\} = \frac{1}{EM'};$$

$$\therefore EM = EM'.$$

Draw EA equal to EA' making an angle equal to $\angle MEM'$ with EA': then through the variable points A and M an ellipse can always be drawn having its focus in E and equal in dimensions to the ellipse on aA'. Hence this inequality shews, that, if we neglect all the other terms, the Moon's motion may be represented by supposing that it moves in an ellipse, the perigee of which revolves about the Earth with an angular velocity =

* In the figure E is, by mistake, placed nearer to a than A'.

$$\frac{d \cdot AEA'}{dt} = \frac{d \cdot MEM'}{dt} = (1-c)\frac{d\theta}{dt} = \frac{3m^2}{4}\frac{d\theta}{dt} \text{ nearly.}$$

Principia, Lib. I. Prop. 66. Cor. 7.

The two terms of the longitude

$$2e\sin(cnt-\alpha) + \tfrac{5}{4}e^2\sin 2(cnt-\alpha)$$

correspond to the above term in the reciprocal of the radius-vector; as may be seen by comparing the form of the terms with those in the expansion of $\theta - \varpi$ in Art. 277.

The second term in the reciprocal of the radius-vector is

$$\text{b}m^2\cos\{(2-2m)\theta - 2\beta\}.$$

Now $(1-m)\theta - \beta = \theta - (m\theta + \beta) = \text{long}^{\text{e}}.\text{of Moon} - \text{long}^{\text{e}}.\text{of Sun}$

$$= \text{angular distance of Moon from the Sun.}$$

Hence this inequality has its greatest positive value when the Moon is in syzygies, and its greatest negative value when the Moon is in quadratures. Hence its effect is to diminish the distance of the Moon when in syzygies, and to increase it when in quadratures. This agrees with the *Principia*, Prop. 66. Cor. 5. and also with Art. 303.

The term $\frac{11}{8}m^2\sin\{(2-2m)nt - 2\beta\}$ in the longitude corresponds to the above term: and is the inequality called the *Variation*, discovered by Tycho Brahe (See Arts. 291, 306).

The third term in the reciprocal of the radius-vector is

$$\tfrac{15}{8}\text{b}me\cos\{(2-2m-c)\theta - 2\beta + \alpha\}.$$

Since $2-2m-c$ nearly equals unity it will be seen that this term is nearly analogous to the first term, though of much less importance because of the smallness of its coefficient. We shall take it in conjunction with that term (see *Airy's Tracts, Lunar Theory*), neglecting the motion of the perigee and other small quantities.

Then $\dfrac{1}{r} = \text{b}\left\{1 + e\cos(\theta-\alpha) + \tfrac{15}{8}me\cos(\theta - 2\beta + \alpha)\right\}$,

neglecting the other terms

$$= b\left\{1 + e\cos(\theta - a) + \tfrac{15}{8}me\cos\left[\theta - a + 2(a - \beta)\right]\right\}$$

$$= b\left\{1 + \left[e + \tfrac{15}{8}me\cos 2(a - \beta)\right]\cos(\theta - a)\right.$$

$$\left. - \tfrac{15}{8}me\sin 2(a - \beta)\sin(\theta - a)\right\}$$

$$= b\left\{1 + e\left[1 + \tfrac{15}{8}m\cos 2(a - \beta)\right]\cos\left[\theta - a + \tfrac{15}{8}m\sin 2(a - \beta)\right]\right\},$$

as will easily be seen upon expanding this latter expression and neglecting small quantities of the third order.

Hence the effect of this third term in the reciprocal of the Moon's distance is to increase the eccentricity of the elliptic orbit by $\frac{15}{8}me\cos 2(a - \beta)$; and to diminish the longitude of the perigee by $\frac{15}{8}m\sin 2(a - \beta)$.

If we suppose the Sun to be stationary during one revolution of the Moon, β = longitude of the Sun: therefore

$$\text{eccentricity} = e\left\{1 + \tfrac{15}{8}m\cos 2(\text{long. perigee} - \text{long. Sun})\right\}$$

$$\text{long. perigee (corrected)} = a - \tfrac{15}{8}m\sin 2(\text{long. perigee} - \text{long. Sun}).$$

The term $\frac{15}{4}me\sin\{(2 - 2m - c)nt - 2\beta + a\}$ in the longitude exactly corresponds with the term above. It is called the *Evection*, and was discovered by Ptolemy (Art. 290). When the perigee is in syzygies, then $a - \beta = 0$ or π, and the eccentricity is increased by $\frac{15}{8}me$: and when the perigee is in quadratures the eccentricity is diminished by that quantity: *Principia*, Lib. I. Prop. 66. Cor. 9.

The last term in the reciprocal of the radius-vector is

$$-\tfrac{3}{2}bm^2e'\cos(m\theta + \beta - a').$$

This is of the third order: but the corresponding term in the longitude, *viz.* $-3me'\sin(mnt + \beta - a')$, is of the second order. This inequality in the longitude depends upon the Sun's mean anomaly: when the Sun is in perigee and apogee then $(mnt + \beta) - a' = 0$ and π, and this term vanishes: when the Sun is moving from perigee to apogee the term is negative, and positive as the Sun moves from apogee to perigee: hence the Moon is behind or before her mean place (in consequence of this inequality) according as the Sun is moving from perigee to apogee or from apogee to perigee. This is the *Annual*

Equation: (Art. 292, 300). Also see *Principia*, Lib. I. Prop. 66. Cor. 6. and Lib. III. *Scholium to Lunar Theory*.

There is another term $-\frac{1}{4}k^2 \sin(2gnt - 2\gamma)$ in the longitude. This depends upon the Moon's distance from the mean place of her node, and nearly equals the difference between her longitudes measured on the ecliptic and her orbit: hence it is called the ***Reduction***.

PROP. *To explain the physical meaning of the terms in the analytical expression for the inclination of the Moon's orbit to the ecliptic.*

338. The first term is $k \sin(g\theta - \gamma)$.

Let N be the ascending node when $\theta - \dfrac{\gamma}{g} = 0$, fig. 93. Take $\angle NEM' = \theta - \dfrac{\gamma}{g}$: and $\angle M'En = g \,.\, \angle M'EN$: also let M be the Moon, $\tan MEM' = s$: then n is the node, moving backwards. For in the right-angled triangle $MM'n$, we have

$$\sin M'n = \tan MM' \cot MnM', \quad \tan MnM' = \tan AB = k;$$

$$\therefore\ s = \tan MM' = k \sin g \angle M'EN = k \sin(g\theta - \gamma).$$

Hence the meaning of this term is that the node regresses with an angular velocity $= (g-1)\dfrac{d\theta}{dt} = \dfrac{3m^2}{4}\dfrac{d\theta}{dt}$.

The second term $\frac{3}{8}mk \sin\{(2-2m-g)\theta - 2\beta + \gamma\}$ is best considered in connexion with the first, as we did the Evection: (Airy's *Tracts*).

Neglecting the motion of the Node

$$s = k\{\sin(\theta-\gamma) + \tfrac{3}{8}m \sin(\theta - \gamma + 2\gamma - 2\beta)\}$$

$$= k\{1 + \tfrac{3}{8}m\cos 2(\gamma-\beta)\}\sin(\theta-\gamma) + \tfrac{3}{8}mk\sin 2(\gamma-\beta)\cos(\theta-\gamma)$$

$$= k\{1 + \tfrac{3}{8}m\cos 2(\gamma-\beta)\}\sin\{\theta-\gamma + \tfrac{3}{8}m\sin 2(\gamma-\beta)\}.$$

Hence the effect of the second term in s is to increase the tangent of inclination of the lunar orbit by $\frac{3}{8}mk\cos 2(\gamma-\beta)$ or

$\frac{3}{8}mk\cos 2$ (long. node − long. Sun), and to diminish the longitude of the node, calculated on the supposition of its uniformly regressing, by the angle $\frac{3}{8}m\sin 2(\gamma-\beta)$ or $\frac{3}{8}m\sin 2$ (long. node − long. Sun). *Principia*, Lib. III. Props. 33. and 35.

The inclination of the orbit is greatest when the node is in syzygies, and least when in quadratures: see Art. 317, and *Principia*, Lib. I. Prop. 66. Cor. 10.

339. The angle described by the perigee during a revolution of the Moon, as calculated in Art. 317, equals $\frac{3}{4}m^2.2\pi = \frac{3}{2}m^2\pi = 1^0.30'$ nearly: but its true value as proved by observation is about twice this. This apparent discrepancy between theory and observation shook Clairaut's belief in Newton's law of gravitation, and induced him to propose a new and more complicated law; pamphlets were already printed and about to be circulated by Clairaut, when he discovered, that by extending the approximation the value of c is

$$1-\tfrac{3}{4}m^2-\tfrac{225}{32}m^3,$$

the third term of which, owing to the largeness of the coefficient, nearly equals the second term: and therefore reconciles the apparent difference.

340. The value of g is $1+\frac{3}{4}m^2-\frac{9}{32}m^3$, and therefore the ratio of the motion of the perigee to that of the node

$$=(\tfrac{3}{4}m^2+\tfrac{225}{32}m^3)\div(\tfrac{3}{4}m^2-\tfrac{9}{32}m^3)=(1+\tfrac{75}{8}m)(1+\tfrac{3}{8}m)=2 \text{ nearly.}$$

This ratio is much larger than for one of Jupiter's satellites, because for that system m is very small indeed. *Principia*, Lib. III. Prop. 23.

341. If m_1 be the ratio of the mean motion of Jupiter to that of one of his satellites; then the progression of the perijove and regression of the node during a revolution of the satellite each $=\frac{3}{2}\pi m_1^2$. Hence the regression of the node of this satellite *during a given time* equals the regression of the Moon's node $\times$ ($m_1^2\div$ per. time of satellite) $\div$ ($m^2\div$ per. time of Moon)

$$=\left(\frac{\text{mean motion of Jupiter}}{\text{mean motion of Earth}}\right)^2\left(\frac{\text{mean motion of Moon}}{\text{mean motion of satellite}}\right)\text{ regress}^{\text{n}}\text{. of Moon's node.}$$

The same formula is true for the satellites of Saturn.

The progression of the perijove $= \frac{3}{4}\pi m_1^2$, and that of the Moon nearly $= 3\pi m^2$ (Art. 339): hence the progression of the perijove *during a given time*

$$=\tfrac{1}{4}\left(\frac{\text{mean motion of Jupiter}}{\text{mean motion of Earth}}\right)^2\left(\frac{\text{mean motion of Moon}}{\text{mean motion of satellite}}\right)\text{ regress}^{\text{n}}\text{. of Moon's perigee.}$$

The same is true for the satellites of Saturn.

If the series for c were more converging (Art. 339), then the $\frac{1}{2}$ which multiplies this expression would be 1. *Principia*, Lib. III. Prop. 23. Newton omits the $\frac{1}{2}$ and says "diminui tamen debet motus augis sic inventus in ratione 9 ad 5 vel 2 ad 1 circiter, ob causam hîc exponere non vacat." So it seems that Newton had some way of accounting for this apparent anomaly.

The reader that wishes to enter more deeply into the calculation of the lunar inequalities must consult a memoir by Baron Damoiseau in the *Mémoires présentés par divers savans à l'Académie Royale des Sciences*; Tom. I: the *Lunar Theory* of Messrs Plana and Carlini; and that of Mr Lubbock. In these works the approximation is carried so far as to enable us to deduce all the inequalities from theory alone.

342. In this Chapter we have given the inequalities of the first and second order: those of the third order are fifteen in number, these and some of the inequalities of the fourth and higher orders will be found in the *Méc. Cél.* Liv. VII. We shall mention some of the more interesting results.

Among the periodical inequalities of the Moon's motion in longitude, that which depends on the simple angular distance of the Sun and Moon is important on account of the great light it throws upon the Sun's parallax. The parallax is found to be 8.56 seconds, being the same as several astronomers have found from the last transit but one of Venus over the Sun: *Méc. Cél.* Liv. VIII. § 24.

343. An inequality, not less important, is that which depends upon the longitude of the Moon's node: as it did not

appear to depend on the theory of gravity, it was neglected by most astronomers; till a more thorough examination led Laplace to discover that its cause is the oblateness of the Earth: it gives an oblateness $= \frac{1}{305.05}$: *Méc. Cél.* Liv. VII. § 24.

344. There is also an inequality in the Moon's latitude, which Laplace discovered by theory: he shewed that it arises from the oblateness of the Earth's figure: it gives the oblateness $= \frac{1}{304.6}$: *Méc. Cél.* Liv. VII. § 25. See Art. 556.

345. These two inequalities prove that the Moon's gravity to the Earth arises from the attraction of all the particles of the Earth, and not of the centre alone. (Art. 260.)

346. By examining the records of ancient eclipses of the Moon it was found, that the Moon's mean motion was continually accelerated. The cause of this was long sought for in vain; till Laplace discovered by theory, that it depends upon the variation (the secular variation, see Art. 377.) of the eccentricity of the Earth's orbit. All the observations which have been made during the last century and a half, have put beyond a doubt this result of analysis. When the acceleration of the Moon's mean motion was known, but not accounted for, conjectures were started as to its depending on the resistance of a medium, or the transmission of gravity; but analysis shews that neither of these causes produces any sensible alteration. *Méc. Cél.* Liv. VII. § 23.

PROP. *To prove that the centre of gravity of the Earth and Moon very nearly describes an ellipse about the Sun.*

347. Let $x'\,y'\,z'$ be the co-or. of the Earth from the Sun,

$x_1 y_1 z_1$ .... Moon from the Sun,

$x\ y\ z$ .... Moon from the Earth.

$\bar{x}\ \bar{y}\ \bar{z}$ .... centre of grav. of Earth and Moon from the Sun.

m', E, M the masses of the Sun, Earth, and Moon.

$r'\,r_1$ the distances of the Earth and Moon from the Sun.

r the distance of the Moon from the Earth.

Then $x_1 - x' = x$, $y_1 - y' = y$, $z_1 - z' = z$.

The ratio of E to m' equals 1 : 356354 and may therefore be neglected. (Art. 391.)

The equations of motion of the Earth about the Sun, the Moon being the disturbing body, are (Art. 319.)

$$\left.\begin{aligned} \frac{d^2x'}{dt^2} + \frac{m'x'}{r'^3} + \frac{dR'}{dx'} = 0 \\ \frac{d^2y'}{dt^2} + \frac{m'y'}{r'^3} + \frac{dR'}{dy'} = 0 \\ \frac{d^2z'}{dt^2} + \frac{m'z'}{r'^3} + \frac{dR'}{dz'} = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(1);$$

$$R' = \frac{M(x'x_1 + y'y_1 + z'z_1)}{\{x_{\prime}^2 + y_{\prime}^2 + z_{\prime}^2\}^{\frac{3}{2}}} - \frac{M}{\sqrt{(x_1 - x')^2 + (y_1 - y')^2 + (z_1 - z')^2}}.$$

The equations of motion of the Moon about the Earth, the Sun being the disturbing body, are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{(E+M)x}{r^3} + \frac{dR}{dx} = 0 \\ \frac{d^2y}{dt^2} + \frac{(E+M)y}{r^3} + \frac{dR}{dy} = 0 \\ \frac{d^2z}{dt^2} + \frac{(E+M)z}{r^3} + \frac{dR}{dz} = 0 \end{aligned}\right\} \ldots\ldots\ldots(2);$$

$$R = -\frac{m'(x'x + y'y + z'z)}{\{x'^2 + y'^2 + z'^2\}^{\frac{3}{2}}} - \frac{m'}{\sqrt{(x'+x)^2 + (y'+y)^2 + (z'+z)^2}};$$

since $-x'$, $-y'$, $-z'$ are the co-ordinates of the Sun from the Earth.

$$\text{Now } \bar{x} = \frac{Ex' + Mx_1}{E+M}, \; = x' + \frac{Mx}{E+M}, \; \text{also} = x_1 - \frac{Ex}{E+M};$$

$$\therefore \frac{d^2\bar{x}}{dt^2} = \frac{d^2x'}{dt^2} + \frac{M}{E+M}\frac{d^2x}{dt^2}, \text{ by equations (1) (2),}$$

$$= -\frac{m'x'}{r'^3} - \frac{Mx}{r^3} - \frac{dR'}{dx'} - \frac{M}{E+M}\frac{dR}{dx}$$

$$= -\frac{m'x'}{r'^3} - \frac{Mx}{r^3} - \frac{Mx_1}{r_1^3} + \frac{M(x_1 - x')}{r^3} + \frac{M}{E+M}\left\{\frac{m'x'}{r'^3} - \frac{m'(x+x')}{r_1^3}\right\}$$

$$= -\frac{E}{E+M}\frac{m'x'}{r'^3} - \frac{M}{E+M}\frac{m'x_1}{r_1^3}, \text{ neglecting } \frac{Mx_1}{r_1^3},$$

substituting for x' and x_1 in terms of x and $\bar{x}$,

$$= -\frac{m'}{r'^3}\frac{E}{E+M}\left(\bar{x} - \frac{Mx}{E+M}\right) - \frac{m'}{r_1^3}\frac{M}{E+M}\left(\bar{x} + \frac{Ex}{E+M}\right).$$

Now $$\frac{1}{r'^3} = \left\{\left(\bar{x} - \frac{Mx}{E+M}\right)^2 + \left(\bar{y} - \frac{My}{E+M}\right)^2 + \left(\bar{z} - \frac{Mz}{E+M}\right)^2\right\}^{-\frac{3}{2}}$$

$$= \frac{1}{\bar{r}^3}\left\{1 + \frac{3M}{E+M}\frac{x\bar{x} + y\bar{y} + z\bar{z}}{\bar{r}^2} + \ldots\ldots\right\}.$$

Also $$\frac{1}{r_1^3} = \left\{\left(\bar{x} + \frac{Ex}{E+M}\right)^2 + \left(\bar{y} + \frac{Ey}{E+M}\right)^2 + \left(\bar{z} + \frac{Ez}{E+M}\right)^2\right\}^{-\frac{3}{2}}.$$

$$= \frac{1}{\bar{r}^3}\left\{1 - \frac{3E}{E+M}\frac{x\bar{x} + y\bar{y} + z\bar{z}}{\bar{r}^2} + \ldots\ldots\right\};$$

$$\therefore \frac{d^2\bar{x}}{dt^2} = -\frac{m'\bar{x}}{\bar{r}^3}$$

$$-\frac{3EMm'}{(E+M)^2}\frac{x\bar{x} + y\bar{y} + z\bar{z}}{\bar{r}^5}\left\{\bar{x} - \frac{Mx}{E+M} - \bar{x} - \frac{Ex}{E+M}\right\} + \ldots$$

$$= -\frac{m'\bar{x}}{\bar{r}^3} + \frac{3EMm'}{(E+M)^2}\frac{x^2\bar{x} + xy\bar{y} + xz\bar{z}}{\bar{r}^5} + \ldots$$

$= -\dfrac{m'\bar{x}}{\bar{r}^3} +$ terms multiplied by the products and powers of $\dfrac{x}{\bar{r}}$, $\dfrac{y}{\bar{r}}$ and $\dfrac{z}{\bar{r}}$ higher than the first.

Now $\dfrac{r}{\bar{r}} = \dfrac{1}{400}$ nearly, and x, y, z cannot be greater than r: hence if we neglect small quantities of the fourth order we have

$$\frac{d^2\bar{x}}{dt^2} + \frac{m'\bar{x}}{\bar{r}^3} = 0, \text{ and similarly } \frac{d^2\bar{y}}{dt^2} + \frac{m'\bar{y}}{\bar{r}^3} = 0, \ \frac{d^2\bar{z}}{dt^2} + \frac{m'\bar{z}}{\bar{r}^3} = 0.$$

These equations shew, that the path of the centre of gravity is a conic section in one plane, the Sun being in the focus, (Arts. 241, 246, 252); it evidently must be an ellipse.

348. Mr Airy, Astronomer Royal, has proposed a method for determining the mass of the Moon, which depends upon this Proposition. Since the centre of gravity of the Earth and Moon describes an ellipse about the Sun, it follows that the Earth does not describe an ellipse about the Sun: this deviation from elliptic motion depends upon the mass of the Moon, and can easily be calculated by theory: and thence can be determined the error in the Sun's right ascension and declination on the supposition of the orbit of the Earth being an ellipse. Now when Venus is near inferior conjunction she is only a third of the distance of the Earth from the Sun, and consequently the errors in her right ascension and declination will be much greater than in the Sun's. Some observations for this end will be found in the *Memoirs of the Royal Astronomical Society*, Vol. V. p. 223.

CHAPTER VI.

PLANETARY THEORY.

349. We have already stated that the perturbations of the Moon are far larger than those of the planets, because the Sun, the mass of which is enormous and distance not proportionably great, is one of the disturbing bodies.

The perturbations of the planets, on the other hand, are very minute; and are not detected in short periods of time. These might, however, be calculated in the manner pursued in calculating the longitude, latitude, and radius-vector of the Moon: but since the approximation is made by means of series which proceed by powers of the ratio of the distances of the disturbed and disturbing bodies from the central one, and since this fraction is much smaller in the Lunar Theory than in the Planetary Theory, it is necessary to retain many more terms in the calculation of the perturbations of the planets than in that of the perturbations of the Moon; and consequently the process is much slower in the former than in the latter calculation. For this reason R should be expanded in powers of the eccentricities and inclinations of the orbits of the planets, instead of the ratio of the distances of the disturbed and disturbing bodies, and the calculation then conducted as in the last Chapter.

350. But we shall make use of an entirely different mode of calculation. It is to Lagrange that we are indebted for the method we are about to lay before our readers.

If at any instant the disturbing forces were to cease acting, the planet would move in an exact ellipse; and this ellipse and the actual orbit of the planet would manifestly have a common tangent, and the actual velocity of the planet and

that calculated for the motion in this ellipse according to the elliptic theory would be the same. For this reason this ellipse is called the *ellipse of curvature* to the orbit at that instant; it is also denominated the *instantaneous ellipse* of the planet.

From what precedes it is evident that the motion of the planet may be represented by supposing it to move in an ellipse of which the elements are continually and slowly changing. If we know the elements of the instantaneous ellipse at any proposed instant, we have nothing to do but to calculate the position of the planet in this ellipse by the ordinary formulæ in Chap. III.

351. Since the perturbations of the planetary motions are very small, it follows from the Principle of the superposition of small motions, that the perturbations will be the sum of the perturbations produced by the several disturbing bodies considered separately: (Art. 288). We shall therefore in the following calculations consider only one disturbing body.

PROP. *To explain the process of integrating the equations of motion of a disturbed planet.*

352. The equations of motion of a disturbed planet are by Art. 319,

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0,$$

$$\text{and } \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0;$$

$$\text{where } R = \frac{m'(xx'+yy'+zz')}{\{x'^2+y'^2+z'^2\}^{\frac{3}{2}}} - \frac{m'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

μ = mass of Sun + mass of disturbed planet

m' = mass of the disturbing planet.

We shall first integrate these equations of motion omitting the disturbing forces: by this process we shall obtain six integrals of the first order, containing six arbitrary constants.

These six constants must be determined in terms of the six elements of the planet's orbit (Art. 270.); the inclination, the longitude of the node, the mean distance, the eccentricity, the longitude of the perihelion, and the epoch. By eliminating the three quantities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ from these six integrals we have the three final integrals of the equations of motion.

We shall then proceed to the integration of the equations of motion taking into consideration the disturbing forces.

The six integrals of the first order obtained on the supposition that there were no disturbing forces, contain the six arbitrary constants and also the quantities $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$.

Now it will be easily seen, that by supposing the six constants to be six unknown functions of x, y, z, and t, we may assume these six integrals of the first order to be the integrals of the equations of motion of the disturbed planet, and then determine the values of the unknown functions by substituting the integrals in the equations of motion. And in this way we shall have six equations for calculating the six unknown functions. The variable parts of these functions will be small, because the terms by which the equations of the disturbed planet differ from those of the undisturbed planet are small. When we know these small variations of the arbitrary constants (as they are sometimes called), we can calculate the small variations in the elements of the instantaneous orbit, by differentiating the equations which connect the arbitrary constants and elements of an undisturbed planet: so that if at any epoch these elements are known, they may be calculated for any other epoch near the former. Then these elements being put in the series of Arts. 278, 280, we know the position of the planet at any given time.

We proceed now to the investigation of the several Propositions necessary for these results.

Prop. *To integrate the equations of motion of an undisturbed planet.*

353. These equations are

$$\frac{d^2x}{dt^2}+\frac{\mu x}{r^3}=0,\quad \frac{d^2y}{dt^2}+\frac{\mu y}{r^3}=0,\quad \frac{d^2z}{dt^2}+\frac{\mu z}{r^3}=0\ldots\ldots(1),$$

where $\mu = M + m$ = mass of Sun + mass of planet.

Multiply the first by y and the second by x and take their difference; then

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=0;$$

$$\left.\begin{aligned} \therefore\ x\frac{dy}{dt}-y\frac{dx}{dt}&=\text{const.}=h,\\ \text{similarly } z\frac{dx}{dt}-x\frac{dz}{dt}=h_1,\quad y\frac{dz}{dt}-z\frac{dy}{dt}&=h_2 \end{aligned}\right\}\ldots(2).$$

These are three of the first integrals, and they contain the three arbitrary constants h, h_1, h_2.

Again, multiply the equations of motion by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, $2\frac{dz}{dt}$ respectively, add them, and integrate: then

$$\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}-\frac{2\mu}{r}=\text{const.}=c\ldots\ldots\ldots\ldots(3).$$

This is a fourth integral of the first order and contains the arbitrary constant c.

Again, multiply the first and second of the equations of motion by the second and third of (2) respectively: then by subtraction we have

$$h_1\frac{d^2x}{dt^2}-h_2\frac{d^2y}{dt^2}=-\frac{\mu x}{r^3}\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)+\frac{\mu y}{r^3}\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)$$

$$=\frac{\mu}{r^3}(x^2+y^2+z^2)\frac{dz}{dt}-\frac{\mu z}{r^3}\left(x\frac{dx}{dt}+y\frac{dy}{dt}+z\frac{dz}{dt}\right)$$

$$=\frac{\mu}{r}\frac{dz}{dt}-\frac{\mu z}{r^2}\frac{dr}{dt}=\mu\frac{d\left(\frac{z}{r}\right)}{dt};$$

$$\left.\begin{aligned} \therefore\ h_1\frac{dx}{dt} - h_2\frac{dy}{dt} &= \frac{\mu z}{r} + f, \\ \text{so } h_2\frac{dz}{dt} - h\frac{dx}{dt} = \frac{\mu y}{r} + f_1, \text{ and } h\frac{dy}{dt} - h_1\frac{dz}{dt} &= \frac{\mu x}{r} + f_2 \end{aligned}\right\}\ldots(4).$$

Thus we have three more integrals of the first order, containing the arbitrary constants f, f_1, f_2.

It would appear, then, that we have seven, and not only six, integrals of the first order: but we can shew, that any one of these seven is a consequence of the other six: and the constants h, h_1, h_2, f, f_1, f_2 are connected by an equation.

For multiplying equations (4) by h, h_1, h_2 respectively, and adding,

$$\therefore\ fh + f_1h_1 + f_2h_2 + \frac{\mu}{r}(hz + h_1y + h_2x) = 0,$$

and then by equations (2)

$$hf + h_1f_1 + h_2f_2 = 0 \ldots\ldots\ldots\ldots (5),$$

or the arbitrary constants have a necessary relation, and therefore the seven integrals found above are not independent integrals.

And moreover, since the seven integrals contained in (2) (3) (4) do not involve the time t *explicitly* it would appear, that by eliminating $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ we should obtain three final integrals functions of xyz without t: but this evidently cannot be the case. It follows, then, that the seven integrals must be equivalent to only five independent integrals: and the constants $h, h_1, h_2, c, f, f_1, f_2$ are connected by another relation. This relation is found as follows. Add the squares of equations (4);

$$\therefore\ f^2 + f_1^2 + f_2^2 + \frac{2\mu}{r}(fz + f_1y + f_2x) + \mu^2$$

$$= (h^2 + h_1^2 + h_2^2)\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\} - \left\{h\frac{dz}{dt} + h_1\frac{dy}{dt} + h_2\frac{dx}{dt}\right\}^2$$

$$= (h^2 + h_1^2 + h_2^2)\left(\frac{2\mu}{r} + c\right) \text{ by equations (3) and (2).}$$

But by equations (4) $fz + f_1 y + f_2 x + \mu r$

$$= h\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + h_1\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + h_2\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$= h^2 + h_1^2 + h_2^2;$$

$$\therefore\ f^2 + f_1^2 + f_2^2 = \mu^2 + (h^2 + h_1^2 + h_2^2)\,c\ldots\ldots(6),$$

this is the relation sought for.

We are unable to obtain a sixth integral of the first order by direct integration: and must therefore integrate the integrals already obtained to get a relation involving the time: this we shall do presently; the result will be one of the final integrals.

To obtain the other two final integrals we must eliminate the differential coefficients $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ from the integrals of the first order: to effect this multiply the equations (2) respectively by z, y, x and add;

$$\therefore\ hz + h_1 y + h_2 x = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

this proves that the undisturbed planet moves in a plane.

Again, multiply equations (4) by z, y, x respectively and add: then

$$fz + f_1 y + f_2 x + \mu\sqrt{x^2 + y^2 + z^2}$$

$$= h\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + h_1\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) + h_2\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)$$

$$= h^2 + h_1^2 + h_2^2 \text{ by (2)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

This is the equation to a surface of revolution of the second order, the origin of co-ordinates being in the focus: the equation to the plane generated by the directrix of the generating conic section is

$$fz' + f_1 y' + f_2 x' = h^2 + h_1^2 + h_2^2.$$

For the perpendicular from any point (xyz) of the surface on this plane

$$= \frac{fz + f_1 y + f_2 x - h^2 - h_1^2 - h_2^2}{\sqrt{f^2 + f_1^2 + f_2^2}} = -\frac{\mu r}{\sqrt{f^2 + f_1^2 + f_2^2}}.$$

Now r is the same for all points equally distant from the origin. Hence the surface must be one of revolution about an axis perpendicular to the plane of which the equation is

$$fz' + f_1 y' + f_2 x' = h^2 + h_1^2 + h_2^2.$$

Also the ratio of the perpendicular to the distance r is constant: and this is a property peculiar to the focus of conic sections. Hence the surface is a surface of the second order from the focus: and by combining this with the equation (7) we learn, that the planet moves in a conic section, the Sun being in the focus: (Art. 252).

To obtain the third integral add the squares of equations (2): then

$$(x^2+y^2+z^2)\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\} - \left\{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}\right\}^2 = h^2+h_1^2+h_2^2,$$

$$\text{and } r^2 = x^2 + y^2 + z^2;$$

$$\therefore \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} - \frac{dr^2}{dt^2} = \frac{h^2 + h_1^2 + h_2^2}{r^2} \;\ldots\ldots\; (9).$$

Let θ be the longitude of the planet at the time t measured on the plane of its orbit: then

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = r^2\frac{d\theta^2}{dt^2} + \frac{dr^2}{dt^2};$$

$$\therefore \frac{dt}{d\theta} = \frac{r^2}{\sqrt{h^2 + h_1^2 + h_2^2}}.$$

In this we must substitute for r in terms of θ by means of the two other integrals: and in integrating we shall introduce the sixth independent arbitrary constant: this constant is called the *epoch*, since it depends upon the epoch of the planet's perihelion passage.

Having integrated the equations of motion for an undisturbed planet we proceed to the following Proposition.

PROP. *To calculate the elements of the orbit in terms of the arbitrary constants introduced by the integration.*

354. Let i be the inclination of the plane of the orbit to the plane of the ecliptic; the ecliptic we shall take to be the plane of xy; Ω the longitude of the node, the axis of x being drawn through the first point of Aries; ϖ the longitude of the perihelion projected on the ecliptic; $2a$ the axis-major of the orbit; e the eccentricity; ϵ the epoch.

The equation to the plane of the orbit is $zh + yh_1 + xh_2 = 0$:

$$\therefore \cos i = \frac{h}{\sqrt{h^2 + h_1^2 + h_2^2}}, \quad \text{and } \tan i = \sqrt{\frac{h_1^2 + h_2^2}{h^2}}.$$

By putting $z = 0$ in the equation to the plane of the orbit we have $h_1 y + h_2 x = 0$, the equation to the line of nodes:

$$\therefore \tan \Omega = -\frac{h_2}{h_1}.$$

At the perihelion r is a minimum; $\therefore x\dfrac{dx}{dt} + y\dfrac{dy}{dt} + z\dfrac{dz}{dt} = 0$,

$$\text{also } \tan \varpi = \frac{y}{x} \text{ at that point:}$$

we must therefore find the value of this ratio at the perihelion: for this end we have

$$\frac{\mu y}{r} = h_2 \frac{dz}{dt} - h\frac{dx}{dt} - f_1 \text{ by (4) of last Article}$$

$$= y\left(\frac{dz^2}{dt^2} + \frac{dx^2}{dt^2}\right) - \frac{dy}{dt}\left(z\frac{dz}{dt} + x\frac{dx}{dt}\right) - f_1 \text{ by (2)}$$

$$= y\left(\frac{dz^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dx^2}{dt^2}\right) - f_1 \text{ at the perihelion;}$$

$$\text{so also } \frac{\mu x}{r} = x\left(\frac{dz^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dx^2}{dt^2}\right) - f_2 \text{ at the perihelion;}$$

$$\therefore \tan \varpi = \frac{y}{x} \text{ at perihelion} = \frac{f_1}{f_2}.$$

At the extremities of the axis-major $\dfrac{dr}{dt} = 0$, and therefore equations (3) and (9) of last Article give

$$\frac{h^2 + h_1^2 + h_2^2}{r^2} = \frac{2\mu}{r} + c;$$

$$\therefore\ r = -\frac{\mu}{c} \pm \frac{\sqrt{\mu^2 + (h^2 + h_1^2 + h_2^2)\,c}}{c};$$

$$\therefore\ a = -\frac{\mu}{c},$$

$$\text{and } e = \frac{\sqrt{\mu^2 + (h^2 + h_1^2 + h_2^2)\,c}}{\mu} = \frac{\sqrt{f^2 + f_1^2 + f_2^2}}{\mu}$$

by equation (6) of last Article.

Lastly, to find the *epoch* (ϵ) we must integrate the equation $\frac{dt}{d\theta} = \frac{r^2}{\sqrt{h^2 + h_1^2 + h_2^2}}$ after having substituted for r.

Having thus obtained the elements of the undisturbed orbit in terms of the constants, we will proceed to shew the importance of these expressions in determining the perturbations of a disturbed planet.

Prop. *To integrate the equations of motion of a disturbed planet.*

355. The equations of motion are

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} + \frac{dR}{dx} = 0, \quad \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} + \frac{dR}{dy} = 0,$$

$$\text{and } \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} + \frac{dR}{dz} = 0.$$

In conformity with the method of the variation of parameters invented by Lagrange, and explained in Art. 352, we shall assume that the following integrals (taken from Art. 353.) satisfy these equations, $h h_1 h_2\, c f f_1 f_2$ being *variables*,

$$h = x\frac{dy}{dt} - y\frac{dx}{dt}, \quad h_1 = z\frac{dx}{dt} - x\frac{dz}{dt}, \quad h_2 = y\frac{dz}{dt} - z\frac{dy}{dt},$$

$$c + \frac{2\mu}{r} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2},$$

$$f + \frac{\mu z}{r} = h_1 \frac{dx}{dt} - h_2 \frac{dy}{dt}, \quad f_1 + \frac{\mu y}{r} = h_2 \frac{dz}{dt} - h \frac{dx}{dt},$$

$$f_2 + \frac{\mu x}{r} = h \frac{dy}{dt} - h_1 \frac{dz}{dt},$$

and we now proceed to shew how to determine the values of the variables $h h_1 h_2 \, c f f_1 f_2$ in order that this may be the case.

Differentiate all these equations with respect to t and eliminate $\frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2}$ by the equations of motion: then

$$\frac{dh}{dt} = y \frac{dR}{dx} - x \frac{dR}{dy}, \quad \frac{dh_1}{dt} = x \frac{dR}{dz} - z \frac{dR}{dx}, \quad \frac{dh_2}{dt} = z \frac{dR}{dy} - y \frac{dR}{dz},$$

$$\frac{dc}{dt} = -2 \left\{ \frac{dR}{dx}\frac{dx}{dt} + \frac{dR}{dy}\frac{dy}{dt} + \frac{dR}{dz}\frac{dz}{dt} \right\} = -2 \frac{d(R)}{dt},$$

the brackets surrounding R implying that the *total* differential coefficient with respect to t is to be taken, *but this only in so far as* R *is a function of* xyz*.

$$\frac{df}{dt} = \frac{dh_1}{dt}\frac{dx}{dt} - \frac{dh_2}{dt}\frac{dy}{dt} - h_1 \frac{dR}{dx} + h_2 \frac{dR}{dy},$$

$$\frac{df_1}{dt} = \frac{dh_2}{dt}\frac{dz}{dt} - \frac{dh}{dt}\frac{dx}{dt} - h_2 \frac{dR}{dz} + h \frac{dR}{dx},$$

$$\frac{df_2}{dt} = \frac{dh}{dt}\frac{dy}{dt} - \frac{dh_1}{dt}\frac{dz}{dt} - h \frac{dR}{dy} + h_1 \frac{dR}{dz}.$$

356. The inclinations of the planes of the planetary orbits to the ecliptic are very small; the asteroids (of which the masses, however, are very small) being excepted. This is the case also with the eccentricities. We shall consequently neglect powers of these quantities higher than the square.

* R is also a function of t in consequence of being a function of $x'y'z'$, but the bracket is meant to imply that R is to be differentiated only in so far as it is a function of xyz.

By referring to the value of R (Art. 352.) it will be seen that $\frac{dR}{dx}$, $\frac{dR}{dy}$, $\frac{dR}{dz}$ all vary as m' the mass of the disturbing planet, which in our system is always extremely small in comparison with that of the Sun: we shall therefore neglect these quantities when they have small multipliers, and also their squares and higher powers.

The difference between all angles and distances measured on the plane of the orbit and their projections on the ecliptic varies as the ver. sine of the orbit's inclination, and therefore as the square of the angle of inclination nearly. This shews that in calculating the perturbations of the mean distance, the eccentricity, the longitude of the perihelion, and the epoch, we may neglect i and therefore $h_1^2 + h_2^2$ and consequently h_1 and h_2, and also f, Art. 353, equation (5): hence the equations of last Article become

$$\frac{dh}{dt} = y\frac{dR}{dx} - x\frac{dR}{dy}, \quad \frac{dc}{dt} = -2\frac{d(R)}{dt},$$

$$\frac{df_1}{dt} = \frac{dx}{dt}\left\{x\frac{dR}{dy} - y\frac{dR}{dx}\right\} + h\frac{dR}{dx},$$

$$\frac{df_2}{dt} = -\frac{dy}{dt}\left\{x\frac{dR}{dy} - y\frac{dR}{dx}\right\} - h\frac{dR}{dy}.$$

357. When we have expanded the function R then we must calculate the terms of these equations which involve the partial differential coefficients of R. After this we shall obtain the variations of the elements of the instantaneous orbit of the planet in terms of these variations of the arbitrary quantities $hh_1h_2cff_1f_2$. Then by integrating these we shall know the elements of the instantaneous orbit. Let $a_{,}e_{,}\varpi_{,}\epsilon_{,}i_{,}\Omega_{,}$ be these elements at the time t; the subscript accents being used to denote that *the elements are variable*. Then by substituting these in

$$r = a_{,}\left\{1 + \tfrac{1}{2}e_{,}^2 - e_{,}\cos(n_{,}t + \epsilon_{,} - \varpi_{,}) - \tfrac{1}{2}e_{,}^2\cos 2(n_{,}t + \epsilon_{,} - \varpi_{,}) - \ldots\right\}$$

$$\theta = n_{,}t + \epsilon_{,} + 2e_{,}\sin(n_{,}t + \epsilon_{,} - \varpi_{,})$$
$$+ \tfrac{5}{4}e_{,}^2\sin 2(n_{,}t + \epsilon_{,} - \varpi_{,}) + \ldots\ldots$$

we know the position of the planet in its orbit; the position of the orbit being known by $i_{,}$ and $\Omega_{,}$.

At present, however, we shall proceed to the transformation of R to polar co-ordinates.

PROP. *To determine* $\dfrac{dR}{dx}$, $\dfrac{dR}{dy}$ *in terms of* $\dfrac{dR}{d\theta}$, $\dfrac{dR}{dr}$.

358. In calculating these disturbing forces we may suppose r and θ the same as their projections on the plane of xy: for otherwise we should be retaining quantities varying as the product of the square of the inclination and disturbing force;

$$\therefore\ x = r\cos\theta,\quad y = r\sin\theta,\quad r^2 = x^2 + y^2,\quad \tan\theta = \frac{y}{x};$$

$$\therefore\ \frac{dR}{dx} = \frac{dR}{dr}\frac{dr}{dx} + \frac{dR}{d\theta}\frac{d\theta}{dx} = \frac{dR}{dr}\cos\theta - \frac{dR}{d\theta}\frac{\sin\theta}{r},$$

$$\frac{dR}{dy} = \frac{dR}{dr}\frac{dr}{dy} + \frac{dR}{d\theta}\frac{d\theta}{dy} = \frac{dR}{dr}\sin\theta + \frac{dR}{d\theta}\frac{\cos\theta}{r}.$$

359. But, since R is to be expanded in terms of t and the elements (Art. 349), we must still further transform these partial differential coefficients.

Upon examining the expansions of r and θ we see that $\epsilon_{,}$ and $\varpi_{,}$ are remarkably connected with $n_{,}t$: r is a function of $n_{,}t + \epsilon_{,} - \varpi_{,}$, and θ equals $n_{,}t + \epsilon_{,} +$ a function of $n_{,}t + \epsilon_{,} - \varpi_{,}$; and $\epsilon_{,}$ and $\varpi_{,}$ occur in no other way in r and θ and consequently in R. Hence by an analytical artifice we may consider R as a function of $\epsilon_{,}$ and $\varpi_{,}$ in consequence of its being a function of r and θ, and may change the variables from r and θ to $\epsilon_{,}$ and $\varpi_{,}$: this will be better understood by reading the next Proposition.

PROP. *To obtain* $\dfrac{d(R)}{dt}$, $\dfrac{dR}{d\theta}$, $\dfrac{dR}{dr}$ *in terms of the partial differential coefficients of* R *with respect to the elements.*

360. In $\frac{d(R)}{dt}$ R is supposed to be differentiated only inasmuch as it depends on the co-ordinates of the disturbed planet (Art. 355); viz. r and θ. Now by examining the expansions of r and θ we see that wherever t occurs $\epsilon_,$ is connected with it in the expression $n_, t + \epsilon_,$, and $\epsilon_,$ occurs in no other place in r and θ: hence

$$\frac{d(R)}{dt} = n_, \frac{dR}{d(n_, t + \epsilon_,)} = n_, \frac{dR}{d\epsilon_,}.$$

Again, to obtain $\frac{dR}{d\theta}$ and $\frac{dR}{dr}$ we observe, as before, that R is a function of $\epsilon_,$ and $\varpi_,$ solely because it is a function of r and θ;

$$\therefore \frac{dR}{d\epsilon_,} = \frac{dR}{d\theta}\frac{d\theta}{d\epsilon_,} + \frac{dR}{dr}\frac{dr}{d\epsilon_,}, \quad \frac{dR}{d\varpi_,} = \frac{dR}{d\theta}\frac{d\theta}{d\varpi_,} + \frac{dR}{dr}\frac{dr}{d\varpi_,}.$$

Now by referring to the expansions of r and θ we have

$$\frac{d\theta}{d\epsilon_,} + \frac{d\theta}{d\varpi_,} = 1 \text{ and } \frac{dr}{d\epsilon_,} + \frac{dr}{d\varpi_,} = 0;$$

in consequence of these the above equations give by addition

$$\frac{dR}{d\theta} = \frac{dR}{d\epsilon_,} + \frac{dR}{d\varpi_,}.$$

361. Again, to obtain $\frac{dR}{dr}$ we observe that r is a function of $\epsilon_,$ solely because it is a function of θ; for $\epsilon_,$ does not occur in the equation $\frac{1}{r} = \frac{1 + e_, \cos(\theta - \varpi_,)}{a_, (1 - e_,^2)}$;

$$\therefore \frac{dr}{d\epsilon_,} = \frac{dr}{d\theta}\frac{d\theta}{d\epsilon_,} = \frac{r^2 e_, \sin(\theta - \varpi_,)}{a_, (1 - e_,^2)} \frac{d\theta}{d\epsilon_,}$$

and $$\frac{d\theta}{d\epsilon_,} = \frac{d\theta}{d(n_, t + \epsilon_,)} = \frac{1}{n_,}\frac{d\theta}{dt} = \frac{\sqrt{a_, \mu (1 - e_,^2)}}{n_, r^2} \text{ (Art 273.)}$$

$$= \frac{a_,^2 \sqrt{1 - e_,^2}}{r^2}.$$

Substitute these in the formula

$$\frac{dR}{d\epsilon_{,}} = \frac{dR}{d\theta}\frac{d\theta}{d\epsilon_{,}} + \frac{dR}{dr}\frac{dr}{d\epsilon_{,}}$$

transpose and divide by $\frac{dr}{d\epsilon_{,}}$, and we have

$$\frac{dR}{dr} = \frac{\sqrt{1-e_{,}^{2}}}{a_{,}e_{,}\sin(\theta-\varpi_{,})}\left\{\frac{dR}{d\epsilon_{,}} - \frac{a_{,}^{2}\sqrt{1-e_{,}^{2}}}{r^{2}}\frac{dR}{d\theta}\right\}.$$

We shall find the following Proposition of use hereafter in reducing our formulæ.

PROP. *To obtain* $\frac{dR}{de_{,}}$ *in terms of* $\frac{dR}{d\epsilon_{,}}$ *and* $\frac{dR}{d\varpi_{,}}$.

362. Since R is a function of $e_{,}$ solely because it is a function of r and θ;

$$\therefore \frac{dR}{de_{,}} = \frac{dR}{d\theta}\frac{d\theta}{de_{,}} + \frac{dR}{dr}\frac{dr}{de_{,}}.$$

We must therefore calculate $\frac{d\theta}{de_{,}}$ and $\frac{dr}{de_{,}}$.

$$\text{Now } \theta = n_{,}t + \epsilon_{,} + (2e_{,} - \tfrac{1}{4}e_{,}^{3})\sin(n_{,}t + \epsilon_{,} - \varpi_{,}) + \ldots$$

$$\text{and } r = a_{,}\left\{1 + \tfrac{1}{2}e_{,}^{2} - e_{,}\cos(n_{,}t + \epsilon_{,} - \varpi_{,}) - \ldots\ldots\right\},$$

and from these we should obtain $\frac{d\theta}{de_{,}}$ and $\frac{dr}{de_{,}}$: but since we do not know the *law* of these series we must refer to the functions from which they were developed, viz: (Arts. 273, 279.)

$$n_{,}t = u - e_{,}\sin u,\ \tan\tfrac{1}{2}u = \sqrt{\frac{1-e_{,}}{1+e_{,}}}\tan\tfrac{1}{2}(\theta-\varpi_{,}),$$

$$\text{and } r = \frac{a_{,}(1-e_{,}^{2})}{1+e_{,}\cos(\theta-\varpi_{,})} = a_{,}(1-e_{,}\cos u),$$

θ is expanded from the first and second of these, and then substituted in the value of r, and r is expanded.

To calculate $\frac{d\theta}{de_{,}}$ differentiate the first of the above equations, and also the logarithm of the second, with respect to $e_{,}$;

$$\therefore\ 0 = (1 - e_{,} \cos u)\frac{du}{de_{,}} - \sin u,$$

and $$\frac{1}{\sin u}\frac{du}{de_{,}} = -\frac{1}{1-e_{,}^2} + \frac{1}{\sin(\theta - \varpi_{,})}\frac{d\theta}{de_{,}}.$$

Eliminate $\dfrac{1}{\sin u}\dfrac{du}{de_{,}}$ from these;

$$\therefore\ \frac{1}{\sin(\theta-\varpi_{,})}\frac{d\theta}{de_{,}} = \frac{1}{1-e_{,}^2} + \frac{1+e_{,}\cos(\theta-\varpi_{,})}{1-e_{,}^2};$$

$$\therefore\ \frac{d\theta}{de_{,}} = \frac{\sin(\theta-\varpi_{,})}{1-e_{,}^2}\left\{2 + e_{,}\cos(\theta-\varpi_{,})\right\}.$$

Also, since the series for r is obtained by developing the function $\dfrac{a_{,}(1-e_{,}^2)}{1+e_{,}\cos(\theta-\varpi_{,})}$ after substituting for θ,

$$\therefore\ \frac{dr}{de_{,}} = \left(\frac{dr}{de_{,}}\right) + \frac{dr}{d\theta}\frac{d\theta}{de_{,}}$$

$$= \frac{a_{,}\left\{-2e_{,} - (1+e_{,}^2)\cos(\theta-\varpi_{,})\right\}}{\left\{1+e_{,}\cos(\theta-\varpi_{,})\right\}^2} + \frac{a_{,}e_{,}\sin^2(\theta-\varpi_{,})\left\{2+e_{,}\cos(\theta-\varpi_{,})\right\}}{\left\{1+e_{,}\cos(\theta-\varpi_{,})\right\}^2}$$

$$= -a_{,}\cos(\theta-\varpi_{,});$$

$$\therefore\ \frac{dR}{de_{,}} = \frac{dR}{d\theta}\frac{\sin(\theta-\varpi_{,})\left\{2+e_{,}\cos(\theta-\varpi_{,})\right\}}{1-e_{,}^2} - a_{,}\frac{dR}{dr}\cos(\theta-\varpi_{,}).$$

We now proceed to obtain the formulæ for calculating the variations of the elements.

PROP. *To calculate the variations of the mean distance, the eccentricity, and the longitude of the perihelion of the instantaneous orbit of the disturbed planet.*

363. Let $a_{,}$, $e_{,}$, $\varpi_{,}$ be these elements; the subscript accents indicating that the elements are functions of t. Then by Art. 354,

1. The mean distance $a_{,} = -\dfrac{\mu}{c}$; $\therefore\ \dfrac{1}{a_{,}} = -\dfrac{c}{\mu}$;

$$\therefore \frac{da_{\prime}}{dt} = \frac{a_{\prime}^2}{\mu}\frac{dc}{dt} = -\frac{2a_{\prime}^2}{\mu}\frac{d(R)}{dt} \text{ by Art. 356.}$$

$$= -\frac{2n_{\prime}a_{\prime}^2}{\mu}\frac{dR}{d\epsilon_{\prime}} \text{ by Art. 360.}$$

2. The eccentricity

$$e_{\prime} = \frac{1}{\mu}\sqrt{f^2 + f_1^2 + f_2^2} = \frac{1}{\mu}\sqrt{f_1^2 + f_2^2} \text{ (Art. 356)};$$

$$\therefore \frac{de_{\prime}}{dt} = \frac{1}{\mu\sqrt{f_1^2 + f_2^2}}\left\{f_1\frac{df_1}{dt} + f_2\frac{df_2}{dt}\right\}$$

$$= \frac{1}{\mu}\left\{\sin\varpi_{\prime}\frac{df_1}{dt} + \cos\varpi_{\prime}\frac{df_2}{dt}\right\}; \because \tan\varpi_{\prime} = \frac{f_1}{f_2}.$$

But by Arts. 356, 358,

$$\frac{df_1}{dt} = \left\{\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right\}\frac{dR}{d\theta} - h\left\{\frac{dR}{d\theta}\frac{\sin\theta}{r} - \frac{dR}{dr}\cos\theta\right\},$$

$$\left\{\text{but in small terms } h = \sqrt{a_{\prime}\mu(1-e_{\prime}^2)}, \text{ and } = r^2\frac{d\theta}{dt}\right\}$$

$$= \sqrt{a_{\prime}\mu(1-e_{\prime}^2)}\left\{\left(\frac{\cos\theta}{r^2}\frac{dr}{d\theta} - \frac{2\sin\theta}{r}\right)\frac{dR}{d\theta} + \cos\theta\frac{dR}{dr}\right\}$$

$$\frac{df_2}{dt} = -\left\{\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right\}\frac{dR}{d\theta} - h\left\{\frac{dR}{d\theta}\frac{\cos\theta}{r} + \frac{dR}{dr}\sin\theta\right\}$$

$$= -\sqrt{a_{\prime}\mu(1-e_{\prime}^2)}\left\{\left(\frac{\sin\theta}{r^2}\frac{dr}{d\theta} + \frac{2\cos\theta}{r}\right)\frac{dR}{d\theta} + \sin\theta\frac{dR}{dr}\right\};$$

$$\therefore \frac{de_{\prime}}{dt} = \sqrt{\frac{a_{\prime}(1-e_{\prime}^2)}{\mu}}$$

$$\times\left\{-\left(\frac{\sin(\theta-\varpi_{\prime})}{r^2}\frac{dr}{d\theta} + \frac{2\cos(\theta-\varpi_{\prime})}{r}\right)\frac{dR}{d\theta} - \sin(\theta-\varpi_{\prime})\frac{dR}{dr}\right\},$$

{putting $\frac{1}{r} = \frac{1 + e_{\prime}\cos(\theta-\varpi_{\prime})}{a_{\prime}(1-e_{\prime}^2)}$ in small terms, and using the properties proved in Arts. 360, 361.}

$$= \sqrt{\frac{a_{,}(1-e_{,}^2)}{\mu}}$$

$$\times \left\{\frac{-e_{,}^2 \sin^2(\theta-\varpi_{,}) - e_{,}^2\cos^2(\theta-\varpi_{,}) + 1}{a_{,}e_{,}(1-e_{,}^2)} \frac{dR}{d\theta} - \frac{\sqrt{1-e_{,}^2}}{a_{,}e_{,}} \frac{dR}{d\epsilon_{,}}\right\}$$

$$= \sqrt{\frac{1-e_{,}^2}{a_{,}e_{,}^2\mu}} \left(\frac{dR}{d\epsilon_{,}} + \frac{dR}{d\varpi_{,}}\right) - \frac{1-e_{,}^2}{e_{,}\sqrt{\mu a_{,}}} \frac{dR}{d\epsilon_{,}}.$$

3. For the longitude of the perihelion $\tan \varpi_{,} = \dfrac{f_1}{f_2}$;

$$\therefore \frac{d\varpi_{,}}{dt} = \frac{1}{f_1^2 + f_2^2}\left\{f_2 \frac{df_1}{dt} - f_1 \frac{df_2}{dt}\right\}$$

$$= \frac{1}{\mu e_{,}}\left\{\cos \varpi_{,} \frac{df_1}{dt} - \sin \varpi_{,} \frac{df_2}{dt}\right\}$$

$$= \sqrt{\frac{a_{,}(1-e_{,}^2)}{\mu e_{,}^2}}$$

$$\times \left\{\left(\frac{\cos(\theta-\varpi_{,})}{r^2}\frac{dr}{d\theta} - \frac{2\sin(\theta-\varpi_{,})}{r}\right)\frac{dR}{d\theta} + \cos(\theta-\varpi_{,})\frac{dR}{dr}\right\}$$

$$= \sqrt{\frac{a_{,}(1-e_{,}^2)}{\mu e_{,}^2}}$$

$$\times \left\{-\frac{\sin(\theta-\varpi_{,})\{2+e_{,}\cos(\theta-\varpi_{,})\}}{a_{,}(1-e_{,}^2)}\frac{dR}{d\theta} + \cos(\theta-\varpi_{,})\frac{dR}{dr}\right\}$$

$$= -\sqrt{\frac{1-e_{,}^2}{a_{,}\mu e_{,}^2}}\frac{dR}{de_{,}} \text{ by Art. 362.}$$

PROP. *To find the variations of the inclination and the longitude of the node.*

364. We have by Art. 354 the formulæ

$$\tan i_{,} = \frac{\sqrt{h_1^2 + h_2^2}}{h}, \quad \tan \Omega_{,} = -\frac{h_2}{h_1};$$

$$\therefore \tan i_{,} \sin \Omega_{,} = -\frac{h_2}{h}, \quad \tan i_{,} \cos \Omega_{,} = \frac{h_1}{h}.$$

Hence $\dfrac{d(\tan i_{,}\sin\Omega_{,})}{dt} = \dfrac{1}{h^2}\left\{h_2\dfrac{dh}{dt} - h\dfrac{dh_2}{dt}\right\}.$

Substituting by the equations of Art. 355, and neglecting z and $\dfrac{dz}{dt}$ as being small,

$$\sin\Omega_{,}\frac{d\tan i_{,}}{dt} + \cos\Omega_{,}\tan i_{,}\frac{d\Omega_{,}}{dt} = \frac{y}{h}\frac{dR}{dz}.$$

In a similar manner by differentiating $\tan i_{,}\cos\Omega_{,} = \dfrac{h_1}{h}$,

$$\cos\Omega_{,}\frac{d\tan i_{,}}{dt} - \sin\Omega_{,}\tan i_{,}\frac{d\Omega_{,}}{dt} = \frac{x}{h}\frac{dR}{dz}.$$

Multiplying these equations by $\sin\Omega_{,}$ and $\cos\Omega_{,}$ respectively and adding

$$\frac{d\tan i_{,}}{dt} = \frac{1}{h}(y\sin\Omega_{,} + x\cos\Omega_{,})\frac{dR}{dz}.$$

Multiplying by $\cos\Omega_{,}$, $\sin\Omega_{,}$ and subtracting

$$\tan i_{,}\frac{d\Omega_{,}}{dt} = \frac{1}{h}(y\cos\Omega_{,} - x\sin\Omega_{,})\frac{dR}{dz}.$$

Now $\dfrac{dR}{di_{,}} = \dfrac{dR}{dx}\dfrac{dx}{di_{,}} + \dfrac{dR}{dy}\dfrac{dy}{di_{,}} + \dfrac{dR}{dz}\dfrac{dz}{di_{,}} = \dfrac{dR}{dz}\dfrac{dz}{di_{,}}$ nearly;

and similarly $\dfrac{dR}{d\Omega_{,}} = \dfrac{dR}{dz}\dfrac{dz}{d\Omega_{,}}$ nearly;

since for a given alteration in the inclination or longitude of the node the alteration in z is much greater than in x or y.

Also $z = -\dfrac{h_2}{h}x - \dfrac{h_1}{h}y$ by Art. 353, equation (7).

$$= \tan i_{,}(x\sin\Omega_{,} - y\cos\Omega_{,}),$$

$$\therefore\ \frac{dz}{di_{,}} = \sec^2 i_{,}(x\sin\Omega_{,} - y\cos\Omega_{,}) \text{ nearly;}$$

$$\frac{dz}{d\Omega_{,}} = \tan i_{,}(x\cos\Omega_{,} + y\sin\Omega_{,}) \text{ nearly.}$$

Hence $\dfrac{d\tan i_{,}}{dt} = \dfrac{1}{h\tan i_{,}}\dfrac{dR}{d\Omega_{,}}$;

$$\frac{d\Omega_{,}}{dt} = -\frac{1}{h\tan i_{,}}\frac{dR}{di_{,}}, \text{ neglecting } \tan^2 i_{,} \ldots$$

PROP. *To find the variation of the epoch.*

365. Now R is a function of $a_{,} e_{,} \varpi_{,} n_{,} t + \epsilon_{,} i_{,} \Omega_{,}$, and since the instantaneous ellipse is an ellipse of curvature to the orbit described of the first order (Art. 350), it follows that the first differential coefficients of r and θ with respect to t will be the same in the real orbit and in the instantaneous ellipse. The same will be the case with the first differential coefficient of any function of r and θ, as R.

Now in the ellipse $\dfrac{d(R)}{dt} = n_{,}\dfrac{dR}{d\epsilon_{,}}$; and in the real orbit, since R is a function of the variable quantities $a_{,}$, $e_{,}$, $\varpi_{,}$, $n_{,}t + \epsilon_{,}$, $i_{,}$, and $\Omega_{,}$;

$$\therefore \frac{d(R)}{dt} = \frac{dR}{da_{,}}\frac{da_{,}}{dt} + \frac{dR}{de_{,}}\frac{de_{,}}{dt} + \frac{dR}{d\varpi_{,}}\frac{d\varpi_{,}}{dt}$$

$$+ \frac{dR}{d\epsilon_{,}}\left\{n_{,} + t\frac{dn_{,}}{dt} + \frac{d\epsilon_{,}}{dt}\right\} + \frac{dR}{di_{,}}\frac{di_{,}}{dt} + \frac{dR}{d\Omega_{,}}\frac{d\Omega_{,}}{dt}.$$

Equating these values of $\dfrac{d(R)}{dt}$ and substituting for $\dfrac{da_{,}}{dt}$, $\dfrac{de_{,}}{dt}$, $\dfrac{d\varpi_{,}}{dt}$, $\dfrac{dn_{,}}{dt}\left(= -\dfrac{3n_{,}}{2a_{,}}\dfrac{da_{,}}{dt}\right)$, $\dfrac{di_{,}}{dt}$, $\dfrac{d\Omega_{,}}{dt}$ the values found in Arts. 363, 364, and transposing $\dfrac{d\epsilon_{,}}{dt}$ and dividing by $\dfrac{dR}{d\epsilon_{,}}$, we have

$$\frac{d\epsilon_{,}}{dt} = -\frac{3n_{,}^2 a_{,}}{\mu}\frac{dR}{d\epsilon_{,}}t + \frac{2n_{,}a_{,}^2}{\mu}\frac{dR}{da_{,}} - \frac{n_{,}a_{,}}{\mu e_{,}}\left\{\sqrt{1-e_{,}^2} - (1-e_{,}^2)\right\}\frac{dR}{de_{,}}.$$

366. We shall now bring together the variations of the elliptic elements obtained in the last three Articles, and present them under one point of view.

(1) $$\frac{da_{,}}{dt} = -\frac{2a_{,}^2 n_{,}}{\mu}\frac{dR}{d\epsilon_{,}}.$$

(2) $$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}\sqrt{1-e_{,}^2}}{\mu_{,}e_{,}}\left(\frac{dR}{d\epsilon_{,}} + \frac{dR}{d\varpi_{,}}\right) - \frac{n_{,}a_{,}(1-e_{,}^2)}{\mu e_{,}}\frac{dR}{d\epsilon_{,}}.$$

(3) $$\frac{d\varpi_{,}}{dt} = -\frac{n_{,}a_{,}\sqrt{1-e_{,}^2}}{\mu e_{,}}\frac{dR}{de_{,}}.$$

(4) $$\frac{d\epsilon_{,}}{dt} = -\frac{3n_{,}^2 a_{,}}{\mu}\frac{dR}{de_{,}}t + \frac{2n_{,}a_{,}^2}{\mu}\frac{dR}{da_{,}}$$
$$-\frac{n_{,}a_{,}}{\mu e_{,}}\left\{\sqrt{1-e_{,}^2} - (1-e_{,}^2)\right\}\frac{dR}{de_{,}}.$$

(5) $$\frac{d\tan i_{,}}{dt} = \frac{n_{,}a_{,}}{\mu\tan i_{,}\sqrt{1-e_{,}^2}}\frac{dR}{d\Omega_{,}}.$$

(6) $$\frac{d\Omega_{,}}{dt} = -\frac{n_{,}a_{,}}{\mu\tan i_{,}\sqrt{1-e_{,}^2}}\frac{dR}{di_{,}}.$$

Before we can make use of these formulæ we must explain how R is to be developed.

PROP. *To explain the manner in which* R *is to be developed.*

367. If we recur to Art. 352 we see that

$$R = \frac{m'(xx' + yy' + zz')}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} - \frac{m'}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}}.$$

Let r and θ be the radius-vector and longitude of m measured on the plane of xy as far as the node and then on the plane of m's orbit: $r_{,}$ and $\theta_{,}$ the rad. vect. and longitude projected on plane xy: $\Omega_{,}$ and $i_{,}$ the longitude of the node and inclination of m's orbit to the plane xy: λ the latitude of m.

Then we have

$$x = r_{,}\cos\theta_{,},\ \ y = r_{,}\sin\theta_{,},\ \ z = r_{,}\tan\lambda = r_{,}\sin(\theta_{,} - \Omega_{,})\tan i_{,},$$

Similar expressions are true for m'.

Hence R

$$= \frac{m'\{r_{,}r_{,}'\cos(\theta_{,}-\theta_{,}') + zz'\}}{(r_{,}'^2 + z'^2)^{\frac{3}{2}}} - \frac{m'}{\sqrt{r_{,}^2 + r_{,}'^2 - 2r_{,}r_{,}'\cos(\theta_{,}-\theta_{,}') + (z-z')^2}}.$$

Now $r_{,} = r\cos\lambda = r\left\{1 - \frac{1}{2}\tan^2\lambda\right\}$ very nearly

$$= r\left\{1 - \tfrac{1}{2}\tan^2 i_{,}\sin^2(\theta_{,} - \Omega_{,})\right\}$$

$$= r\left\{1 - \tfrac{1}{4}\tan^2 i_{,} + \tfrac{1}{4}\tan^2 i_{,}\cos 2(\theta_{,} - \Omega_{,})\right\}.$$

Also $\tan(\theta_{,} - \Omega_{,}) = \cos i_{,}\tan(\theta - \Omega_{,})$;

$$\therefore\ \theta_{,} - \Omega_{,} = \theta - \Omega_{,} - \tan^2\tfrac{1}{2}i_{,}\sin 2(\theta - \Omega_{,}) \text{ nearly.}$$

Substitute in these the values of r and θ given by Art. 357, and we have

$$r_{,} = a_{,}\left\{1 + \tfrac{1}{2}e_{,}^2 - e_{,}\cos(n_{,}t + \epsilon_{,} - \varpi_{,}) - \tfrac{1}{2}e_{,}^2\cos 2(n_{,}t + \epsilon_{,} - \varpi_{,})\right.$$
$$\left. - \tfrac{1}{4}\tan^2 i_{,} + \tfrac{1}{4}\tan^2 i_{,}\cos 2(n_{,}t + \epsilon_{,} - \Omega_{,}) + \dots\right\}$$
$$= a_{,}\{1 + u\}$$

and $\theta_{,} = n_{,}t + \epsilon_{,} + 2e_{,}\sin(n_{,}t + \epsilon_{,} - \varpi_{,}) + \frac{5}{4}e_{,}^2\sin 2(n_{,}t + \epsilon_{,} - \varpi_{,})$

$$- \tan^2\tfrac{1}{2}i_{,}\sin 2(n_{,}t + \epsilon_{,} - \Omega_{,}) + \dots$$
$$= n_{,}t + \epsilon_{,} + \nu \text{ suppose.}$$

Now let R' be the value of R when $a_{,}$ and $a_{,}'$ are put for $r_{,}$ and $r_{,}'$ and suppose $r_{,} = a_{,}(1 + u)$ and $r_{,}' = a_{,}'(1 + u')$ u and u' are small quantities because the orbits of the planets are nearly circular: then by Taylor's Theorem

$$R = R' + \frac{dR'}{da_{,}}a_{,}u + \frac{dR'}{da_{,}'}a_{,}'u' + \dots$$

also $R' = \dfrac{m'\{a_{,}a_{,}'\cos(\theta_{,} - \theta_{,}') + a_{,}a_{,}'\tan i_{,}\tan i_{,}'\sin(\theta_{,} - \Omega_{,})\sin(\theta_{,}' - \Omega_{,}')\}}{\{a_{,}'^2 + a_{,}'^2\tan^2 i_{,}'\sin^2(\theta_{,}' - \Omega_{,}')\}^{\frac{3}{2}}}$

$$- \frac{m'}{\sqrt{a_{,}^2 + a_{,}'^2 - 2a_{,}a_{,}'\cos(\theta_{,} - \theta_{,}') + \{a_{,}\tan i_{,}\sin(\theta_{,} - \Omega_{,}) - a_{,}'\tan i_{,}'\sin(\theta_{,}' - \Omega_{,}')\}^2}}$$

$$= \frac{m'a_{,}\cos(\theta_{,} - \theta_{,}')}{a_{,}'^2} - \frac{m'}{\sqrt{a_{,}^2 + a_{,}'^2 - 2a_{,}a_{,}'\cos(\theta_{,} - \theta_{,}')}}$$

$$+ \frac{m'a_{,}\tan i_{,}\tan i_{,}'\sin(\theta_{,} - \Omega_{,})\sin(\theta_{,}' - \Omega_{,}')}{a_{,}'^2}$$

$$- \frac{3m'a_{,}\tan^2 i_{,}'\sin^2(\theta_{,}' - \Omega_{,}')\cos(\theta_{,} - \theta_{,}')}{2a_{,}'^2}$$

$$+ \frac{m'\{a_{,}\tan i_{,}\sin(\theta_{,} - \Omega_{,}) - a_{,}'\tan i_{,}'\sin(\theta_{,}' - \Omega_{,}')\}^2}{2\{a_{,}^2 + a_{,}'^2 - 2a_{,}a_{,}'\cos(\theta_{,} - \theta_{,}')\}^{\frac{3}{2}}} + \dots$$

$$\text{Let } \frac{1}{\sqrt{a_{,}^2 + a_{,}'^2 - 2a_{,}a_{,}'\cos(\theta_{,} - \theta_{,}')}}$$

$$= \tfrac{1}{2}C_0 + C_1\cos(\theta_{,} - \theta_{,}') + C_2\cos 2(\theta_{,} - \theta_{,}') + \ldots\ldots$$

$$\frac{1}{\{a_{,}^2 + a_{,}'^2 - 2a_{,}a_{,}'\cos(\theta_{,} - \theta_{,}')\}^{\frac{3}{2}}}$$

$$= \tfrac{1}{2}D_0 + D_1\cos(\theta_{,} - \theta_{,}') + D_2\cos 2(\theta_{,} - \theta_{,}') + \ldots\ldots$$

These coefficients should be calculated and then R' may be arranged in a series. When we have thus calculated R' we must find $\dfrac{dR'}{da_{,}}$, $\dfrac{dR'}{da_{,}'}$ ......and substitute them in

$$R' + \frac{dR'}{da_{,}}a_{,}u + \frac{dR'}{da_{,}'}a_{,}'u' + \ldots\ldots\ldots$$

and we shall have R expressed in a series of terms depending on the time and the elements of the instantaneous orbits.

It is not our object to enter into the numerical calculation of the coefficients of the expansion of R: for this we refer the reader to M. Pontécoulant's *Theorie Analytique du Système du Monde*, Tom. I. p. 340, *Mécanique Céleste*, Tom. III. and Mr. Lubbock's Papers in the *Philos. Trans.* and *Astron. Trans.*

We proceed to demonstrate some Propositions relative to the general nature of the terms.

PROP. *To prove that the terms of* R *which depend on the mean anomalies* ($n_{,}t$ *and* $n_{,}'t$) *of the planets are of the form* $P\cos\{(pn_{,} - qn_{,}')t + Q\}$ *or* $P\cos\{(pn_{,} + qn_{,}')t + Q\}$, *where* P *is a function of the mean distances, eccentricities, and inclinations of the orbits, and* Q *is a function of the longitudes of the perihelia and nodes and of the epochs; and* p *and* q *are positive integers.*

368. We shall make use of the following elementary trigonometrical formulæ:

I. $\quad \text{Cos}\, a\cos b = \tfrac{1}{2}\cos(a - b) + \tfrac{1}{2}\cos(a + b)$.

II. $\quad \text{Sin}\, a\sin b = \tfrac{1}{2}\cos(a - b) - \tfrac{1}{2}\cos(a + b)$.

III. $\quad \text{Sin}\, a\cos b = \tfrac{1}{2}\sin(a + b) + \tfrac{1}{2}\sin(a - b)$.

$$\text{Now } \theta_{,} - \theta_{,}' = (n_{,}t + \epsilon_{,}) - (n_{,}'t + \epsilon_{,}')$$
$$+ (2e_{,} \ldots\ldots) \sin(n_{,}t + \epsilon_{,} - \varpi_{,}) + (\tfrac{5}{4}e_{,}^2 + \ldots) \sin 2(n_{,}t + \epsilon_{,} - \varpi_{,}) + \ldots$$
$$- (2e_{,}' + \ldots) \sin(n_{,}'t + \epsilon_{,}' - \varpi_{,}') - (\tfrac{5}{4}e_{,}'^2 + \ldots) \sin 2(n_{,}'t + \epsilon_{,}' - \varpi_{,}') + \ldots$$
$$- \tan^2 \tfrac{1}{2} i_{,} \sin 2(n_{,}t + \epsilon_{,} - \Omega_{,}) + \ldots\ldots$$
$$+ \tan^2 \tfrac{1}{2} i_{,}' \sin 2(n_{,}'t + \epsilon_{,}' - \Omega_{,}') + \ldots\ldots$$
$$= (n_{,}t + \epsilon_{,}) - (n_{,}'t + \epsilon_{,}') + T \text{ suppose;}$$
$$\therefore \cos k(\theta_{,} - \theta_{,}') = \cos\{k(n_{,}t + \epsilon_{,}) - k(n_{,}'t + \epsilon_{,}')\} \cos kT$$
$$- \sin\{k(n_{,}t + \epsilon_{,}) - k(n_{,}'t + \epsilon_{,}')\} \sin kT,$$
$$\text{and } \cos kT = 1 - \tfrac{1}{2}k^2T^2 + \ldots\ldots$$
$$\sin kT = kT - \frac{1}{1 \cdot 2 \cdot 3} k^3T^3 + \ldots\ldots$$

Now by formulæ II. and I, the even powers of T, and $\therefore \cos kT$, will involve only simple cosines; and by formulæ II. and III. the odd powers of T, and therefore $\sin kT$, only simple sines.

Hence by formulæ I. and II. the expansion of $\cos k(\theta_{,} - \theta_{,}')$ given by the above formula will contain only simple cosines.

In the same manner we might shew that $\sin(\theta_{,} - \Omega_{,})$ and $\sin(\theta_{,}' - \Omega_{,}')$ will equal a series of simple sines (with no constant term), and therefore by formula II. the squares or product of these will contain only simple cosines.

We see, then, that when the complete development of R' given in Art. 367. is worked out and arranged in a series, it will consist only of simple cosines.

Again by Art. 367. we see, that u and u' consist of a series of terms involving the simple cosines of angles. Hence, by formula I. each of the quantities u, u', u^2 uu', u'^2... will consist of a series of simple cosines of angles formed by combining the arguments* of the terms of u and u' in endless variety by addition and subtraction.

It follows, then, finally, that the series into which R is to be developed (Art. 367.) will by formula I. consist only of terms of the form

* In an expression $a\cos(pnt+q)$, the angle $pnt+q$ is called the *argument* of the term $a\cos(pnt+q)$.

$$P\cos\{(pn_{,}-qn_{,}')t+Q\} \text{ and } P\cos\{(pn_{,}+qn_{,}')t+Q\},$$

p and q being positive integers, P a function of the mean distances, eccentricities, and inclinations of the instantaneous orbits; and Q a function of the longitudes of the perihelia and nodes and the epochs.

369. We have already frequently remarked, that the eccentricities and inclinations of the planets are so small, that their higher powers will be of almost imperceptible magnitude. It becomes important then to search for some means of determining the relative magnitude of P in reference to the argument $(pn_{,}-qn_{,}')t+Q$ for this will materially shorten the calculation of R, by pointing out at once those terms of the infinite series into which R is developed, which are of sufficient importance to be retained.

In the following Article we shall prove a principle which answers our purpose.

Prop. *The lowest dimension of the quantities* $e_{,}$, $e_{,}'$, $\tan i_{,}$, $\tan i_{,}'$ *in the coefficient of* $P\cos\{(pn_{,}-qn_{,}')t+Q\}$ *is of the order* $p\sim q$.

370. We have by Art. 367,

$$R = R' + \frac{dR'}{da_{,}}a_{,}u + \frac{dR'}{da_{,}'}a_{,}'u' + \ldots\ldots$$

(1) A remarkable law prevails in the expansions of u and u'. It is this (Art. 367). The number which multiplies $n_{,}t+\epsilon_{,}$ in the argument of any term in these expansions represents the dimensions in $e_{,}$, $e_{,}'$, $\tan i_{,}$, $\tan i_{,}'$ of the principal part of the coefficient of that term; the *principal* part being that which is of lowest dimension.

Now the same holds good in any power of u or u'. Thus in u^2 a term $P\cos(pn_{,}t+P')$ can arise only in the following ways; partly from the multiplication of two terms in u of which the arguments are $ln_{,}t+L$ and $mn_{,}t+M$, where $l+m=p$; and partly from such as have the arguments $l'n_{,}t+L$ and $m'n_{,}t+M$, where $l'-m'=p$. In the former case the dimension of the principal part of the coefficient will be $l+m=p$, in the latter it will be $l'+m'$ and this is greater than p.

Hence the principal part of the coefficient of a term $P \cos (pn_{,}t + P')$ in u^2 will be of the dimension p.

The same is evidently true of $u'^2, u^3, u'^3, \ldots\ldots$

(2) In the product of any powers of u and u' as $u^{\alpha}u'^{\beta}$, the dimension is the *sum* of the multipliers of nt and $n't$.

For let us consider a term $N \cos \{(ln_{,} \pm l'n_{,}')\,t + M\}$. Now this must evidently have arisen from the multiplication of $\cos (ln_{,}t + L)$ and $\cos (l'n_{,}'t + L')$ in u^{α} and u'^{β} respectively. The principal coefficient of this is of the order $l + l'$.

(3) Let us next consider the law of the coefficients in $\cos k\,(\theta_{,} - \theta_{,}')$.

If we turn to Art. 368. and examine the expansions of $\cos kT$ and $\sin kT$, we shall find that the laws (1) and (2) hold equally in them. But in $\cos k\,(\theta_{,} - \theta_{,}')$, since it is equal to

$$\cos \{k\,(n_{,}t + \epsilon_{,}) - k\,(n_{,}'t + \epsilon_{,}')\} \cos kT$$
$$- \sin \{k\,(n_{,}t + \epsilon_{,}) - k\,(n_{,}'t + \epsilon_{,}')\} \sin kT,$$

the dimension of the coefficient of any term calculated by the laws (1), (2) will be higher or lower by $2k$ than it ought to be according as the argument is formed by addition or subtraction.

If, then, we turn to Art. 367. and examine the expression given for R we see that the laws (1), (2) just proved hold for R, if we leave out of consideration all the multipliers which are of the form $\cos k\,(\theta_{,} - \theta_{,}')$. Bearing this in mind we shall be able to prove our Proposition.

Any term $P \cos \{(pn_{,} - qn_{,}')\,t + Q\}$ in R has partly arisen from the multiplication of $\cos \{(kn_{,} - kn_{,}')\,t + Q'\}$ with $\cos \{[(p - k)\,n_{,} - (q - k)\,n_{,}']\,t + Q''\}$ and partly from multiplication with $\cos \{[(p + k)\,n_{,} - (q + k)\,n_{,}']\,t + Q'''\}$, and in no other way can it have been formed: k being any number of the series 0, 1, 2, 3,......

First: suppose k intermediate to p and q. Then the first of these cosines becomes

$$\cos \{[(p - k)\,n_{,} + (k - q)\,n_{,}']\,t + Q''\},$$

and the dimension of the principal coefficient of this and therefore of $\cos\{(pn_{,} - qn_{,}')t + Q\}$ is by law (2) equal to $(p \sim k) + (k \sim q) = p \sim q$, since k is intermediate to p and q.

Second: suppose k is not greater than the smaller of p and q. Then the dimension of the principal coefficient of

$$\cos\{[(p-k)n_{,} - (q-k)n_{,}']t + Q''\} \text{ is } p+q-2k:$$

and therefore the dimension of the principal coefficient of $\cos\{(pn' - qn_{,}')t + Q\}$ is the least value of which $p + q - 2k$ is susceptible, and that is $p \sim q$.

Third: suppose k is not less than the greater of p and q. Then the dimension of the principal coefficient of

$$\cos\{[(k-p)n_{,} - (k-q)n_{,}']t - Q''\} \text{ is } 2k-p-q,$$

and therefore the dimension of the principal coefficient of $\cos\{(pn_{,} - qn_{,}')t + Q\}$ is the least value of $2k - p - q$, and this is $p \sim q$, as before.

Hence the Proposition is true.

PROP. *To prove that the principal coefficient of the term* $P\cos\{(pn_{,} + qn_{,}')t + Q\}$ *in* R *is of the dimension* $p + q$ *in* $e_{,}$, $e_{,}'$, $\tan i_{,}$, $\tan i_{,}'$.

371. This term arises from the multiplication of such terms as $P'\cos\{(kn_{,} - kn_{,}')t + Q'\}$ with

$$P''\cos\{[(p-k)n_{,} + (q+k)n_{,}']t + Q''\}$$

$$\text{and } P''\cos\{[(p+k)n_{,} + (q-k)n_{,}']t + Q''\}.$$

Hence, in both cases, the dimension will be $p + q$, since $(p-k) + (q+k)$ and $(p+k) + (q-k)$ each equals $p + q$: see law (2) of Art. 370. We have here supposed k is not greater than p and q: but if k be greater than p or q it will be easily seen that the dimension will be greater than $p + q$. Hence the Proposition is true.

PROP. *To determine the part of* R *which is independent of the periodic terms.*

372. We have

$$R = R' + \frac{dR'}{da_{,}} a_{,} u + \frac{dR'}{da_{,}'} a_{,}' u'$$
$$+ \frac{d^2R'}{da_{,}^2} \frac{a_{,}^2 u^2}{2} + \frac{d^2R'}{da_{,} da_{,}'} a_{,} a_{,}' u u' + \frac{d^2R'}{da_{,}'^2} \frac{a_{,}'^2 u'^2}{2}$$
$$+ \ldots\ldots\ldots\ldots$$

We shall neglect small quantities of the third order; hence we need calculate the first differential coefficients of R' only to the first order: and in the second differential coefficients we may neglect all small quantities.

Let us turn to the expression for R' and that for u in Art. 367, and it will be seen (after reduction) that the constant part of R' is

$$+\frac{m' a_{,} e_{,} e_{,}'}{a_{,}'^2} \cos(\varpi_{,} - \varpi_{,}') \qquad \{\text{from the first term of } R'\}$$
$$-\tfrac{1}{2} m' C_0 - m' e_{,} e_{,}' C_1 \cos(\varpi_{,} - \varpi_{,}') \quad \{\text{from the second term of } R'\}$$
$$+\tfrac{1}{8} m' (a_{,}^2 \tan^2 i_{,} + a_{,}'^2 \tan^2 i_{,}') D_0 \qquad \{\text{from the fifth term of } R'\}$$
$$-\tfrac{1}{4} m' a_{,} a_{,}' \tan i_{,} \tan i_{,}' D_1 \cos(\Omega_{,} - \Omega_{,}') \; \{\text{from the fifth term of } R'\}.$$

The constant part of $\frac{dR'}{da_{,}} a_{,} u$ is

$$+\frac{m' a_{,} e_{,} e_{,}'}{2a_{,}'^2} \cos(\varpi_{,} - \varpi_{,}') \qquad \{\text{from the first term of } R'\}$$
$$-\frac{dC_0}{da_{,}} \frac{m' a_{,}}{4} (e_{,}^2 - \tfrac{1}{2} \tan^2 i_{,}) \qquad \{\text{from the second term of } R'\}$$
$$-\frac{dC_1}{da_{,}} \frac{m' a_{,} e_{,} e_{,}'}{2} \cos(\varpi_{,} - \varpi_{,}') \quad \{\text{from the second term of } R'\}.$$

The constant part of $\frac{dR'}{da_{,}'} a_{,}' u'$ is

$$-\frac{m' a_{,} e_{,} e_{,}'}{a_{,}'^2} \cos(\varpi_{,} - \varpi_{,}') \qquad \{\text{from the first term of } R'\}$$
$$-\frac{dC_0}{da_{,}'} \frac{m' a_{,}'}{4} (e_{,}'^2 - \tfrac{1}{2} \tan^2 i_{,}') \qquad \{\text{from the second term of } R'\}$$
$$-\frac{dC_1}{da_{,}'} \frac{m' a_{,}' e_{,} e_{,}'}{2} \cos(\varpi_{,} - \varpi_{,}') \quad \{\text{from the second term of } R'\}.$$

The constant part of $\dfrac{d^2R'}{da_{\prime}^2}\dfrac{a_{\prime}^2u^2}{2}$ is

$$-\frac{d^2C_0}{da_{\prime}^2}\frac{m'a_{\prime}^2e_{\prime}^2}{8} \text{ (from the second term of } R').$$

The constant part of $\dfrac{d^2R'}{da_{\prime}da_{\prime}'}a_{\prime}a_{\prime}'uu'$ is

$$-\frac{m'a_{\prime}e_{\prime}e_{\prime}'}{2a_{\prime}'^2}\cos(\varpi_{\prime}-\varpi_{\prime}') \quad \{\text{from the first term of } R'\}$$

$$-\frac{d^2C_1}{da_{\prime}da_{\prime}'}\frac{m'a_{\prime}a_{\prime}'e_{\prime}e_{\prime}'}{4}\cos(\varpi_{\prime}-\varpi_{\prime}') \ \{\text{from the second term of } R'\}.$$

The constant part of $\dfrac{d^2R'}{da_{\prime}'^2}\dfrac{a_{\prime}'^2u_{\prime}^2}{2}$ is

$$-\frac{d^2C_0}{da_{\prime}'^2}\frac{m'a_{\prime}'^2e_{\prime}'^2}{8} \text{ (from the second term of } R').$$

The part of R which is independent of periodic terms equals the sum of these parts. We shall call this sum F;

$$\therefore F = -\tfrac{1}{2}m'C_0$$

$$+\tfrac{1}{8}m'\left(a'^2D_0 + a_{\prime}\frac{dC_0}{da_{\prime}}\right)\tan^2 i_{\prime} + \tfrac{1}{8}m'\left(a_{\prime}'^2D_0 + a_{\prime}'\frac{dC_0}{da_{\prime}'}\right)\tan^2 i_{\prime}'$$

$$-\tfrac{1}{4}m'a_{\prime}a_{\prime}'\tan i_{\prime}\tan i_{\prime}'D_1\cos(\Omega_{\prime}-\Omega_{\prime}')$$

$$-\tfrac{1}{4}m'\left(a_{\prime}\frac{dC_0}{da_{\prime}}+\tfrac{1}{2}a_{\prime}^2\frac{d^2C_0}{da_{\prime}^2}\right)e_{\prime}^2 - \tfrac{1}{4}m'\left(a_{\prime}'\frac{dC_0}{da_{\prime}'}+\tfrac{1}{2}a_{\prime}'^2\frac{d^2C_0}{da_{\prime}'^2}\right)e_{\prime}'^2$$

$$-\tfrac{1}{4}m'\left(4C_1+2a_{\prime}\frac{dC_1}{da_{\prime}}+a_{\prime}a_{\prime}'\frac{d^2C_1}{da_{\prime}da_{\prime}'}+2a_{\prime}'\frac{dC_1}{da_{\prime}'}\right)e_{\prime}e_{\prime}'\cos(\varpi_{\prime}-\varpi_{\prime}').$$

Now $\tfrac{1}{2}C_0 + C_1\cos(\theta_{\prime}-\theta_{\prime}')+\ldots = \{a_{\prime}^2+a_{\prime}'^2-2a_{\prime}a_{\prime}'\cos(\theta_{\prime}-\theta_{\prime}')\}^{-\frac{1}{2}}$;

$$\therefore \tfrac{1}{2}\frac{dC_0}{da_{\prime}}+\ldots = -\{\tfrac{1}{2}D_0+D_1\cos(\theta_{\prime}-\theta_{\prime}')+\ldots\}.\{a_{\prime}-a_{\prime}'\cos(\theta_{\prime}-\theta_{\prime}')\}$$

$$= -(\tfrac{1}{2}a_{\prime}D_0 - \tfrac{1}{2}a_{\prime}'D_1) + \ldots\ldots$$

$$\therefore a_{\prime}^2D_0 + a_{\prime}\frac{dC_0}{da_{\prime}} = a_{\prime}a_{\prime}'D_1.$$

Similarly, $a_{\prime}'^2 D_0 + a_{\prime}' \dfrac{dC_0}{da_{\prime}'} = a_{\prime}' a_{\prime} D_1$.

Wherefore putting the coefficients of the last three terms of F equal to B, B', C, we have

$$\begin{aligned} F_{\prime} = &- \tfrac{1}{2} m' C_0 + \tfrac{1}{8} m' a_{\prime} a_{\prime}' D_1 (\tan^2 i_{\prime} + \tan^2 i_{\prime}') \\ &- \tfrac{1}{4} m' a_{\prime} a_{\prime}' \tan i_{\prime} \tan i_{\prime}' D_1 \cos (\Omega_{\prime} - \Omega_{\prime}') \\ &- m' B e_{\prime}^2 - m' B' e_{\prime}'^2 - m' C e_{\prime} e_{\prime}' \cos (\varpi_{\prime} - \varpi_{\prime}')^* \end{aligned}$$

in which we observe, that C is symmetrical with respect to $a_{\prime}$ and $a_{\prime}'$.

373. In Art. 366. we collected the formulæ for calculating the elements of the instantaneous ellipse at any time. Since the object of the present work is only to explain the theory, and not to enter into the numerical calculations of the perturbations, we shall proceed to demonstrate a few of the most important and interesting results to which these equations conduct us.

PROP. *To shew that the effect of all the terms of* R *(after the first) upon the elements of the planetary orbits is periodical.*

374. Any term $P \cos \{(pn_{\prime} \pm qn_{\prime}') t + Q\}$ will produce a similar term in $\dfrac{dR}{da_{\prime}}$, $\dfrac{dR}{de_{\prime}}$ and $\dfrac{dR}{di_{\prime}}$; but a term of the form $P \sin \{(pn_{\prime} \pm qn_{\prime}') t + Q\}$ in $\dfrac{dR}{d\epsilon_{\prime}}$, $\dfrac{dR}{d\varpi_{\prime}}$ and $\dfrac{dR}{d\Omega_{\prime}}$: since Q is independent of $a_{\prime}$, $e_{\prime}$, $i_{\prime}$; and P is independent of $\epsilon_{\prime}$, $\varpi_{\prime}$, $\Omega_{\prime}$. If then these be substituted in the equations of Art. 366. and the integrations be effected, the elements $a_{\prime}$, $e_{\prime}$, $\varpi_{\prime}$, $i_{\prime}$, $\Omega_{\prime}$ will receive, in consequence, a term of the form

$$\frac{P}{pn_{\prime} \pm qn_{\prime}'} \begin{matrix} \cos \\ \sin \end{matrix} \{(pn_{\prime} \pm qn_{\prime}') t + Q\};$$

Since the formula for $\dfrac{d\epsilon_{\prime}}{dt}$ contains a term multiplied by t, the

* We might shew that $B = B' = \frac{1}{8} a_{\prime} a_{\prime}' D_1$; but there is no occasion for this in what follows.

element $\epsilon_{,}$ will receive, besides this, terms of the form (as may be shewn by integrating by parts)

$$\frac{Pt}{(pn_{,} \pm qn_{,}')} \cos \{(pn_{,} \pm qn_{,}')\, t + Q\}$$
$$+ \frac{P}{(pn_{,} \pm qn_{,}')^2} \sin \{(pn_{,} \pm qn_{,}')\, t + Q\}.$$

It follows, then, that after a period of time $= \dfrac{360^0}{pn_{,} \pm qn_{,}'}$, the perturbations of the elements, arising from the above term in R, will have gone through their changes.

These variations of the elements are therefore termed *Periodic Variations.*

It will be remarked that if $pn_{,} + qn_{,}'$ or $pn_{,} - qn_{,}'$ be a very small quantity the integration described above will increase the corresponding terms considerably: and therefore it may happen that terms in R, of which the coefficients are so small as to appear of no consequence, may rise to importance by receiving in the process of integration a small divisor.

PROP. *To find what terms in the development of* R *will be much increased by the process of integration in determining the elliptic elements.*

375. By reference to the last Article we see that either *First*, $pn_{,} + qn_{,}'$ must be a small quantity: hence, since p and q are positive integers or zero, $n_{,}$ and $n_{,}'$ must be small. By reference to the first Table in Art. 392 we see, that this is not the case with any of the planets.

Or *Secondly*: $pn_{,} \sim qn_{,}'$ must be small.

Hence p and q must be in the ratio $n_{,}' : n_{,}$ as nearly as possible. Now the lowest dimension of the coefficient in terms of small quantities is $p \sim q$, Art. 370. If, then, we can find two integers p and q nearly in the ratio $n_{,}' : n_{,}$ and having a small difference, the corresponding term of R will rise into importance by the integration. If we turn, now, to the first Table in Art. 392, and by continued fractions find the convergents which express the ratio of the values of $n_{,}$ for any two planets, and choose those of them which have a small difference between

the numerator and denominator, we shall be able to detect terms of importance in the development of R, which would otherwise have escaped notice.

For Jupiter and Saturn $n_{,} : n_{,}' :: 5 : 2$ nearly, and $5 - 2 = 3$: hence the dimension of the coefficient of a term $P \cos \{(2n_{,} - 5n_{,}')t + Q\}$ will be of the third order and will be divided by the small quantities $(2n_{,} - 5n_{,}')$ and $(2n_{,} - 5n_{,}')^2$.

For the Earth and Venus $n_{,} : n_{,}' :: 8 : 13$ nearly and $13 - 8 = 5$: hence the order of small quantities in the coefficient will be of the fifth degree and the argument

$$(13n_{,} - 8n_{,}')t + Q.$$

376. These two examples present very remarkable instances of the agreement of theory with observation.

The observations upon Jupiter and Saturn from the times of the Chinese and Arabian Astronomers down to the present day prove, that for ages the mean motions of these planets have been affected by an inequality of long period. This formed an apparent anomaly in the Planetary Theory till Laplace pointed out the real cause of the inequality, and rescued Newton's doctrine of Gravity from the reproach, which had long attached to it in consequence of its inability to assign the cause of so remarkable a phenomenon. Laplace proved that the inequality depends upon the near commensurability of the mean motions of the planets (as explained in Art. 375), and succeeded in calculating its period and amount.

Mr Airy has discovered a similar inequality in the motion of the Earth and Venus. In the *Phil. Trans.* for 1832 he shews, that it amounts to no more than a few seconds at its maximum, though its period is no less than 240 years. Mr Airy had detected an error in the solar tables, and this induced him to seek for the cause, which is so satisfactorily shewn to arise from the near commensurability of the mean motions of the Earth and Venus.

PROP. *To explain the difference between Periodic Variations and Secular Variations.*

377. In Art. 374 we have supposed the elements which are involved in the right-hand side of the equations to be con-

stant, while they are in fact functions of t. The only effect, however, which would result from this consideration would be that the *period* of the variations would be slightly altered.

But if we consider the effect of the first part of the expansion of R, which is independent of the periodic terms, and which we call F, and suppose the elements involved in F *constant*, it is evident, that by the integration of the equations of Art. 366, the elements will receive additions which continually increase or decrease with the time, unless in any instance the right-hand side of the equation vanishes, when F is put for R. If, however, we make a nearer approximation, and suppose that the elements in F are variable, and then integrate the equations of Art. 366, the integrals *may* give periodical values for the elements. If this be the case in any instance the variation is not called a *Periodic Variation*, though in fact it is periodical, but a *Secular Variation;* since it arises from a cause quite different from that, which produces the periodic variations. In short a periodic variation arises from the fact of R involving r and θ the co-ordinates of the planet disturbed: but a secular variation arises from the fact that the elements of the orbit vary. And since they vary very slowly, the period in which they perform their secular variations is of immense duration*. Perhaps the following observations may throw light upon this subject.

The magnitude of the forces, which disturb the elliptic motion of the planets, depends solely upon the relative positions of the Sun and planets, and not on their velocities and the directions of their motion. When therefore, after a lapse of years, the planets return to the same relative positions, that they occupied at the commencement of that period, the disturbing forces and the perturbations in the places of the planets will have gone through a series of changes, compensating in one part of this period for the errors they have caused in some other part. The inequalities produced during this interval of time are termed Periodical Variations. But although the configuration of the planetary system may become the same,

* The *periodic variation* of longest duration among those that are of sufficient importance to be calculated has a period equal to 929 years. But some of the *secular variations* have a period of 70000 and even more years.

yet, as was before mentioned, the velocities and directions of the motion of the planets will not necessarily become the same also; the original and final orbits intersecting respectively in those points, which the planets occupy at the beginning and end of the time, which the periodic variations have taken to go through their changes. The inequalities produced in this way are termed Secular Variations in consequence of their very slow variation.

We proceed now to the examination of the Secular Variations.

PROP. *To obtain the equations for calculating the Secular Variations of the elliptic elements of a planet's orbit.*

378. We must first find the differential coefficients of F (the part of R independent of the periodic terms) with respect to the elements: hence by Art. 372,

$$\frac{dF}{d\epsilon_{,}} = 0, \quad \frac{dF}{d\varpi_{,}} = m'Ce_{,}e_{,}'\sin(\varpi_{,} - \varpi_{,}'),$$

$$\frac{dF}{de_{,}} = -2m'Be_{,} - m'Ce_{,}'\cos(\varpi_{,} - \varpi_{,}'),$$

$$\frac{dF}{d\Omega_{,}} = \tfrac{1}{4}m'a_{,}a_{,}'\tan i_{,}\tan i_{,}' D_1 \sin(\Omega_{,} - \Omega_{,}'),$$

$$\frac{dF}{di_{,}} = \tfrac{1}{4}m'a_{,}a_{,}' D_1 \tan i_{,} - \tfrac{1}{4}m'a_{,}a_{,}'\tan i_{,}' D_1 \cos(\Omega_{,} - \Omega_{,}').$$

Substituting these in the equations of Art. 366,

$$\frac{da_{,}}{dt} = 0, \quad \frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu}\sin(\varpi_{,} - \varpi_{,}'),$$

$$\frac{d\varpi_{,}}{dt} = \frac{n_{,}a_{,}m'\sqrt{1-e_{,}^2}}{\mu e_{,}}\{2Be_{,} + Ce_{,}'\cos(\varpi_{,} - \varpi_{,}')\},$$

$$\frac{d\tan i_{,}}{dt} = \frac{n_{,}a_{,}m'a_{,}a_{,}'\tan i_{,}' D_1}{4\mu}\sin(\Omega_{,} - \Omega_{,}'),$$

$$\frac{d\Omega_{,}}{dt} = \frac{n_{,}a_{,}^2 a_{,}' m' D_1}{4\mu}\left\{-1 + \frac{\tan i_{,}'}{\tan i_{,}}\cos(\Omega_{,} - \Omega_{,}')\right\}.$$

We have retained the variable values of the elements on the right hand of these equations: but should it be necessary, we may use the constant values in approximating.

Prop. *To prove the stability of the mean distances of the planets from the Sun: and of their mean motions.*

379. By Art. 378 $\dfrac{da_{,}}{dt} = 0$; $\therefore\ a_{,} = \text{const.}$

This shews, that the axis-major of any of the planets is susceptible of no secular variation; and will suffer no permanent change: the changes it undergoes in consequence of the mutual attraction of the planets are wholly periodical. The same is true of the mean motion $n_{,}$, since it $= \sqrt{\dfrac{\mu}{a_{,}^{3}}}$, and μ does not alter. We are hereby assured of the impossibility of any of the bodies of our system ever leaving it, in consequence of the disturbances it may experience from the other bodies; and this secures the general permanence of the whole by keeping the mean distances and periodic times perpetually fluctuating between certain limits (very restricted ones) which they can never exceed, nor fall short of.

Prop. *To prove the stability of the eccentricities of the planetary orbits.*

380. By Art. 378 we have

$$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu}\sin(\varpi_{,} - \varpi_{,}').$$

$$\text{Similarly } \frac{de_{,}'}{dt} = \frac{n_{,}'a_{,}'mCe_{,}}{\mu}\sin(\varpi_{,}' - \varpi_{,}).$$

Multiply these by $\dfrac{m}{n_{,}a_{,}}e_{,}$ and $\dfrac{m'}{n_{,}'a_{,}'}e_{,}'$ and add them,

$$\therefore\ \frac{m}{n_{,}a_{,}}e_{,}\frac{de_{,}}{dt} + \frac{m'}{n_{,}'a_{,}'}e_{,}'\frac{de_{,}'}{dt} = 0;$$

$$\therefore\ \frac{m}{n_{,}a_{,}}e_{,}^{2} + \frac{m'}{n_{,}'a_{,}'}e_{,}'^{2} = \text{constant}.$$

If we had considered three planets we should have had the following equations.

$$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu}\sin(\varpi_{,} - \varpi_{,}') + \frac{n_{,}a_{,}m''C'e_{,}''}{\mu}\sin(\varpi_{,} - \varpi_{,}''),$$

$$\frac{de_{,}'}{dt} = \frac{n_{,}'a_{,}'mCe_{,}}{\mu}\sin(\varpi_{,}' - \varpi_{,}) + \frac{n_{,}'a_{,}'m''C''e_{,}''}{\mu}\sin(\varpi_{,}' - \varpi_{,}''),$$

$$\frac{de_{,}''}{dt} = \frac{n_{,}''a_{,}''mC'e_{,}}{\mu}\sin(\varpi_{,}'' - \varpi_{,}) + \frac{n_{,}''a_{,}''m'C''e_{,}'}{\mu}\sin(\varpi_{,}'' - \varpi_{,}').$$

These equations give

$$\frac{m}{n_{,}a_{,}}e_{,}\frac{de_{,}}{dt} + \frac{m'}{n_{,}'a_{,}'}e_{,}'\frac{de_{,}'}{dt} + \frac{m''}{n_{,}''a_{,}''}e_{,}''\frac{de_{,}''}{dt} = 0;$$

$$\therefore \frac{m}{n_{,}a_{,}}e_{,}^2 + \frac{m'}{n_{,}'a_{,}'}e_{,}'^2 + \frac{m''}{n_{,}''a_{,}''}e_{,}''^2 = \text{constant}.$$

And the same formula would be true of any number of planets.

Now observation shews, that the eccentricities of the orbits of the planets at present are very small indeed, with the exception of the Asteroids, the masses of which are very small. Hence the above constant must be small. Since, then, all the terms of the first side of the last equation are positive and their sum always equals a small constant it follows, that the terms are always small, and therefore the eccentricities are always small.

Hence the eccentricities of the orbits of the planets are confined within very restricted limits: and therefore the forms of the orbits are said to be stable.

381. The only quantities in the above equation, subject to a change of sign in applying it to a system of bodies, are the mean motions $n_{,}$, $n_{,}'$, $n_{,}''$,... But observation shews, that all the planets revolve round the Sun in the same direction: and consequently the terms are all positive.

Prop. *To prove the stability of the inclinations of the planets of the Solar System.*

382. By Art. 378 we have

$$\frac{d\tan i_{,}}{dt} = \frac{n_{,}a_{,}^{2}a_{,}'m'\tan i_{,}'D_1}{4\mu}\sin(\Omega_{,} - \Omega_{,}').$$

Similarly $$\frac{d\tan i_{,}'}{dt} = \frac{n_{,}'a_{,}'^{2}a_{,}m\tan i_{,}D_1}{4\mu}\sin(\Omega_{,}' - \Omega_{,}),$$

$$\therefore \frac{m}{n_{,}a_{,}}\tan i_{,}\frac{d\tan i_{,}}{dt} + \frac{m'}{n_{,}'a_{,}'}\tan i_{,}'\frac{d\tan i_{,}'}{dt} = 0;$$

$$\therefore \frac{m}{n_{,}a_{,}}\tan^2 i_{,} + \frac{m'}{n_{,}'a_{,}'}\tan^2 i_{,}' = \text{constant}.$$

The same equation would (as in the eccentricities) be true for any number of planets: and we see that the inclinations must always be small. The certainty of this fact depends, as before, upon the fact that the planets all revolve in the same direction; Art. 381.

383. We are thus led to the following remarkable conclusion: *The fact that the planets revolve about the Sun in the same direction ensures the stability of the planetary system.*

The converse of this would not necessarily be true, as we shall see in Arts. 385, 387: the numerical relations of the dimensions and positions of the orbits of the planets might be such as to ensure stability although they revolved in opposite directions. But the above is independent of particular numerical relations.

384. We have given the two foregoing Propositions because of the simplicity of their demonstrations as well as the beauty of the results. We shall, however, in the following Articles obtain formulæ for calculating the magnitude of the variations of the orbits in dimension and position.

PROP. *To find the secular variation of the eccentricity of the planetary orbits.*

385. By Art. 378 we have for the planet m,

$$\frac{de_{,}}{dt} = \frac{n_{,}a_{,}m'Ce_{,}'}{\mu}\sin(\varpi_{,} - \varpi_{,}');$$

$$\frac{d\varpi_{,}}{dt} = \frac{n_{,}a_{,}m'\sqrt{1-e_{,}^2}}{\mu e_{,}}\{2Be_{,} + Ce_{,}'\cos(\varpi_{,} - \varpi_{,}')\}.$$

And for the other planet m',

$$\frac{de_{,}'}{dt} = \frac{n_{,}'a_{,}'mCe_{,}}{\mu}\sin(\varpi_{,}' - \varpi_{,}),$$

$$\frac{d\varpi_{,}'}{dt} = \frac{n_{,}'a_{,}'m\sqrt{1-e_{,}'^2}}{\mu e_{,}'}\{2B'e_{,}' + Ce_{,}\cos(\varpi_{,}' - \varpi_{,})\};$$

observing that C is the same for m and m', (Art. 372).

To integrate these equations assume

$$r = e_{,}\sin\varpi_{,},\quad s = e_{,}\cos\varpi_{,},\quad r' = e_{,}'\sin\varpi_{,}',\quad s' = e_{,}'\cos\varpi_{,}';$$

$$\therefore\ \frac{dr}{dt} = e_{,}\cos\varpi_{,}\frac{d\varpi_{,}}{dt} + \frac{de_{,}}{dt}\sin\varpi_{,}$$

$$= \frac{n_{,}a_{,}m'}{\mu}\{2Be_{,}\cos\varpi_{,} + Ce_{,}'\cos\varpi_{,}'\} = \frac{n_{,}a_{,}m'}{\mu}\{2Bs + Cs'\},$$

$$\frac{ds}{dt} = -e_{,}\sin\varpi_{,}\frac{d\varpi_{,}}{dt} + \frac{de_{,}}{dt}\cos\varpi_{,} = -\frac{n_{,}a_{,}m'}{\mu}\{2Br + Cr'\},$$

$$\frac{dr'}{dt} = \frac{n_{,}'a_{,}'m}{\mu}\{2B's' + Cs\},\qquad \frac{ds'}{dt} = -\frac{n_{,}'a_{,}'m}{\mu}\{2B'r' + Cr\}.$$

These four are linear equations and their solutions are of the form

$$r = D\sin(gt + k) + E\sin(ht + l)$$
$$s = D\cos(gt + k) + E\cos(ht + l)$$
$$r' = D'\sin(gt + k) + E'\sin(ht + l)$$
$$s' = D'\cos(gt + k) + E'\cos(ht + l).$$

If we put these values in the differential equations, we arrive at the four following conditions connecting the eight constants, four of which are consequently arbitrary and depend upon the configuration of the planetary system.

$$Dg = \frac{n_{,}a_{,}m'}{\mu}\{2BD + CD'\},\qquad Eh = \frac{n_{,}a_{,}m'}{\mu}\{2BE + CE'\},$$

$$D'g = \frac{n_{,}'a_{,}'m}{\mu}\{2B'D' + CD\},\qquad E'h = \frac{n_{,}'a_{,}'m}{\mu}\{2B'E' + CE\}.$$

By eliminating D' from the first and third of these, we have

$$\left\{g - \frac{2n_, a_, m' B}{\mu}\right\} . \left\{g - \frac{2n_,' a_,' m B'}{\mu}\right\} = \frac{n_, n_,' a_, a_,' m m' C^2}{\mu^2};$$

$$\therefore\ g = \frac{n_, a_, m' B + n_,' a_,' m B'}{\mu} \pm \frac{1}{\mu}\sqrt{(n_, a_, m' B - n_,' a_,' m B')^2 + n_, n_,' a_, a_,' m m' C^2}.$$

In a similar way we might shew that h has the same values.

Now these values of g and h are possible when $n_,$ and $n_,'$ have the same sign; that is, when the planets revolve in the same direction about the Sun. But even if they do not revolve in the same direction and $n_, a_, m' B + n_,' a_,' m B'$ be not less than $\sqrt{n_, n_,' a_, a_,' m' m C}$, then g and h are still possible.

Now $e_,^2 = r^2 + s^2 = D^2 + E^2 + 2DE \cos\{(g-h)\,t + k - l\}$, and a similar expression is true for $e_,'^2$.

This shews that the eccentricity of m's orbit fluctuates between the limits $D + E$ and $D \sim E$. Hence the *form* of the orbit will be stable: the same is true of m''s orbit.

The values of D and E are very small in all the planets, this is shewn by observation.*

The periods of the changes in the eccentricities of the orbits of the two planets are the same in each, being $\dfrac{360^0}{g-h}$. In the case of Jupiter and Saturn this equals 70414 years! The greatest and least eccentricities which Jupiter's orbit can attain are 0.06036 and 0.02606, and those of Saturn 0.08409 and 0.01345; the maximum of each taking place at the time of the minimum of the other, and vice versâ.

PROP. *To find the secular variation of the longitude of the perihelion.*

* Sir John Herschel finds that

$$D = -0.01715,\quad E = 0.04321 \text{ for Jupiter.}$$
$$D' = \ \ 0.04877,\quad E' = 0.03532 \text{ for Saturn.}$$
$$g = 21''.9905,\quad h = 3''.5851,\quad k = 306^0\,34'\,40'',\quad l = 210^0\,16'\,40'',$$

t being the number of *years* since the year A.D. 1700. See Article *Physical Astronomy* in *Encyclop. Metrop.*

386. By Art. 385, $\tan \varpi_{,} = \frac{r}{s}$

$$= \frac{D \sin (gt + k) + E \sin (ht + l)}{D \cos (gt + k) + E \cos (ht + l)}.$$

The maxima and minima values of $\varpi_{,}$, or the greatest deviations of the perihelion, from its mean place are found by the equation

$$gD^2 + hE^2 + DE(g + h) \cos \{(g - h)t + (k - l)\} = 0,$$

$$\text{or } \cos \{(g - h)t + (k - l)\} = -\frac{gD^2 + hE^2}{DE(g + h)},$$

which is obtained by equating to zero the differential coefficient of $\tan \varpi_{,}$.

If this (disregarding the sign) be not greater than unity, the perihelion will vibrate: but if, as is the case with Jupiter and Saturn, this be greater than unity the longitude of the perihelion has no maximum or minimum and therefore the mean motion of the perihelion is continually in one direction.

PROP. *To find the secular variation of the inclination.*

387. By Art. 378. we have for m

$$\frac{d \tan i_{,}}{dt} = \frac{n_{,} a_{,} m' a_{,} a_{,}' \tan i_{,}' D_1}{4\mu} \sin (\Omega_{,} - \Omega_{,}')$$

$$\frac{d\Omega_{,}}{dt} = \frac{n_{,} a_{,}' a_{,}^2 m' D_1}{4\mu} \left\{-1 + \frac{\tan i_{,}'}{\tan i_{,}} \cos (\Omega_{,} - \Omega_{,}')\right\},$$

and for the planet m'

$$\frac{d \tan i_{,}'}{dt} = \frac{n_{,}' a_{,}' m a_{,} a_{,}' a_{,} \tan i_{,} D_1}{4\mu} \sin (\Omega_{,}' - \Omega_{,})$$

$$\frac{d\Omega_{,}'}{dt} = \frac{n_{,}' a_{,}'^2 a_{,} m D_1}{4\mu} \left\{-1 + \frac{\tan i_{,}}{\tan i_{,}'} \cos (\Omega_{,}' - \Omega_{,})\right\}.$$

To integrate these, assume

$$p = \tan i_{,} \sin \Omega_{,}, \quad q = \tan i_{,} \cos \Omega_{,}$$

$$p' = \tan i_{,}' \sin \Omega_{,}', \quad q' = \tan i_{,}' \cos \Omega_{,}';$$

$$\therefore \frac{dp}{dt} = \tan i_, \cos \Omega_, \frac{d\Omega_,}{dt} + \sin \Omega_, \frac{d \tan i_,}{dt}$$

$$= \frac{n_, a_,^2 a_,' m' D_1}{4\mu} (q' - q)$$

$$\frac{dq}{dt} = \frac{n_, a_,^2 a_,' m' D_1}{4\mu} (p - p'),$$

$$\frac{dp'}{dt} = \frac{n_,' a_, a_,'^2 m D_1}{4\mu} (q - q'), \quad \frac{dq'}{dt} = \frac{n_,' a_, a_,'^2 m D_1}{4\mu} (p' - p).$$

The integrals are of the form

$$p = G \sin(at+\gamma) + H \sin(\beta t + \delta), \quad q = G \cos(at+\gamma) + H \cos(\beta t+\delta),$$
$$p' = G' \sin(at+\gamma) + H' \sin(\beta t+\delta), \quad q' = G' \cos(at+\gamma) + H' \cos(\beta t+\delta).$$

Substituting in the differential equations we have

$$Ga = \frac{n_, a_,^2 a_,' m' D_1}{4\mu} (G' - G), \quad H\beta = \frac{n_, a_,^2 a_,' m' D_1}{4\mu} (H' - H),$$

$$G'a = \frac{n_,' a_, a_,'^2 m D_1}{4\mu} (G - G'), \quad H'\beta = \frac{n_,' a_, a_,'^2 m D_1}{4\mu} (H - H').$$

Eliminating G from the first and third

$$\left(a + \frac{n_, a_,^2 a_,' m' D_1}{4\mu}\right)\left(a + \frac{n_,' a_, a_,'^2 m D_1}{4\mu}\right) = \frac{n_, n_,' a_,^3 a_,'^3 m m' D_1^2}{16\mu^2};$$

$$\therefore a^2 + \frac{(n_, a_, m' + n_,' a_,' m)\, a_, a_,' D_1}{4\mu} a = 0.$$

We should arrive at the same equation for β: hence

$$a = -\frac{n_, a_, m' + n_,' a_,' m}{4\mu} a_, a_,' D_1 \text{ and } \beta = 0,$$

$$\tan^2 i_, = p^2 + q^2 = G^2 + H^2 + 2GH \cos\{at + \gamma - \delta\}.$$

Observation proves that G and H are very small for all the planets (except the Asteroids). Hence the tangent of inclination fluctuates between the small limits $G + H$ and $G \sim H$*.

* Sir John Herschel shews that when Jupiter and Saturn are the two planets,

$G = -0.00661$, $H = -0.02905$ for Jupiter.
$G' = 0.01537$, $H' = 0.02905$ for Saturn.
$a = -25''.5756$, $\gamma = 125^\circ\ 15'\ 40''$, $\delta = 103^\circ\ 38'\ 40''$,

t being the number of years since A.D. 1700.

The period of the changes in the inclination equals $360^0 \div a^0$ years. In the case of Jupiter and Saturn the number of years is 50673! The maximum and minimum inclinations of Jupiter's orbit to the ecliptic are $2^0\ 2'\ 30''$ and $1^0\ 17'\ 10''$: and those of Saturn are $2^0\ 32'\ 40''$ and $0^0\ 47'$. The maximum of each takes place at the time of the minimum of the other, and vice versâ.

Prop. *To find the secular variation of the longitude of the node.*

388. By Art. 378, we have

$$\tan \Omega_{\prime} = \frac{p}{q} = \frac{G \sin (at + \gamma) + H \sin \delta}{G \cos (at + \gamma) + H \cos \delta}.$$

When $\Omega_{\prime}$ attains a maximum or minimum value the differential coefficient of $\tan \Omega_{\prime}$ equals zero: hence

$$0 = aG^2 + GHa \cos (at + \gamma - \delta);$$

$$\therefore \cos (at + \gamma - \delta) = -\frac{G}{H}.$$

If this (disregarding the sign) be not greater than unity, then the node fluctuates, the period of its fluctuation being $360^0 \div a^0$ years. But if this be greater than unity then there cannot be any stationary positions; but the node continually moves in one direction.

In the case of Saturn and Jupiter the node oscillates, the extent of oscillation being about $13^0\ 9'\ 40''$ in Jupiter's orbit, and $31^0\ 56'\ 20''$ in Saturn's, on either side of their mean positions; the plane of the ecliptic being supposed immoveable.

389. The conclusions at which we have arrived in Arts. 379—383, with regard to the stability of the planetary system are of especial interest. In consequence of the changes in the elements we might have fancied that in the lapse of ages the orbits would undergo such alterations in their dimensions as to bring the planets into collision or hurry them into boundless space. But we are assured that this can never be the case, (unless by the action of a resisting medium); since analysis shews us, that the orbits will continually fluctuate within very

small limits, never departing considerably from circles; and the inclinations of the orbits will never change much: a striking illustration of GEN. viii. 22.

390. Our calculations have not included the square of the disturbing forces. But the same conclusions are found to hold when the approximation is carried so far as analysts have at present advanced: see the *Mécanique Céleste*, Liv. VI; Pontécoulant's *Système du Monde*, Tom. III; Plana's *Planetary Theory* in the *Memoirs of the Astronomical Society*, Vol. II.; also a Memoir by Professor Hansen of Seeberg, the title of this Memoir is *Untersuchung üeber die gegenseitigen Störungen des Jupiters und Saturns.* In this method the true longitude is computed by means of the elements corresponding to the *invariable ellipse at the time of the epoch*, taking a function of t, instead of t, which corrects for the perturbations. See M. Pontécoulant's remarks on this in the *Connaissance des Tems* for 1837. And lastly Mr. Lubbock's papers in the Transactions of the Royal Society and of the Astronomical Society may be consulted.

PROP. *To shew how the masses of the planets may be discovered.*

391. There are in general two methods of determining the masses of the planets; either by observing the elongations of a satellite, when the planet is accompanied by a satellite; or by comparing the inequalities produced in their motion by their mutual action. The secular variations are best adapted to give the most exact results; but these are not yet known with sufficient accuracy to allow of this use. We are therefore obliged to recur to the periodic variations, and, by combining a vast number of observations, gather from them the most probable results. It is by these means that Astronomers have obtained the following results.

Mass of Sun.........	1	mass of Mars......	$\frac{1}{2680337}$
.... Mercury...	$\frac{1}{1909706}$	.... Jupiter...	$\frac{1}{1053.924}$
.... Venus......	$\frac{1}{401839}$	.... Saturn ...	$\frac{1}{3512}$
.... Earth......	$\frac{1}{356354}$	.... Herschel..	$\frac{1}{17918}$.

We have taken these from M. Pontécoulant, *Système du Monde*, Tom. III. p. 341. The following is the formula for calculating the mass when the planet has a satellite.

Let 1, M, m be the masses of the Sun, the planet, and the satellite: T, t the periodic times of the planet about the Sun, and the satellite about the planet: A, a the mean distances of the planet from the Sun, and the satellite from the planet. Hence by Art. 269,

$$T = \frac{2\pi A^{\frac{3}{2}}}{\sqrt{1+M}}, \quad t = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}}; \quad \therefore \frac{M+m}{1+M} = \frac{a^3}{A^3}\,\frac{T^2}{t^2};$$

$$\text{therefore (if we neglect } m\text{) } M = \frac{1}{\dfrac{A^3}{a^3}\,\dfrac{t^2}{T^2} - 1}.$$

In the case of Jupiter and his fourth satellite, we find by this formula $M = \frac{1}{1048.69}$: this is more properly the mass of Jupiter with that of his fourth satellite.

The first value of the mass of Jupiter determined by Laplace (*Méc. Cél.* Liv. VI. §. 21.) is $\frac{1}{1067.09}$, and is founded on the observed elongations of the satellites by Pound. These elongations have been lately observed with much greater accuracy by Mr Airy at the Observatory of the University of Cambridge, the result of his measures gives $\frac{1}{1048.69}$; *Astronomische Nachrichten*, Vol. x. p. 304. Nicolai makes the mass $\frac{1}{1053.924}$ by observing the perturbations of Juno. Encke makes it $\frac{1}{1050.117}$ by observing the motion of Vesta, and $\frac{1}{1054.4}$ by observations on the comet which bears his name. All these concur in proving that the mass of Jupiter assumed by Laplace is too small by about $\frac{1}{70}$th part. The observations of Bouvard, however, are at variance with this: he gives $\frac{1}{1070.5}$. The mass of the Earth may be determined as follows.

The attraction of the Earth on a body at its surface, in the parallel of which the square of the sine of the latitude is $\frac{1}{3}$, is very nearly the same as if the Earth were condensed into its centre: as we shall see in *the Figure of the Earth* in a subsequent Chapter. Let $\sin^2 l = \frac{1}{3}$, g = gravity in latitude l, b the mean radius of the Earth, 1 and E the masses of the Sun

and Earth, T the length of the year, a the mean radius of the Earth's orbit: hence

$$g = \frac{E}{b^2} \text{ and } T = 2\pi a^{\frac{3}{2}}; \quad \therefore E = \frac{T^2 g}{4\pi^2} \frac{b^2}{a^3},$$

$$\frac{b}{a} = \sin \text{Sun's parallax} = \sin 8''.7.$$

The mass of the Moon $= \frac{1}{74}$ nearly. *Méc. Cél.* Liv. VI. §. 44.

392. We extract the following Table from M. Pontécoulant's *Système du Monde*. These results are obtained by the methods mentioned in Art. 270.

Epoch is Jan. 1. 1800.	Mean Motions in a Year of 365¼ Days.	Mean Distance from Sun.	Eccentricities.	Longitudes of Epochs.	Longitudes of Perihelia.	Inclinations.	Longitudes of Ascending Node.
Mercury	5323416″.79	0.38709812	0.2055149	110° 13′ 17″.9	74° 21′ 41″	7° 00′ 9″	45° 57′ 39″
Venus	2106641 .52	0.72333230	0.0068531	145 56 52 .1	128 43 6	3 23 29	74 52 39
Earth	1295977 .35	1.00000000	0.0168536	100 23 32 .6	99 29 53	0 00 00	0 00 00
Mars	689051 .12	1.52369352	0.0933061	232 49 50 .5	332 23 40	1 50 6	48 00 26
Jupiter	109256 .29	5.20116636	0.0481621	81 52 10 .3	11 7 36	1 18 52	98 25 45
Saturn	43996 .72	9.53787090	0.0561505	123 5 29 .4	89 8 20	2 29 38	111 56 7
Herschel	15424 .54	19.18330500	0.0466108	173 30 16 .6	167 30 24	46 26	72 59 21

Table of Secular Inequalities of the Planets calculated for the beginning of the Year 1801.

	In the Eccentricity.	In the Long. of Perihelion.	In the Long. of the Node.	In the Inclin. of Orbit to Ecliptic.
Mercury	0.000003867	9′ 43″.5	− 13′ 22″	19″.8
Venus	0.000062711	4′ 28″	− 31′ 10″	4″.5
Earth	0.000041200	11″.9496		
Mars	0.000090176	26″.22	− 38′ 48″	1″.5
Jupiter	0.00015935	11′ 4″	− 26′ 17″	23″
Saturn	0.000312402	13′ 17″	− 37′ 54″	15′ 5″
Herschel	0.000025072	4′	− 59′ 57″	3″.7

To obtain equations for calculating the effect of a resisting medium upon a comet we must refer the reader to the *Mécanique Céleste*, and also to Mr. Airy's translation of the dissertation on Encke's Comet in the *Astronomische Nachrichten*.

Also for a very interesting paper on the orbits of revolving double stars the reader is referred to Vol. v. of the *Memoirs of the Astronomical Society* in which Sir John F. W. Herschel has treated the subject in a very original manner.

The following are Tables of the elements of the four small planets Vesta, Juno, Pallas, and Ceres: and of the four known periodical comets. The comet of Olbers has been observed only once, at the time of its return to the perihelion in 1815: the others have been observed in several successive revolutions. It must be remarked that the elements of the small planets given in the Table are not their *mean* values, but their values at the specified epoch.

Epoch 1831 July 23d. 0h. Mean Time at Berlin.	Mean Longitude.	Mean Anomaly.	Longitude of Perihelion.	Longitude of Asc. Node.	Inclination.	Eccentricity.	Mean Motion.	Mean Distance.	Period.
Vesta	$84^d 47^m 03^s$	$195^d 35^m 26^s$	$249^d 11^m 37^s$	$103^d 20^m 28^s$	$7^d 07^m 57^s$	0.0885601	$977^s.75540$	2.361484	$1325^d.5$
Juno	74 39 44	20 22 31	54 17 13	170 52 34	13 02 10	0.2555592	813 .52533	2.669464	1593 .1
Pallas	290 38 12	169 33 11	121 05 01	172 38 30	34 35 49	0.2419986	768 .54421	2.772631	1686 .3
Ceres	307 03 26	159 22 02	147 41 23	80 53 50	10 36 56	0.0767379	769 .26059	2.770907	1684 .7

Name of the Comet.	Period.	Time of Perihelion Passage.	Longitude of Perihelion on the Orbit.	Longitude of Asc. Node.	Inclination.	Eccentricity.	Mean Distance.
Halley's	76 years	Nov. 7, 1835	$304^d 31^m 43^s$	$55^d 30^m$	$17^d 44^m 24^s$	0.9675212	17.98705
Olbers's	74 years	Ap. 26, 1815	149 02	83 29	44 30	0.9313	17.7
Encke's	1204 days	Jan. 10, 1829	157 18 35	$334\ 24\ 15^s$	13 22 34	0.8446862	2.224346
Biela's	6.7 years	Nov. 27, 1832	109 56 45	248 12 24	13 13 13	0.751748	3.58683

CHAPTER VII.

MOTION OF A PARTICLE ON CURVES AND SURFACES. SIMPLE PENDULUM.

PROP. *A material particle moves on a curve in a vertical plane, and acted upon by gravity: required to determine the motion.*

R

393. Let A be the lowest point of the curve (fig. 94.) Ax the axis of x drawn vertically upwards: P the position of the body on the curve AP at the time t: $AM = x$, $MP = y$: let R be the pressure of the curve against the body, this acts in the normal line PG: M the mass of the body: then $\frac{R}{M}$ is the accelerating force resulting from the action of R (Art. 217): g the force of gravity.

Now the forces acting vertically are g downwards and $\frac{R}{M}\cos PGM$ or $\frac{R}{M}\frac{dy}{ds}$ upwards; and $\frac{R}{M}\frac{dx}{ds}$ is the only horizontal force.

Hence, attending to the *directions* of the forces, we have the following equations of motion:

$$\frac{d^2x}{dt^2} = -g + \frac{R}{M}\frac{dy}{ds} \ldots\ldots(1), \quad \frac{d^2y}{dt^2} = -\frac{R}{M}\frac{dx}{ds} \ldots\ldots(2).$$

Multiply these respectively by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, and add, then

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} + 2\frac{dy}{dt}\frac{d^2y}{dt^2} = -2g\frac{dx}{dt} + \frac{2R}{M}\left(\frac{dx}{dt}\frac{dy}{ds} - \frac{dy}{dt}\frac{dx}{ds}\right)$$

$$= -2g\frac{dx}{dt};$$

$$\therefore \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} \text{ or } \frac{ds^2}{dt^2} = \text{const.} - 2gx.$$

When the motion begins let $x = h$; then, const. $= 2gh$:

$$\therefore \frac{ds^2}{dt^2} = 2g(h - x).$$

This shews, that the velocity at any time depends not on the form of the curve on which the body moves; but solely on the vertical space through which it passes.

Extracting the square root and inverting,

$$-\frac{dt}{ds} = \frac{1}{\sqrt{2g}} \frac{1}{\sqrt{h - x}},$$

the negative sign being taken because s diminishes as t increases (Note in Art. 231).

$$\therefore t = -\frac{1}{\sqrt{2g}} \int \frac{dx}{\sqrt{h - x}} \frac{ds}{dx}.$$

We must determine $\dfrac{ds}{dx}$ from the equation to the curve by the formula $\dfrac{ds}{dx} = \sqrt{1 + \dfrac{dy^2}{dx^2}}$, then by integration we shall know t in terms of x and therefore x in terms of t. In this manner, then, we shall know the velocity and position of the body at every assigned instant.

PROP. *To find the pressure upon the curve.*

394. The equations of motion being

$$\frac{d^2x}{dt^2} = -g + \frac{R}{M}\frac{dy}{ds}, \quad \frac{d^2y}{dt^2} = -\frac{R}{M}\frac{dx}{ds},$$

we multiply them respectively by $\dfrac{dy}{dt}, \dfrac{dx}{dt}$ and subtract;

$$\therefore \frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2} = -g\frac{dy}{dt} + \frac{R}{M}\left(\frac{dy}{ds}\frac{dy}{dt} + \frac{dx}{ds}\frac{dx}{dt}\right)$$

$$= -g\frac{dy}{dt} + \frac{R}{M}\frac{ds}{dt}, \quad \because \frac{dy^2}{ds^2} + \frac{dx^2}{ds^2} = 1.$$

Let ρ be the radius of curvature of the curve, on which the body moves, at the point $(x\,y)$; then

$$\frac{1}{\rho}\frac{ds^3}{dt^3} = \frac{dy}{dt}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2},$$

t being a function of x and y, as is the case here:

$$\therefore\ \frac{R}{M} = g\frac{dy}{ds} + \frac{v^2}{\rho};\quad v = \text{velocity} = \frac{ds}{dt}.$$

This expression shews that the pressure consists of two parts; one, the part of the forces which act upon the body resolved along the normal; and the other, the centrifugal force arising from the motion. (Art. 254.)

PROP. *A body moves on a cycloid, the axis of the cycloid being vertical: required to find the time of an oscillation, and to shew that it is independent of the extent of the vibration.*

395. We have shewn that $\dfrac{ds^2}{dt^2} = 2g(h-x)$;

$$\therefore\ -\frac{dt}{ds} = \frac{1}{\sqrt{2g}}\frac{1}{\sqrt{h-x}},$$

the negative sign being taken because the arc decreases as the time increases.

The equation to the cycloid from the lowest point is

$$y = \sqrt{2ax - x^2} + a\,\text{vers}^{-1}\frac{x}{a};$$

$$\therefore\ \frac{dy}{dx} = \frac{a-x}{\sqrt{2ax-x^2}} + \frac{a}{\sqrt{2ax-x^2}} = \sqrt{\frac{2a-x}{x}};$$

$$\therefore\ \frac{ds}{dx} = \sqrt{1+\frac{dy^2}{dx^2}} = \sqrt{\frac{2a}{x}}.$$

Hence $\dfrac{dt}{dx} = \dfrac{dt}{ds}\dfrac{ds}{dx} = -\sqrt{\dfrac{a}{g}}\dfrac{1}{\sqrt{hx-x^2}}$;

$$\therefore\ t = C - \sqrt{\frac{a}{g}}\,\text{vers}^{-1}\frac{2x}{h}$$

when $t = 0$, $x = h$; $\therefore\ 0 = C - \sqrt{\dfrac{a}{g}}\,\pi$

$$t = \sqrt{\frac{a}{g}}\left\{\pi - \text{vers}^{-1}\frac{2x}{h}\right\}$$

and, whenever the body stops, the velocity, or $\dfrac{ds}{dt}$, $= 0$; and therefore $x = h$, and the values of $\text{vers}^{-1}\dfrac{2x}{h}$ when $x = h$ are

$$\pm\pi,\ \pm 3\pi,\ \pm 5\pi, \ldots\ldots$$

and therefore the values of t, arranged in order, are

$$\ldots\ldots - 4\pi\sqrt{\frac{a}{g}},\ -2\pi\sqrt{\frac{a}{g}},\ 0,\ 2\pi\sqrt{\frac{a}{g}},\ 4\pi\sqrt{\frac{a}{g}}, \ldots\ldots$$

which shew, that the body will oscillate backwards and forwards, the interval of time in which each oscillation is performed being $2\pi\sqrt{\dfrac{a}{g}}$.

This expression is independent of h and therefore points out the remarkable fact, that however large the arc of vibration be the time of oscillation is the same in all.

For this reason the cycloid is called a Tautochronous Curve.*

* It may be interesting to ascertain whether there are any other tautochronous curves when gravity is the force acting.

We have $\dfrac{dt}{dx} = -\dfrac{1}{\sqrt{2g}}\,\dfrac{1}{\sqrt{h-x}}\,\dfrac{ds}{dx}$

$$= -\frac{1}{\sqrt{2g}}\frac{ds}{dx}\left\{\frac{1}{h^{\frac{1}{2}}} + \frac{1}{2}\frac{x}{h^{\frac{3}{2}}} + \frac{1.3}{2.4}\frac{x^2}{h^{\frac{5}{2}}} + \ldots\ldots + \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\frac{x^n}{h^{\frac{2n+1}{2}}} + \ldots\ldots\right\}.$$

Now $\dfrac{ds}{dx}$ is independent of h: and consequently the integral of the general term

$$-\frac{1}{\sqrt{2g}}\cdot\frac{1.3\ldots(2n-1)}{2.4\ldots 2n},\ \frac{x^n}{h^{\frac{2n+1}{2}}}\frac{ds}{dx}$$

must

PROP. *A particle moves on a circular arc acted upon by gravity; required the time of oscillating through a given portion of the arc.*

396. As before $\frac{ds^2}{dt^2} = 2g(h - x)$ and the equation to the circle from the lowest point is $y^2 = 2ax - x^2$;

$$\therefore \frac{dy}{dx} = \frac{a - x}{\sqrt{2ax - x^2}}, \quad \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{a}{\sqrt{2ax - x^2}};$$

$$\therefore \frac{dt}{dx} = -\frac{a}{\sqrt{2g}} \frac{1}{\sqrt{(h - x)(2ax - x^2)}}.$$

We are not able to integrate this function of x: it is reducible to one of the class called Elliptic Transcendents, the properties of which Legendre has discussed in his *Traité des Fonctions Elliptiques*: tables are given of the approximate values of the integral for given values of x.*

By means of series, however, the integral can be obtained approximately.

must be of the form $c.\left(\frac{x}{h}\right)^{\frac{2n+1}{2}}$, c being a constant, in order that when taken between the limits $x = 0$ and $x = h$ the result may be independent of h: then

$$\int x^n \frac{ds}{dx} dx = \frac{A}{2n+1} x^{\frac{2n+1}{2}}, \quad A \text{ a constant};$$

$$\therefore \frac{ds}{dx} = \frac{A}{2} x^{-\frac{1}{2}},$$

$$\therefore s = A x^{\frac{1}{2}}$$

$$s^2 = A^2 x,$$

and this is the equation to the cycloid and therefore this is the only tautochronous curve for gravity.

* Let $x = h \sin^2\theta$: then $\theta = \frac{\pi}{2}$ when $x = h$ or $t = 0$;

$$\therefore \int \frac{dx}{\sqrt{(h - x)(2ax - x^2)}} = \int \frac{2 \sin\theta \cos\theta \, d\theta}{\sqrt{\cos^2\theta \, (2a - h \sin^2\theta) \sin^2\theta}}$$

$$= \frac{2}{\sqrt{2a}} \int \frac{d\theta}{\sqrt{1 - \frac{h}{2a} \sin^2\theta}};$$

which is an elliptic function of the first order.

$$\frac{dt}{dx} = -\tfrac{1}{2}\sqrt{\frac{a}{g}}\frac{1}{\sqrt{hx-x^2}}\left(1-\frac{x}{2a}\right)^{-\frac{1}{2}}$$

$$= -\tfrac{1}{2}\sqrt{\frac{a}{g}}\frac{1}{\sqrt{hx-x^2}}$$

$$\times\left\{1+\tfrac{1}{2}\frac{x}{2a}+\frac{1.3}{2.4}\left(\frac{x}{2a}\right)^2+\ldots+\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\left(\frac{x}{2a}\right)^n+\ldots\right\}.$$

Now $\displaystyle\int\frac{x^n dx}{\sqrt{hx-x^2}} = \frac{2n-1}{2n}h\int\frac{x^{n-1}dx}{\sqrt{hx-x^2}} - \frac{x^{n-1}\sqrt{hx-x^2}}{n}$,

and between the limits $x = h$ and $x = 0$, we have

$$\int_h^0\frac{x^n dx}{\sqrt{hx-x^2}} = \frac{2n-1}{2n}h\int_h^0\frac{x^{n-1}dx}{\sqrt{hx-x^2}};$$

$$\therefore \int_h^0\frac{x\,dx}{\sqrt{hx-x^2}} = \frac{h}{2}\operatorname{vers}^{-1}\frac{2x}{h} + \text{constant} = -\frac{\pi h}{2};$$

$$\int_h^0\frac{x^2 dx}{\sqrt{hx-x^2}} = -\frac{1.3}{2.4}\pi h^2, \quad \int_h^0\frac{x^3 dx}{\sqrt{hx-x^2}} = -\frac{1.3.5}{2.4.6}\pi h^3,$$

and so on;

$$\therefore T = \frac{\pi}{2}\sqrt{\frac{a}{g}}$$

$$\times\left\{1+\left(\frac{1}{2}\right)^2\frac{h}{2a}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2a}\right)^2+\ldots+\left(\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right)^2\left(\frac{h}{2a}\right)^n+\ldots\right\}.$$

When the arc of vibration is very small, then

$$T = \frac{\pi}{2}\sqrt{\frac{a}{g}},$$

and the time of an oscillation $= \pi\sqrt{\dfrac{a}{g}}$, which coincides with

that in a cycloid, observing that the a in this case is four times the a in that.

The next approximation gives a correction of the time $= \frac{\pi}{2}\sqrt{\frac{a}{g}}\,\frac{h}{8a}$; and the ratio this bears to the time of oscillation

$= \frac{h}{8a} = (\frac{1}{4}$ chord of $\frac{1}{2}$ angle of oscillation$)^2$.

Thus if the body oscillate on each side of the vertical through an angle of which the chord is $\frac{1}{10}$, the time of oscillation will be greater by a $\frac{1}{1600}$th part than that calculated by the formula $\pi\sqrt{\frac{a}{g}}$.

397. Instead of supposing the body to move on a curve, we may imagine it suspended by a string of invariable length, or a thin wire considered of no weight. In this case the instrument is called a *Pendulum*, and is of great importance in physical researches. For if l be the length of a pendulum oscillating in a second (or unit of time) then $\pi\sqrt{\frac{l}{g}} = 1$,

$$\text{and } g = \pi^2 l.$$

By this formula we may estimate the relative intensity of the Earth's attraction at different stations on the surface, above, or below it.

PROP. *A seconds pendulum is carried to the top of a mountain; required to find the height of the mountain by observing the change in the time of oscillation.*

398. Let r be the radius of the Earth, considered spherical; h the height of the mountain; l the length of the pendulum: the force of gravity on bodies outside of the Earth varies inversely as the square of the distance from the centre: hence $\frac{gr^2}{(r+h)^2}$ is gravity at the top of the mountain. Let n be

R

the number of oscillations the pendulum makes in a day, or in $24 \times 60 \times 60$ seconds: then time of oscillation $= \dfrac{24 \times 60 \times 60}{n}$:

$$\therefore\ 1 = \pi\sqrt{\frac{l}{g}} \text{ and } \frac{24\times 60\times 60}{n} = \pi\sqrt{\frac{l\,(r+h)^2}{g\,r^2}} = \frac{\pi(r+h)}{r}\sqrt{\frac{l}{g}};$$

$$\therefore\ \frac{h}{r} = \frac{24 \times 60 \times 60}{n} - 1,$$

which gives the height of the mountain. For the sake of example suppose the pendulum loses 5″ a day:

$$\text{then } n = 24 \times 60 \times 60 - 5,$$

$$\frac{h}{r} = \left(1 - \frac{1}{24 \times 12 \times 60}\right)^{-1} - 1 = \frac{1}{24 \times 12 \times 60} \text{ nearly};$$

$$\therefore\ h = \frac{4000}{24 \times 12 \times 60} = \tfrac{1}{4} \text{ mile nearly}.$$

Prop. *To find the depth of a mine by observing the change of oscillation in a seconds pendulum.*

399. The gravity in the interior of the Earth varies directly as the distance from the centre: if, then, h be the depth, $\dfrac{g\,(r-h)}{r}$ is gravity at the bottom of the mine:

$$\therefore\ 1 = \pi\sqrt{\frac{l}{g}},\quad \frac{24 \times 60 \times 60}{n} = \pi\sqrt{\frac{l r}{g\,(r-h)}};$$

$$\therefore\ 1 - \frac{h}{r} = \left(\frac{n}{24 \times 60 \times 60}\right)^2;$$

from which h can be found. If, as before, the pendulum lose 5″ a day

$$\frac{h}{r} = 1 - \left(1 - \frac{1}{24 \times 60 \times 12}\right)^2 = \frac{1}{12 \times 60 \times 12} \text{ nearly};$$

$$\therefore\ h = \tfrac{1}{2} \text{ mile nearly}.$$

400. The results deduced by the pendulum, as far as we have at present explained its construction, would lead to erroneous conclusions; since we have supposed the rod supporting the *bob*, as the lower extremity is termed, to have no weight. We must leave the correction of this to a future part of the work, in which we shall shew that l must not be taken equal to the length of the pendulum; but some other expression which it is unnecessary to give here.

401. Owing to the remarkable property of the cycloid, that its evolute is an equal cycloid, we can easily make the bob of a flexible pendulum move in a cycloidal arc.

For let CA (fig. 95) be the pendulum when remaining at rest: PAP' the cycloid in which the bob is to move, the length of the axis being half that of the pendulum: CQ, CQ' the evolutes of PAP'. Now move the bob to the right, and let the upper portion of the pendulum bend round CQ and the other portion remain straight, touching CQ in Q. Then since CQ is the evolute of AP, the extremity of the pendulum will be in the curve AP: and by this contrivance the bob will be made to describe the cycloid PAP'.

This suggests the following means of correcting a common pendulum which makes small oscillations. Let a small portion of the upper extremity be flexible, (consisting of watch spring): and let it be suspended between two cycloidal cheeks, as in fig. 96. Then the small oscillations of the bob will be in a cycloid, and in the expression for the time of oscillation the correction depending on $\frac{h}{2a}$ is avoided: see Art. 396.

402. The following Table contains the results of experiments with a seconds pendulum on various parts of the Earth. It is extracted from the *Mécanique Céleste*.

Places.	Latitudes.	Lengths of a Seconds Pendulum.
Peru	$0^{\circ}.00$	0.99669
Porto Bello	10.61	0.99689
Pondicherry	13.25	0.99710
Jamaica	20.00	0.99745
Petit-Goave	20.50	0.99728
Cape of Good Hope	37.69	0.99877
Toulouse	48.44	0.99950
Vienna	53.57	0.99987
Paris	54.26	1.00000
Gotha	56.63	1.00006
London	57.22	1.00018
Petersburgh	64.72	1.00074
Arensgberg	66.60	1.00101
Ponoi	74.22	1.00137
Lapland	74.53	1.00148

It should be remarked, that French degrees are used in this Table.

403. Mr. Airy, in a Paper which was read before the Philosophical Society of Cambridge in the year 1826, has reduced the usual theorems for the alteration in the time and extent of vibration produced by the difference between cycloidal and circular arcs, by the resistance of the air, by the friction at the point of suspension, and by other disturbing causes to a very general investigation which leads to results remarkable for their simplicity. Since the principle of the pendulum is of vast importance in physical researches we shall not scruple to introduce large extracts from this valuable communication.

PROP. *A pendulum is acted upon by a small disturbing force: required the alteration in the time and extent of its oscillations.*

404. We shall suppose that the undisturbed pendulum moves with its extremity in a cycloidal arc, since in this case the calculation is not approximate.

Let s be the distance of the pendulum at the time t from the lowest point of the cycloid, s being measured along the arc described, l the length of the pendulum. Then the resolved part of gravity along the tangent is $g\dfrac{dx}{ds}$, x being measured vertically upwards: and $s^2 = 2lx$ is the equation to the cycloid;

$$\therefore\ g\frac{dx}{ds} = \frac{g}{l}s = n^2 s, \text{ putting } \frac{g}{l} = n^2.$$

Wherefore the equation of motion of the bob of the pendulum is

$$\frac{d^2 s}{dt^2} + n^2 s = 0.$$

$$\therefore\ s = a\sin(nt + b),$$

where a and b are arbitrary constant quantities depending on the length of the arc of vibration and the time of passing the lowest point.

The velocity at time $t = \dfrac{ds}{dt} = na\cos(nt + b)$.

We shall now suppose that f is a small disturbing accelerating force resolved along the tangent: the equation of motion then is

$$\frac{d^2 s}{dt^2} + n^2 s = f.$$

The solution of this equation we shall assume to be

$$s = a\sin(nt + b)$$

(conformably to the principle of the variation of parameters; see Art. 352) a and b being considered unknown functions of t, which it is our business now to determine.

Since there are two functions a and b we may assume any relation between them that we please, since we have but one quantity (s) to determine. Let this assumption be, that the velocity is still expressed by $na\cos(nt+b)$: the convenience of this we shall soon discover.

$$\text{Now } s = a\sin(nt+b);$$

$$\therefore\ \frac{ds}{dt} = na\cos(nt+b) + \frac{da}{dt}\sin(nt+b) + a\cos(nt+b)\frac{db}{dt},$$

$$\text{and } \therefore\ \frac{da}{dt}\sin(nt+b) + a\cos(nt+b)\frac{db}{dt} = 0,$$

this is the assumed relation between a and b.

$$\text{Again since } \frac{ds}{dt} = na\cos(nt+b);$$

$$\therefore\ \frac{d^2s}{dt^2} = -n^2a\sin(nt+b) + n\frac{da}{dt}\cos(nt+b) - na\sin(nt+b)\frac{db}{dt},$$

in this substitute for $\frac{d^2s}{dt^2}$ its value;

$$\therefore\ n\frac{da}{dt}\cos(nt+b) - na\sin(nt+b)\frac{db}{dt} = f,$$

this is the second equation between a and b.

Eliminating successively $\frac{db}{dt}$ and $\frac{da}{dt}$ from these, we have

$$\frac{da}{dt} = \frac{f}{n}\cos(nt+b), \quad \frac{db}{dt} = -\frac{f}{na}\sin(nt+b).$$

If we could solve these equations we should have the complete determination of the motion. In few cases is this practicable: in all to which we shall have to apply the investigation an approximation is sufficient.

We suppose f to be a very small force. Hence the variable parts of a and b are of the same order of magnitude as f, and may be neglected on the right-hand side of the above equations if we agree to neglect the square and higher powers of f.

In order to find the alteration in the extent of vibration which takes place in one oscillation we must integrate $\frac{f}{n}\cos(nt+b)$ through the limits of t corresponding to one oscillation: that is, from a value of t which gives $nt+b=a$ to the value of t which gives $nt+b=\pi+a$. Here a may be any quantity: in different cases we shall find it convenient to integrate between different limits.

$$\therefore \textit{ increase of arc of semi-vibration } = \frac{1}{n}\int f\cos(nt+b)\,dt$$

between the above-mentioned limits.

To find the alteration in the time of oscillation, let T, T' be the values of t at two successive arrivals of the pendulum at the lowest point; B, B' the values of b at these times. Then

$$nT+B=m.\pi,\quad nT'+B'=(m+1).\pi;$$

$$\therefore n(T'-T)+B'-B=\pi,$$

$$T'-T=\frac{\pi}{n}-\frac{1}{n}(B'-B).$$

$$\text{Now } B'-B=\int_T^{T'}\frac{db}{dt}\,dt=-\frac{1}{na}\int_T^{T'}f\sin(nt+b)\,dt:$$

$$\therefore \text{ the increase of time of oscillation } = \frac{1}{n^2a}\int_T^{T'}f\sin(nt+b)\,dt,$$

and *the proportionate increase of time of oscillation*

$$=\frac{1}{\pi na}\int_T^{T'}f\sin(nt+b)\,dt.$$

If the circumstances are such that we must integrate through two vibrations, then

$$\text{proportionate increase of time of osc.} = \frac{1}{2\pi na}\int f\sin(nt+b)\,dt.$$

These formulæ are convenient when f can be expressed in terms of t. If however f be expressed in terms of s, as is the case particularly in clock escapements, we must modify the formulæ:

$$\frac{da}{ds} = \frac{da}{dt}\frac{dt}{ds} = \frac{1}{na\cos(nt+b)}\frac{da}{dt} = \frac{f}{n^2a},$$

$$\text{and } \frac{db}{ds} = \frac{1}{na\cos(nt+b)}\frac{db}{dt}$$

$$= -\frac{f}{n^2a^2}\tan(nt+b) = -\frac{f}{n^2a^2}\frac{s}{\sqrt{a^2-s^2}};$$

$$\therefore \textit{increase of arc of semi-vibration} = \frac{1}{n^2a}\int_{-s}^{s} f\,ds,$$

$$\textit{proportionate increase of the time of vib}^n. = \frac{1}{\pi n^2a^2}\int_{-s}^{s}\frac{fs\,ds}{\sqrt{a^2-s^2}}.$$

The limits should strictly be $-s$ and s', where s' differs from s by a quantity, which depends upon the change in the arc of vibration: but we may neglect this difference between s and s', since the terms in which they occur are small.

We shall subjoin a variety of examples.

Ex. 1. *Instead of vibrating in a cycloid let the pendulum vibrate in a circle.*

$$\text{Here the force} = g\sin\frac{s}{l} = \frac{gs}{l} - \frac{gs^3}{6l^3} \text{ nearly};$$

$$\therefore f = \frac{g}{6l^3}s^3 = \frac{ga^3}{6l^3}\sin^3(nt+b);$$

therefore proportionate increase in time of vibration

$$= \frac{ga^2}{6\pi nl^3}\int \sin^4(nt+b)\,dt.$$

$$\text{Now} \int\sin^4(nt+b)\,dt = \frac{1}{8}\int\left\{3 - 4\cos 2(nt+b) + \cos 4(nt+b)\right\}dt$$

$$= \frac{1}{8}\left\{3t - \frac{2}{n}\sin 2(nt+b) + \frac{1}{4n}\sin 4(nt+b)\right\} + C$$

$$= \frac{3}{8}\frac{\pi}{n}, \text{ from } nt+b = 0 \text{ to } \pi;$$

$$\therefore \text{proportionate increase of time} = \frac{ga^2}{16n^2l^3} = \frac{a^2}{16l^2} \text{ since } n^2 = \frac{g}{l}.$$

The increase of arc of vib. $= \dfrac{ga^3}{6nl^3}\displaystyle\int \cos(nt+b)\sin^3(nt+b)\,dt$

$$= \frac{ga^3}{24n^2l^3}\sin^4(nt+b) + C = 0 \text{ between the limits,}$$

as we might easily have foreseen.

Ex. 2. *Suppose the friction at the point of suspension to be constant.*

It will be convenient to take the integrals during that time in which the friction acts in the same direction: that is, from the beginning of a vibration to its end, or from $nt+b=-\frac{1}{2}\pi$ to $nt+b=\frac{1}{2}\pi$. Here $f=-c$, since the friction *retards* the motion.

$$\therefore \text{ increase of arc} = -\frac{c}{n}\int \cos(nt+b)\,dt$$

$$= -\frac{c}{n^2}\sin(nt+b) + C = -\frac{2c}{n^2},$$

$$\text{proportionate increase of time} = -\frac{c}{\pi na}\int \sin(nt+b)\,dt$$

$$= \frac{c}{\pi n^2 a}\cos(nt+b) + C = 0, \text{ between the limits.}$$

Ex. 3. *Suppose the resistance of the air to produce a force varying as the* m^th^ *power of the velocity or* $= \mathrm{k}\mathrm{v}^{\mathrm{m}}$, m *being any whole number.*

The velocity in moving from the lowest point

$= na\cos(nt+b)$; $\therefore f = -kn^m a^m \cos^m(nt+b)$;

therefore increase of arc

$$= -kn^{m-1}a^m\int \cos^{m+1}(nt+b)\,dt \text{ from } nt+b = -\tfrac{1}{2}\pi \text{ to } \tfrac{1}{2}\pi$$

$$= -k\pi n^{m-2}a^m\frac{m(m-2)\ldots\ldots\ldots.1}{(m+1)(m-1)\ldots2}\ (m \text{ odd})$$

$$= -2kn^{m-2}a^m\frac{m(m-2)\ldots\ldots\ldots.2}{(m+1)(m-1)\ldots3}\ (m \text{ even}).$$

When $m=2$ (the law usually taken) the decrease of the arc $=\frac{4}{3}ka^2$.

The proportionate increase of time of oscillation

$$= -\frac{k}{\pi} n^{m-1} a^{m-1} \int \cos^m (nt + b) \sin (nt + b) dt$$

$$= \frac{kn^{m-2} a^{m-1}}{\pi (m+1)} \cos^{m+1} (nt + b) + C = 0, \text{ between limits,}$$

whether m be a positive integer or fraction.

Ex. 4. *Suppose the resistance of the air is expressed by any function of the velocity.*

Here $f = -\phi(v)$ for motion in the positive direction: and the increase of the arc of vibration

$$= \frac{1}{n^3 a} \int \phi(v) \frac{\cos (nt + b)}{\sin (nt + b)} dv = \frac{1}{n^3 a} \int \frac{v\phi(v) dv}{\sqrt{n^2 a^2 - v^2}}$$

from $v = 0$ to $v = 0$ again. But it must be observed that from $v = 0$ to $v = na$ (that is, from $s = -a$ to $s = 0$) the radical must be taken with a negative sign, because $\sin (nt + b)$ is then negative. The increase of the arc is consequently

$$-\frac{1}{n^3 a} \int_0^{na} \frac{v\phi(v) dv}{\sqrt{n^2 a^2 - v^2}} + \frac{1}{n^3 a} \int_{na}^0 \frac{v\phi(v) dv}{\sqrt{n^2 a^2 - v^2}},$$

$$\text{and therefore decrease } = \frac{2}{n^3 a} \int_0^{na} \frac{v\phi(v) dv}{\sqrt{n^2 a^2 - v^2}}.$$

The proportionate increase of time of vibration

$$= -\frac{1}{\pi n a} \int \phi(v) \sin (nt + b) dt = \frac{1}{\pi n^3 a^2} \int \phi(v) dv$$

$$= \frac{1}{\pi n^3 a^2} \psi(v) = 0, \text{ from } v = 0 \text{ to } v = 0.$$

Hence a resistance which is constant, or which depends on the velocity, does not alter the time of vibration.

Ex. 5. *Let the resistance be that produced by a current of air moving in the plane of vibration with a velocity* V *greater than the greatest velocity of the pendulum: and varying as the square of their relative velocity.*

Here $f = \phi(v) = k(V - v)^2$ when the pendulum moves in the direction of the current, which we suppose to be the *positive* direction of s; and $f = \phi(v) = k(V + v)^2$ when it moves in the opposite direction.

By the formula in the last Example, when the pendulum moves in the direction of the current, the arc is increased by $k\left(\frac{2V^2}{n^2} - \frac{Va\pi}{n} + \frac{4a^2}{3}\right)$ and when it returns the arc is diminished by $k\left(\frac{2V^2}{n^2} + \frac{Va\pi}{n} + \frac{4a^2}{3}\right)$.

The diminution in two vibrations $= \frac{2kVa\pi}{n}$. The time is unaffected.

Ex. 6. *Let a force* F *act through a very small space* x *at the distance* c *from the lowest point.*

The increase of the arc $= \frac{1}{n^2 a}\int_c^{c+x} F\,ds = \frac{Fx}{n^2 a}$ nearly.

The proportionate increase of the time of vibration

$$= \frac{1}{\pi n^2 a^2}\int_c^{c+x} \frac{Fs\,ds}{\sqrt{a^2 - s^2}},$$

if the general value of the integral be $\phi(s)$, then the proportionate increase of time $= \phi(c + x) - \phi(c) = \phi'(c)x$

$$= \frac{Fx}{\pi n^2 a^2}\,\frac{c}{\sqrt{a^2 - c^2}}.$$

If, then, an impulse be given when the pendulum is at its lowest point, $c = 0$ and the time of vibration is unaffected.

405. Since the preceding theory is applicable to every case in which a pendulum is acted on by small forces, it can be applied to determine the effect produced on the motion of the pendulum of a clock, or the balance of a watch, by the machinery which serves to maintain that motion.

If a pendulum vibrate uninfluenced by any external forces except that of gravity, the resistance of the air and the friction of the point of suspension gradually reduce the extent of vi-

bration. But this diminution goes on very slowly. A pendulum suspended on knife edges has been observed to vibrate more than seven hours before its arc was reduced from two degrees to $\frac{1}{5}$th of a degree. In order to maintain vibrations of the same or nearly the same length (which for clocks is indispensable) a force must act on the pendulum: this force is generally given by the action of a tooth of the seconds wheel on the inclined surfaces of small arms or pallets carried by the pendulum: and the whole apparatus is called an *escapement*.

Now it appears from Examples 2, 3, 4 and 5 of the last Article, that the friction and the resistance of the air do not affect the time of vibration. The maintaining force, therefore, must be impressed in such a manner as not to alter the time of vibration. The escapements of clocks in general use may be divided into the three following classes: recoil escapements, dead-beat escapements, and the escapements in which the action of the wheels raises a small weight which by its descent accelerates the pendulum: this last is Cumming's escapement. A full discussion of these will be found in Mr Airy's Paper. He comes to the conclusion that the dead-beat escapement is far superior to any other.

406. In this the wheel acts on the pallet for a small space near the middle of the vibration, and during the remainder of the vibration it has no effect except in producing a slight friction. The impact also at the beat does not tend to accelerate or retard the pendulum. Neglecting then the consideration of the friction, we have a constant force F, which begins to act when $x = -c$ and ceases when $x = c'$. Hence by Ex. 6. of last Article, proportionate increase of time

$$= \frac{F}{\pi n^2 a^2} \int_{-c}^{c'} \frac{s\,ds}{\sqrt{a^2 - s^2}} = \frac{F}{\pi n^2 a^2} \left\{\sqrt{a^2 - c^2} - \sqrt{a^2 - c'^2}\right\}$$

$$= \frac{F}{\pi n^2 a^2} \frac{c'^2 - c^2}{\sqrt{a^2 - c^2} + \sqrt{a^2 - c'^2}} = \frac{F}{2\pi n^2 a^3} (c' + c)(c' - c) \text{ nearly;}$$

an extremely small quantity, since c and c' are very small when compared with a, and $c' - c$ may be made almost as small as we please, though it cannot be made absolutely zero; for the

wheel must be so adapted to the pallets, that when it is disengaged from one it may strike the other, not on the acting surface, but a little above it; that is, the instant of disengagement from a pallet must follow the instant at which the pendulum is in its middle position by a rather longer time than that by which the instant of beginning to act preceded it. Hence c' must be rather greater than c. But the difference may be made so small that the effect on the clock's rate shall be almost insensible. This escapement, then, approaches very nearly to absolute perfection: and in this respect theory and practice are in exact agreement.

Mr Airy suggests a construction (*Trans. Cam. Phil. Soc.* Vol. III. p. 125.) for a clock escapement similar in its principles to the best detached escapements of chronometers.

PROP. *To prove that the velocity of a particle moving on a smooth surface is independent of the path described, but depends solely on the co-ordinates of position.*

407. Let R be the normal pressure between the surface and particle at the time t, M the mass of the particle; $\alpha\beta\gamma$ the angles which the direction of R makes with the axes: then, X, Y, Z being the other forces acting on the particle, the equations of motion are

$$\frac{d^2x}{dt^2} = X + \frac{R}{M}\cos\alpha, \quad \frac{d^2y}{dt^2} = Y + \frac{R}{M}\cos\beta,$$

$$\frac{d^2z}{dt^2} = Z + \frac{R}{M}\cos\gamma.$$

Multiply these by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$, $2\frac{dz}{dt}$ and add; then

$$\frac{d\,.\,v^2}{dt} = 2\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)$$

$$+ \frac{2R}{M}\left(\frac{dx}{dt}\cos\alpha + \frac{dy}{dt}\cos\beta + \frac{dz}{dt}\cos\gamma\right).$$

But $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$ are the cosines of the angles which the tangent line to the curve described makes with the axes; hence

$$\frac{dx}{ds}\cos\alpha + \frac{dy}{ds}\cos\beta + \frac{dz}{ds}\cos\gamma$$

equals the cosine of the angle which this tangent makes with the normal, and therefore equals zero;

$$\therefore\ v^2 = 2\int(Xdx + Ydy + Zdz),$$

and X, Y, Z being functions of x, y, z this expression when integrated will be a function of x, y, z, the co-ordinates of position, and does not depend on the path described.

PROP. *A particle moves in a spherical bowl acted on by gravity: required to determine the motion.*

408. The equations of motion are (z being vertical)

$$\frac{d^2x}{dt^2} = -\frac{R}{M}\cos\alpha,\quad \frac{d^2y}{dt^2} = -\frac{R}{M}\cos\beta,\quad \frac{d^2z}{dt^2} = g - \frac{R}{M}\cos\gamma,$$

also $x^2+y^2+z^2=a^2$ is the equation to the surface: in this case,

$$\cos\alpha = \frac{x}{a},\quad \cos\beta = \frac{y}{a},\quad \cos\gamma = \frac{z}{a},$$

then (as in last Article)

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = C + 2gz.$$

Let V and k be the initial values of the velocity and of z: then

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = V^2 - 2g(k-z),$$

$$\text{also}\quad x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore\ x\frac{dy}{dt} - y\frac{dx}{dt} = \text{const.} = h,$$

$$\text{likewise}\quad x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0.$$

By eliminating $\frac{dx}{dt}$ and $\frac{dy}{dt}$ from these, we have

$$t = \int\frac{a\,dz}{\sqrt{(a^2-z^2)\{V^2-2g(k-z)\} - h^2}}.$$

This is an elliptic function, Art. 396. If this could be integrated, then z (and consequently x and y) is known in terms of t, and the motion is determined.

409. We may obtain approximate results by supposing the oscillations to be very small.

In this case, let θ be the angle that the radius drawn to the particle makes with the vertical; ψ the angle, which the vertical plane in which θ is measured, makes with the vertical plane through the centre of the sphere and the point of projection; let the velocity of projection $(V) = \beta\sqrt{ga}$, β being a small numerical quantity, the direction of V horizontal, α the initial value of θ; then

$$k = a - \tfrac{1}{2}a\alpha^2, \quad z = a - \tfrac{1}{2}a\theta^2, \quad h^2 = a^3 g\alpha^2\beta^2,$$

$$y = x\tan\psi, \quad x^2 + y^2 + z^2 = a^2;$$

$$\therefore \frac{dt}{d\theta} = \frac{dt}{dz}\frac{dz}{d\theta} = -a\theta\frac{dt}{dz} = -\sqrt{\frac{a}{g}}\frac{\theta}{\sqrt{(\alpha^2-\theta^2)(\theta^2-\beta^2)}},$$

$$\frac{d\psi}{d\theta} = \frac{d\psi}{dt}\frac{dt}{d\theta} = \frac{1}{x^2+y^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\frac{dt}{d\theta} = -\frac{\alpha\beta}{\theta\sqrt{(\alpha^2-\theta^2)(\theta^2-\beta^2)}}.$$

The first of these equations gives

$$2t = -\sqrt{\frac{a}{g}}\int\frac{2d.\theta^2}{\sqrt{(\alpha^2-\beta^2)^2 - \{2\theta^2-(\alpha^2+\beta^2)\}^2}}$$

$$= \sqrt{\frac{a}{g}}\cos^{-1}\left\{\frac{2\theta^2-(\alpha^2+\beta^2)}{\alpha^2-\beta^2}\right\}, \text{ const.} = 0;$$

$$\therefore \theta^2 = \tfrac{1}{2}(\alpha^2+\beta^2) + \tfrac{1}{2}(\alpha^2-\beta^2)\cos 2\sqrt{\frac{g}{a}}\,t:$$

this shews that the pendulum makes isochronous oscillations in the moveable vertical plane: the extreme angles being α and β, and the time of oscillation being $\frac{\pi}{2}\sqrt{\frac{a}{g}}$, or half the time of oscillation when the plane of motion is constant.

Hence also $$\frac{d\psi}{dt} = \sqrt{\frac{g}{a}}\frac{\alpha\beta}{\alpha^2\cos^2\sqrt{\frac{g}{a}}\,t + \beta^2\sin^2\sqrt{\frac{g}{a}}\,t};$$

$$\therefore\ a \tan \psi = \beta \tan \sqrt{\frac{g}{a}}\, t,$$

from which the azimuth of the plane of oscillation is known at any time.

By substitution we have

$$\frac{x^2}{a^2} + \frac{y^2}{\beta^2} = (a^2 - z^2)\left(\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{\beta^2}\right) = a^2\theta^2\left(\frac{\cos^2\psi}{a^2} + \frac{\sin^2\psi}{\beta^2}\right),$$

and substituting for θ and ψ their values in terms of t,

$$\frac{x^2}{a^2} + \frac{y^2}{\beta^2} = a^2,$$

which shews that the projection of the path on a horizontal plane is an ellipse with its centre in the vertical radius of the sphere.

Cor. If $a = \beta$, then $\theta^2 = a^2$, $\psi = \sqrt{\frac{g}{a}}\, t$, $x^2 + y^2 = a^2 a^2$,

and the pendulum describes a conical surface with a uniform motion.

Prop. *A particle moves on a curve surface, required to find the pressure at any instant.*

410. The equations of motion are

$$\frac{d^2x}{dt^2} = X + \frac{R}{M}\cos a,\quad \frac{d^2y}{dt^2} = Y + \frac{R}{M}\cos\beta,\quad \frac{d^2z}{dt^2} = Z + \frac{R}{M}\cos\gamma.$$

Multiply by $\cos a$, $\cos\beta$, $\cos\gamma$ respectively, and add, then

$$\frac{R}{M} = \frac{d^2x}{dt^2}\cos a + \frac{d^2y}{dt^2}\cos\beta + \frac{d^2z}{dt^2}\cos\gamma$$

$$- \left\{X\cos a + Y\cos\beta + Z\cos\gamma\right\}.$$

To calculate the former part suppose that the co-ordinate planes are so chosen, that, at the instant under consideration, the axis of z is the normal line at the point of contact of the particle: hence $\cos a = 0$, $\cos\beta = 0$, $\cos\gamma = 1$, and this part becomes $\frac{d^2z}{dt^2}$.

Now z is a function of x and y: x and y are functions of t; hence

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt},$$

$$\frac{d^2z}{dt^2} = \frac{d^2z}{dx^2}\frac{dx^2}{dt^2} + 2\frac{d^2z}{dxdy}\frac{dx}{dt}\frac{dy}{dt} + \frac{d^2z}{dy^2}\frac{dy^2}{dt^2} + \frac{dz}{dx}\frac{d^2x}{dt^2} + \frac{dz}{dy}\frac{d^2y}{dt^2}.$$

But $\dfrac{dz}{dx} = 0$, $\dfrac{dz}{dy} = 0$ as the axes are chosen.

Hence $$\frac{d^2z}{dt^2} = \frac{ds^2}{dt^2}\left\{\frac{d^2z}{dx^2}\frac{dx^2}{ds^2} + 2\frac{d^2z}{dxdy}\frac{dx}{ds}\frac{dy}{ds} + \frac{d^2z}{dy^2}\frac{dy^2}{ds^2}\right\}$$

$$= \frac{(\text{velocity})^2}{\text{rad. of curv. in normal plane of motion}} = \frac{v^2}{\rho},$$

and the magnitude of this cannot depend upon the manner of fixing the axis; therefore, in general,

$$\frac{R}{M} = \frac{v^2}{\rho} - (X\cos\alpha + Y\cos\beta + Z\cos\gamma)$$

= centrifugal force − resolved part of the forces along the normal.

Prop. *A particle moves in a groove in the form of a curve of double curvature; required the pressure.*

411. The equations of motion are the same as in the last Article: $\alpha\beta\gamma$ being the angles which the direction of the pressure makes with the axes; this coincides with the radius of absolute curvature.

Let ρ be the radius, and $x_, y_, z_,$ the co-ordinates to the centre of curvature, then

$$x_, = x + \rho^2\frac{d^2x}{ds^2},\quad y_, = y + \rho^2\frac{d^2y}{ds^2},\quad z_, = z + \rho^2\frac{d^2z}{ds^2};$$

$$\therefore \cos\alpha = \frac{x_, - x}{\rho} = \rho\frac{d^2x}{ds^2},\quad \cos\beta = \rho\frac{d^2y}{ds^2},\quad \cos\gamma = \rho\frac{d^2z}{ds^2};$$

$$\therefore \frac{R}{M} = \rho\left\{\frac{d^2x}{dt^2}\frac{d^2x}{ds^2} + \frac{d^2y}{dt^2}\frac{d^2y}{ds^2} + \frac{d^2z}{dt^2}\frac{d^2z}{ds^2}\right\} - \{X\cos\alpha + Y\cos\beta + Z\cos\gamma\},$$

the former part, by changing the independent variable to s (as in Art. 255), becomes

$$\frac{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}{\frac{dt^2}{ds^2}} - \frac{\frac{d^2t}{ds^2}\frac{d}{ds}\left\{\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2}\right\}}{\frac{dt^3}{ds^3}}$$

$$= \frac{1}{\rho^2}(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) \frac{ds^2}{dt^2} = \frac{1}{\rho^2}\frac{ds^2}{dt^2};$$

$$\therefore \frac{R}{M} = \frac{v^2}{\rho} - (X \cos \alpha + Y \cos \beta + Z \cos \gamma)$$

= centrifugal force − resolved part of the forces along the radius of absolute curvature.

CHAPTER VIII.

PROBLEMS ON THE MOTION OF BODIES CONSIDERED AS PARTICLES.

PROB. 1. A BODY is projected vertically upwards and the time between its leaving a given point and returning to it is given: find the velocity of projection, and the whole time of motion.

PROB. 2. Two bodies fall from two given points in space in the same vertical down two straight lines drawn to any point of a surface in the same time, find the form of the surface.

PROB. 3. A semi-cycloid is placed with its axis vertical and vertex downwards, and from different points in it a number of bodies are let fall at the same instant, each moving down the tangent at the point from which it sets out: prove that they will reach the involute (which passes through the vertex) all at the same instant.

PROB. 4. From the top of a tower two bodies are projected with the same given velocity at different given angles of elevation, and they strike the horizon at the same place: find the height of the tower.

PROB. 5. A body acted upon by two central forces, each varying inversely as the square of a distance, is projected from a point between them towards one of the centres: required the velocity of projection that the body may just arrive at the neutral point of attraction and remain at rest there.

PROB. 6. A body, acted on by a force varying inversely as the fifth power of the distance, is projected in any direction with a velocity equal to that which would be acquired in falling from an infinite distance: find the orbit.

PROB. 7. A body, projected in a given direction with a given velocity and attracted towards a given centre of force,

has its velocity at every point: the velocity in a circle at the same distance :: 1 : $\sqrt{2}$; find the orbit described, the position of the apse, the magnitude of its axis, and the law of force.

PROB. 8. Two bodies lie on a horizontal plane connected by a string, which can slip freely through a fixed point in the plane; one of them is projected in any direction in the horizontal plane; find the motion of the bodies, and the curve described on the plane.

PROB. 9. A body is projected in any direction from one extremity of a right line, each particle of which attracts it by a force proportional to the distance; prove that the body will pass through the other extremity.

PROB. 10. A body, projected from a given point in the plane of xy, is attracted by forces $\frac{m}{x^3}$ in the direction of x, and $\frac{m'}{y^3}$ in the direction of y: prove that if the velocity be rightly assumed, it will describe a circle round the origin as centre, and find how the velocity varies in different parts of the orbit.

PROB. 11. A body, urged towards a plane by a force varying as the perpendicular distance from it, is projected at right angles to the plane from a given point in it with a given velocity: find what force must act at the same time on the body parallel to the plane, that it may move in a given parabola having its axis in the plane; and determine the circumstances of the motion.

PROB. 12. A body acted on by a force varying partly as the inverse cube and partly as the inverse fifth power of the distance is projected with the velocity which would be acquired in falling from infinity, at an angle with the distance the tangent of which $= \sqrt{2}$, the forces being equal at the point of projection; determine the motion.

PROB. 13. A body is projected from a point about a centre of force which varies inversely as the square of the distance, in a direction perpendicular to the line joining the point of projection with the centre of force, and so as to describe an ellipse about that centre: shew that the point of projection

will coincide with the nearer or further apse according as the velocity of projection is greater or less than that with which a circle might be described at the same distance.

PROB. 14. If a force vary inversely as the 7^{th} power of the distance, and a body be projected from an apse with a velocity which is to the velocity in a circle at the same distance :: 1 : $\sqrt{3}$; find the polar equation to the curve described, and transform it to rectangular co-ordinates.

PROB. 15. If a body be projected about a centre of force varying inversely as the square of the distance with a velocity equal to n times the velocity in a circle at the same distance, and in a direction making an angle β with the distance; the angle α between the axis-major and this distance may be determined from the equation

$$\tan(\alpha - \beta) = (1 \sim n^2) \tan \beta.$$

PROB. 16. If a be the mean distance of a planet from the Sun, and l the length of the line of nodes, then the time of the planet's passage (supposed undisturbed) from node to node through perihelion is

$$= \frac{a^{\frac{3}{2}} p}{\pi} \left\{ \tan^{-1} \sqrt{\frac{l}{2a - l}} - \frac{l}{2a} \sqrt{\frac{2a - l}{l}} \right\}$$

where $p =$ the length of the year, and $1 =$ mean distance of the Earth from the Sun.

PROB. 17. If a body revolve in an ellipse round the focus prove, that a progressive motion of the apse will be the effect of any continual addition of force in the direction of the radius-vector during the progress of the body from the further to the nearer apse, and point out the effect on the eccentricity.

PROB. 18. A body is acted on by two forces, one repulsive and varying as the distance from a given point, and the other constant and acting in parallel lines: determine the motion of the body.

PROB. 19. If a body can describe a given curve about one centre with one law of force, about another centre with another law of force and so for any number of centres, it is possible to project the body with such a velocity that it may describe the same curve under the action of all those forces.

PROB. 20. A body describes a parabola about a centre of force residing in a point in the circumference of a given ellipse, the foci of which are in the circumference of the parabola, the force varying inversely as the square of the distance: shew that the time of moving from one focus to the other is the same, at whatever point in the circumference of the ellipse the centre of force is placed.

PROB. 21. If P be a central force attracting a catenary, and p be the perpendicular on the tangent at any point from the centre of force; then, the force which would cause a body to revolve in the curve formed by the catenary varies as $P \div p$.

PROB. 22. A body P is projected with a given velocity $a\sqrt{\mu}$ in a direction perpendicular to its distance SA from a centre of force S, which itself moves uniformly with velocity $b\sqrt{\mu}$ in the direction AS produced; the force varies as the distance: determine the equation to the orbit described; and shew that the motions of P and S are parallel when the co-ordinates of P measured from the original position of S are a and $\frac{1}{2}\pi b$.

PROB. 23. If two equal bodies, which attract each other with forces varying inversely as the square of the distance, are constrained to move in two straight lines at right angles to each other; shew that they will arrive together at the point of intersection of the lines, from whatever points their motions commence: and having given their distance at the beginning of the motion, find the time to the point of intersection.

PROB. 24. The times of oscillation of a pendulum are observed at the Earth's surface, and at a given depth below the surface; find from these data the radius of the Earth, supposed spherical.

PROB. 25. If a pendulum oscillating in a small circular arc be acted upon, in addition to the force of gravity, by a small horizontal force (as the attraction of a mountain) in the plane in which it oscillates; having given the number of oscillations gained in a day, find the horizontal force.

PROB. 26. A body oscillates in a cycloid on an inclined plane, and the friction on the plane $= \mu$ times the pressure: shew that the friction will not affect the time of oscillation;

and that the body will stop after it has oscillated a number of times $= \frac{a}{2l\mu} \tan\alpha - \frac{1}{2}$, where a is the original distance from the lowest point and α the inclination of the plane.

PROB. 27. A body acted on by gravity moves on the convex surface of a cycloid, the vertex of which is its highest point; the velocity at the highest point being $\sqrt{2gh}$, determine the point where it will leave the curve, and the latus rectum of the parabola afterwards described.

PROB. 28. A body, moving on the interior surface of a vertical cylinder, was projected with a given velocity, and goes round precisely n times before it begins to descend: find the direction of projection.

PROB. 29. A body acted on by a repulsive central force varying as the distance, moves in a groove of the form of an epicycloid, the pole of which is in the centre of force: prove that the oscillations are isochronous.

PROB. 30. If a body move in an ellipse uniformly, round two centres of force situated in the foci; prove that the forces at any point of the ellipse are equal, and inversely proportional to the square of the corresponding diameter.

PROB. 31. A body moves in a groove under the action of two centres of force each varying inversely as the distance, and of equal intensity at the same distance; the body is projected from the mid-point between the centres: prove that if the velocity be uniform the form of the groove is a lemniscate.

PROB. 32. A body attracted to two centres of force varying inversely as the square of the distance moves in a hyperbolic groove, of which the foci are the centres of force: required to find the pressure on the groove; and to shew that if the particle begin to move from a point where it is equally attracted by the two centres, the pressure on the groove is zero during the whole motion.

CHAPTER IX.

PRELIMINARY ANALYSIS.

412. We now enter upon the calculation of the motion of a rigid body.

In the following Chapters we shall repeatedly meet with the expressions

$$\Sigma . mx, \quad \Sigma . my, \quad \Sigma . mz,$$
$$\Sigma . mxy, \quad \Sigma . mxz, \quad \Sigma . myz,$$
$$\Sigma . mx^2, \quad \Sigma . my^2, \quad \Sigma . mz^2;$$

xyz being the co-ordinates to a particle m of a material system, and Σ being a symbol, which represents that the sum of the quantities, symmetrical with that before which it is placed, is to be taken throughout the system.

It becomes important, then, to enquire whether the axes of co-ordinates may not be so chosen, as to simplify these expressions.

Prop. *The first three may be simplified.*

413. Let $\bar{x}$, $\bar{y}$, $\bar{z}$, be the co-ordinates of the centre of gravity of the system: and let M be the mass of the system. Then by Art. 87, Ex. 25, we have

$$\Sigma . mx = M\bar{x}, \quad \Sigma . my = M\bar{y}, \quad \Sigma . mz = M\bar{z}.$$

If it be allowable in any case to choose for one of the co-ordinate planes a plane passing through the centre of gravity, then, supposing this the plane of xy, we have $\bar{z} = 0$ and therefore $\Sigma . mz = 0$.

If it be allowable to choose for the axis of x a line passing through the centre of gravity, then $\bar{y} = 0$, $\bar{z} = 0$, and these give $\Sigma . my = 0$, $\Sigma . mz = 0$.

Lastly, if it be allowable to choose the origin at the centre of gravity, then $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 0$; and therefore $\Sigma . mx = 0$, $\Sigma . my = 0$, $\Sigma . mz = 0$.

414. The second set of expressions, viz.: $\Sigma . mxy$, $\Sigma . mxz$, $\Sigma . myz$ may be made to vanish by properly choosing the co-ordinates.

This simplification is so important that the axes, which possess this property, are called the *Principal Axes* of the system. They are likewise termed the Natural Axes of Rotation; for a reason hereafter to be assigned: see Art. 439.

Before proceeding to find these axes, we must prove the formulæ by which we pass from one system of axes to another.

PROP. *To prove the formulæ for the transformation of one system of rectangular co-ordinates to another, the origin remaining the same.*

415. Let Ax, Ay, Az be the original axes (fig. 97), $Ax_{,}$, $Ay_{,}$, $Az_{,}$ the new axes.

θ = inclination of plane $x_{,}y_{,}$ to plane xy.

ψ = the angular distance of the line of intersection of these planes from the axis of x; *i. e.* the angle NAx.

ϕ = the angular distance of axis of $x_{,}$ from this line of intersection; *i. e.* the angle $NAx_{,}$.

xyz, $x_{,}y_{,}z_{,}$ the co-ordinates to any point referred to the two systems of axes respectively.

r = the distance of this point from the origin.

Then the cosines of the angles which r makes with the axes of xyz, $x_{,}y_{,}z_{,}$ are respectively $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$; $\frac{x_{,}}{r}, \frac{y_{,}}{r}, \frac{z_{,}}{r}$. Hence

$$\frac{x}{r} = \frac{x_{,}}{r} \cos xx_{,} + \frac{y_{,}}{r} \cos xy_{,} + \frac{z_{,}}{r} \cos xz_{,}$$

$$\frac{y}{r} = \frac{x_{,}}{r} \cos yx_{,} + \frac{y_{,}}{r} \cos yy_{,} + \frac{z_{,}}{r} \cos yz_{,}$$

$$\frac{z}{r} = \frac{x_{,}}{r} \cos zx_{,} + \frac{y_{,}}{r} \cos zy_{,} + \frac{z_{,}}{r} \cos zz_{,}.$$

Let us now suppose all the points where the six axes meet a sphere of radius unity described about A to be joined by arcs of great circles; then we shall have by the formula for the cosine of the side of a spherical triangle in terms of the other sides and opposite angle

$$\begin{aligned}
\cos xx_{,} &= \cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta \\
\cos xy_{,} &= -\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta \\
\cos xz_{,} &= -\sin\psi\sin\theta \\
\cos yx_{,} &= -\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta \\
\cos yy_{,} &= \sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta \\
\cos yz_{,} &= -\cos\psi\sin\theta \\
\cos zx_{,} &= \sin\phi\sin\theta \\
\cos zy_{,} &= \cos\phi\sin\theta \\
\cos zz_{,} &= \cos\theta.
\end{aligned}$$

Hence by substitution

$$\begin{aligned}
x = x_{,}&(\cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta) \\
&- y_{,}(\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta) - z_{,}\sin\psi\sin\theta \\
y = -x_{,}&(\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta) \\
&+ y_{,}(\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta) - z_{,}\cos\psi\sin\theta \\
z = x_{,}&\sin\phi\sin\theta + y_{,}\cos\phi\sin\theta + z_{,}\cos\theta.
\end{aligned}$$

416. In the same manner we should find

$$\begin{aligned}
x_{,} = x&(\cos\phi\cos\psi + \sin\phi\sin\psi\cos\theta) \\
&- y(\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta) + z\sin\phi\sin\theta \\
y_{,} = -x&(\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta) \\
&+ y(\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta) + z\cos\phi\sin\theta \\
z_{,} = -x&\sin\psi\sin\theta - y\cos\psi\sin\theta + z\cos\theta.
\end{aligned}$$

PROP. *To prove that in every body there is a system of rectangular axes, and in general only one system, which will satisfy the conditions* $\Sigma.\mathrm{m}x_{,}y_{,} = 0$, $\Sigma.\mathrm{m}x_{,}z_{,} = 0$, $\Sigma.\mathrm{m}y_{,}z_{,} = 0$, *whatever point be taken as the origin of co-ordinates.*

417. Substitute in the equations $\Sigma . m x_{\prime} y_{\prime} = 0$, $\Sigma . m x_{\prime} z_{\prime} = 0$, $\Sigma . m y_{\prime} z_{\prime} = 0$ the values of $x_{\prime} y_{\prime} z_{\prime}$ given in Art. 416, and putting

$$\Sigma . m(y^2 + z^2) = D,\ \Sigma . m(x^2 + z^2) = E,\ \Sigma . m(x^2 + y^2) = F,$$
$$\Sigma . myz = G,\ \Sigma . mxz = H,\ \Sigma . mxy = K;$$

we have $L \sin 2\phi + M \cos 2\phi = 0 \ldots\ldots\ldots (1)$,

$$N \cos\phi - P \sin\phi = 0,$$
$$N \sin\phi + P \cos\phi = 0,$$

where L, M, N, P are certain functions of θ, ψ, D, E, F, G, H, and K; and are independent of ϕ.

The first of these equations gives ϕ when θ and ψ are known. By eliminating ϕ from the second and third we have $P = 0$, $N = 0$: or, if we replace P and N by their values, we have

$$\left.\begin{array}{l} \sin 2\theta \left\{D \sin^2\psi - 2K \sin\psi \cos\psi + E \cos^2\psi - F\right\} \\ \qquad + 2 \cos 2\theta \left\{G \cos\psi + H \sin\psi\right\} = 0 \\ \sin\theta \left\{(D - E) \sin\psi \cos\psi - K (\cos^2\psi - \sin^2\psi)\right\} \\ \qquad - \cos\theta \left\{G \sin\psi - H \cos\psi\right\} = 0 \end{array}\right\} \ldots\ldots (2).$$

Let $\tan\psi = u$, and $\therefore \sin\psi = \dfrac{u}{\sqrt{1+u^2}}$, $\cos\psi = \dfrac{1}{\sqrt{1+u^2}}$:

also let θ be eliminated from the above equations by the formula $(1 - \tan^2\theta) \tan 2\theta = 2 \tan\theta$; and we have, after all reductions,

$$\left\{(D-E)u - K(1-u^2)\right\}\left\{(GD-GF+HK)u - HE + HF - GK\right\}$$
$$- (Gu - H)^2 (Hu + G) = 0.$$

This equation, being a cubic, must give at least one real value of u, and therefore of ψ: and substituting this in one of the equations (2) we shall have the value of θ; and then ϕ is known from equation (1).

We conclude, then, that we can always find a system of co-ordinate axes which will satisfy our conditions. But, not only so, there is in general only one such system; for although we might fancy that there could be three, since the equation in u is a cubic; yet this will be found not to be the case when it is remarked that this equation, which is to obtain the angle

between the axis of x and the intersection of the planes $x_{,}y_{,}$ and xy, ought likewise to give the angles, which the axis of x makes with the intersections of the two other planes $x_{,}z_{,}$ and $y_{,}z_{,}$ with the plane xy. Hence all three roots of the cubic will be possible and serve to determine the three angles specified above.

Hence the Proposition is true.

COR. 1. The equation in u becomes identical whenever, in any particular case, we have $G = 0$, $H = 0$, $K = 0$. In this case *every* system of rectangular axes is a system of principal axes; as is proved by these three equations: and for this reason the equation in u gives no result.

COR. 2. Again, the equation in u is identical when $G = 0$, $H = 0$. In this case also there is an infinite number of systems of principal axes; but they must all have a common axis, since K does not vanish.

It will be seen that in most cases the difficulty of calculating the position of the principal axes in a body is great. But whenever we know one of them the other two are easily determined, as we shall now shew.

PROP. *To find the principal axes of a body when one of them is known.*

418. Let $Az_{,}$ be the known principal axis, $Ax_{,}$, $Ay_{,}$ the others making an angle ψ with the arbitrary axes Ax, Ay drawn at right angles to $Az_{,}$ (fig. 98).

Let $x_{,}y_{,}z_{,}$, xyz be the co-ordinates to a particle m referred to these two systems of axes: then

$$x_{,} = x\cos\psi + y\sin\psi,\quad y_{,} = y\cos\psi - x\sin\psi.$$

Hence $\Sigma . mx_{,}y_{,} = 0$ gives

$$(\cos^2\psi - \sin^2\psi)\,\Sigma . mxy - \cos\psi\sin\psi\,\Sigma . m(x^2 - y^2) = 0;$$

$$\therefore\ \tan 2\psi = \frac{\sin 2\psi}{\cos 2\psi} = \frac{2\sin\psi\cos\psi}{\cos^2\psi - \sin^2\psi} = \frac{2\Sigma . mxy}{\Sigma . m(x^2 - y^2)}.$$

EX. 1. *One principal axis of a rectangular parallelogram of uniform thickness is perpendicular to its plane through the centre: required the other two.*

Let $2a$, $2b$ be the sides of the parallelogram: M its mass: the sides parallel to the plane xy, and the centre the origin: then the mass of an element $= M\dfrac{dx\,dy}{4ab}$: and therefore

$$\Sigma . mxy = \int_{-a}^{a}\int_{-b}^{b}\frac{M}{4ab}xy\,dx\,dy = \frac{M}{4ab}\int_{-a}^{a}0 . dx = 0,$$

$$\Sigma . m(x^2 - y^2) = \frac{M}{4ab}\int_{-a}^{a}\int_{-b}^{b}(x^2 - y^2)\,dx\,dy = \frac{M}{2ab}\int_{-a}^{a}(x^2 b - \tfrac{1}{3}b^3)$$

$$= \frac{M}{3ab}(a^3 b - b^3 a) = \frac{M}{3}(a^2 - b^2);$$

$\therefore \tan 2\psi = 0$; and $\therefore 2\psi = 0$ and 180^0, or $\psi = 0$ and 90^0, and the other two axes are parallel to the sides of the parallelogram.

COR. If the parallelogram be a square then $a = b$ and $\tan 2\psi = \dfrac{0}{0}$: which shews that in this case any pair of axes x and y are principal axes.

EX. 2. *One principal axis of an elliptic board being perpendicular to its plane through its centre; the other two coincide with the axes of the ellipse.*

419. The last three of the expressions in Art. 412, viz. $\Sigma . mx^2$, $\Sigma . my^2$, $\Sigma . mz^2$, do not admit of much simplification.

The sum of the products of the mass of each particle of the system and the square of its distance from any straight line is called the *Moment of Inertia* of the System about that line.

We proceed to prove certain Propositions connected with the Moment of Inertia.

PROP. *The moment of inertia of a system about any axis is equal to the moment of inertia about an axis through the centre of gravity and parallel to the former, together with the product of the mass of the system and the square of the distance between the two axes.*

420. Let the plane of the paper pass through the centre of gravity G of the system: and be perpendicular to the original axis and cut it in A (fig. 99): Ax, Ay the axes of x and y, and P the projection of any particle of the system m on the plane of the paper: x, y the co-ordinates of P from G; $\bar{x}\,\bar{y}$ the co-ordinates of G from A. Then the moment of inertia

$$= \Sigma . m\,AP^2 = \Sigma . m\,\{(\bar{x}+x)^2 + (\bar{y}+y)^2\}$$

$$= M(\bar{x}^2+\bar{y}^2) + 2\bar{x}\,\Sigma . mx + 2\bar{y}\,\Sigma . my + \Sigma . m\,(x^2+y^2)$$

$$= M(\bar{x}^2+\bar{y}^2) + \Sigma . m\,(x^2+y^2): \text{ see Art. 413.}$$

$= M . GA^2$ + moment of inertia about an axis of which the projection is G.

421. We shall now calculate the moment of inertia in some particular cases.

Let k be such a quantity that the moment of inertia $= Mk^2$. Then k is the distance of the point at which we may suppose the whole mass collected so as not to alter the moment of inertia: k is called the *Radius of Gyration.*

We shall always use the symbol k for this radius when the axis passes through the centre of gravity, and $k_{\prime}$ (with a subscript accent) when it does not.

Ex. 1. *A physical line about an axis through its centre and perpendicular to its length.*

$2a$ = length; r = distance of any particle from the centre;

$$\therefore \text{ mass of a length } dr = M\frac{dr}{2a};$$

$$\therefore \text{ moment of inertia, or } M . k^2 = \int_{-a}^{a} M\frac{r^2}{2a}\,dr = M\frac{a^2}{3};$$

$$\therefore k, \text{ or radius of gyration, } = \frac{a}{\sqrt{3}}.$$

If the axis of rotation be at a distance c from the centre of gravity and parallel to that used above, then

$$k_{\prime}^2 = \tfrac{1}{3}a^2 + c^2 \text{ by Art. 420.}$$

Ex. 2. *A circular body of uniform thickness and density about an axis through its centre and perpendicular to its plane.*

$$a = \text{radius}, \quad \angle BAP = \theta, \quad AP = r \text{ (fig. 25)};$$

$$\text{therefore element of the mass at } P = M\frac{dr \cdot r d\theta}{\pi a^2};$$

$$\therefore Mk^2 = \int_0^a \int_0^{2\pi} M\frac{r^3}{\pi a^2} dr\, d\theta = \int_0^a 2M\frac{r^3}{a^2} dr = M\frac{a^2}{2};$$

$$\therefore k^2 = \tfrac{1}{2}a^2.$$

For an axis parallel to the above at a distance c,

$$k_{\prime}^2 = \tfrac{1}{2}a^2 + c^2 \text{ by Art. 420.}$$

Ex. 3. *The same body about an axis through its centre and in its plane.*

$$\text{Mass of element at } P = M\frac{dr\, r\, d\theta}{\pi a^2},$$

$$Mk^2 = \int_0^a \int_0^{2\pi} M\frac{r^3 \sin^2\theta}{\pi a^2} dr\, d\theta = \frac{M}{2\pi a^2}\int_0^a \int_0^{2\pi} r^3 (1 - \cos 2\theta) dr\, d\theta$$

$$= \frac{M}{a^2}\int_0^a r^3\, dr = M\frac{a^2}{4};$$

$$\therefore k^2 = \tfrac{1}{4}a^2.$$

About an axis parallel to the above at a distance c,

$$k_{\prime}^2 = \tfrac{1}{4}a^2 + c^2.$$

Ex. 4. *A solid of revolution about any axis perpendicular to the axis of the solid.*

Let $DA'E$ be the given axis cutting the axis of the solid in A': let A' be the origin of co-ordinates (fig. 27): $PM = y$, $A'M = x$: $A'A = m$, $A'B = n$, V the volume of the solid;

$$\therefore \text{mass of elementary section } PQ' = M\frac{\pi y^2 dx}{V},$$

$$\text{mom. of iner. of this about } PQ' = \frac{M}{V}\pi y^2 dx \frac{y^2}{4} \text{ (Ex. 3.)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots DA'E = \frac{M}{V}\pi y^2 dx\left(\frac{y^2}{4} + x^2\right) \text{(Art. 420)};$$

$$\therefore\ Mk_{,}^2 = \int_m^h \frac{M}{V}\pi y^2\left(\frac{y^2}{4}+x^2\right)dx: \text{ but } V = \int_m^n \pi y^2 dx:$$

$$\therefore\ k_{,}^2 = \int_m^n(\tfrac{1}{4}y^4 + x^2y^2)dx \div \int_m^n y^2 dx.$$

Ex. 5. *A sphere about a tangent.*

$$y^2 = 2ax - x^2;$$

$$\therefore\ k_{,}^2 = \int_0^{2a}(a^2x^2 + ax^3 - \tfrac{1}{4}x^4)dx \div \int_0^{2a}(2ax - x^2)dx$$

$$= (\tfrac{8}{3} + 4 - \tfrac{24}{5})a^5 \div (4 - \tfrac{8}{3})a^3 = \tfrac{28}{15}a^2 \div \tfrac{4}{3} = \tfrac{7}{5}a^2.$$

$$\text{Also } k^2 = k_{,}^2 - a^2 = \tfrac{2}{5}a^2.$$

PROP. *To find the moment of inertia of a system referred to any axis.*

422. Let AC be the axis (fig. 100): P any particle m of the system: PM perpendicular to AC: Ax, Ay, Az the co-ordinate axes: x, y, z co-ordinates to P; α, β, γ the angles AC makes with the axes;

$$\therefore\ PM^2 = AP^2\sin^2 PAC = AP^2(1-\cos^2 PAC),\quad AP = r$$

$$= r^2 - r^2\left(\frac{x}{r}\cos\alpha + \frac{y}{r}\cos\beta + \frac{z}{r}\cos\gamma\right)^2$$

$$= x^2 + y^2 + z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2$$

$$= x^2\sin^2\alpha + y^2\sin^2\beta + z^2\sin^2\gamma$$

$$-2xy\cos\alpha\cos\beta - 2xz\cos\alpha\cos\gamma - 2yz\cos\beta\cos\gamma.$$

Hence moment of inertia =

$$\sin^2\alpha\Sigma.mx^2 + \sin^2\beta\Sigma.my^2 + \sin^2\gamma\Sigma.mz^2$$

$$-2\cos\alpha\cos\beta\Sigma.mxy - 2\cos\alpha\cos\gamma\Sigma.mxz - 2\cos\beta\cos\gamma\Sigma.myz.$$

If the axes of co-ordinates be principal axes, then, accenting the co-ordinates in accordance with the notation of Art. 417,

$$Mk_{,}^2 = \sin^2\alpha_{,}\Sigma.mx_{,}^2 + \sin^2\beta_{,}\Sigma.my_{,}^2 + \sin^2\gamma_{,}\Sigma.mz_{,}^2.$$

Let A, B, C be the moments of inertia of the system about the principal axes;

$$\therefore\ A = \Sigma.m(y_{,}^2 + z_{,}^2),\quad B = \Sigma.m(x_{,}^2 + z_{,}^2),\quad C = \Sigma.m(x_{,}^2 + y_{,}^2),$$

then $\Sigma.mx_{,}^2 = \frac{1}{2}(B + C - A)$, $\Sigma.my_{,}^2 = \frac{1}{2}(A + C - B)$,

$$\Sigma.mz_{,}^2 = \frac{1}{2}(A + B - C);$$

$$\therefore Mk_{,}^2 = \tfrac{1}{2}A(\sin^2\beta_{,} + \sin^2\gamma_{,} - \sin^2 a_{,})$$

$$+ \tfrac{1}{2}B(\sin^2 a_{,} + \sin^2\gamma_{,} - \sin^2\beta_{,}) + \tfrac{1}{2}C(\sin^2 a_{,} + \sin^2\beta_{,} - \sin^2\gamma_{,})$$

$$= A\cos^2 a_{,} + B\cos^2\beta_{,} + C\cos^2\gamma_{,}.$$

Ex. *Find the moment of inertia of an ellipsoid about any axis.*

Prop. *If* A *and* C *be the greatest and least principal moments, then every other moment of inertia is intermediate to these.*

423. For $Mk_{,}^2 = A - (A - B)\cos^2\beta_{,} - (A - C)\cos^2\gamma_{,}$

and also $= C + (A - C)\cos^2 a_{,} + (B - C)\cos^2\beta_{,}$

since $\cos^2 a_{,} + \cos^2\beta_{,} + \cos^2\gamma_{,} = 1$.

The first is evidently less than A, and the second greater than C.

Prop. *When two of the principal moments are equal to each other, the moments about all axes lying in any right cone described about the principal axis of unequal moment are the same.*

424. For let $B = C$: then

$$Mk_{,}^2 = A\cos^2 a_{,} + B(\cos^2\beta_{,} + \cos^2\gamma_{,}) = A\cos^2 a_{,} + B\sin^2 a_{,},$$

and this is constant when $a_{,}$ remains the same although $\beta_{,}$ and $\gamma_{,}$ may vary.

Prob. *If the three principal moments be equal to each other, every other moment is equal to these.*

425. For $Mk_{,}^2 = A(\cos^2 a_{,} + \cos^2\beta_{,} + \cos^2\gamma_{,}) = A$.

Prop. *To find the points in a system with respect to which the principal moments are equal to each other.*

426. Let the centre of gravity be the origin, and the principal axes the axes of co-ordinates:

$x_, y_, z_,$ co-ordinates to any particle m,

$x_,' y_,' z_,'$ the point which gives the principal moments equal: then from this point the co-ordinates of m are

$$x_, - x_,', \quad y_, - y_,', \quad z_, - z_,';$$

$\therefore$ by Art. 417, Cor. 1. $\Sigma . m(x_, - x_,')(y_, - y_,') = 0$,

$$\Sigma . m(x_, - x_,')(z_, - z_,') = 0, \text{ and } \Sigma . m(y_, - y_,')(z_, - z_,') = 0.$$

Observing the origin and axes we have chosen, we see that these conditions become,

$$Mx_,'y_,' = 0, \quad Mx_,'z_,' = 0, \quad My_,'z_,' = 0;$$

$$\therefore \text{ two of } x_,'y_,'z_,' \text{ must } = 0.$$

Suppose $y_,' = 0$, $z_,' = 0$ and then $x_,'$ remains indeterminate. Hence by Art. 420,

moment about axis parallel to $x_,$ through $(x_,'y_,'z_,') = A$

.... $y_,$ .... $= B + Mx_,'^2$

.... $z_,$ .... $= C + Mx_,'^2$,

and these by hypothesis are all the same;

$$\therefore B = C, \text{ and } x_,'^2 = \frac{A - B}{M}.$$

Hence we derive the following corollaries.

1. If all the moments about the principal axes through the centre of gravity be unequal, there is no point in the system with respect to which the moments are equal.

2. If two of them be equal and the moment of the unequal one be the greatest, there are two points equally distant from the centre of gravity and on the axis of the greatest moment corresponding to which the moments are all equal.

3. When the principal moments are all equal, $x_,' = 0$, and there is no point but the centre of gravity with respect to which the moments are all equal.

CHAPTER X.

MOTION OF ONE RIGID BODY, ACTED ON BY FINITE FORCES.

EQUATIONS OF MOTION.

427. IN considering the equilibrium of a rigid body, (Art. 27) we stated, that, in consequence of our ignorance of the nature and laws of the forces by which the molecules are held together, we are unable to deduce the conditions of equilibrium of a body from those of a single particle. By the aid, however, of the principle of the transmission of force through a body (Art. 28) we deduced certain relations which the impressed forces, that act upon the body when in equilibrium, must satisfy, independently of the molecular forces. It is evident that the molecular forces are themselves in equilibrium, independently of the other forces which act upon the body.

In considering the motion of a rigid body we fall upon the same difficulty. We know nothing of the laws of the molecular forces, and consequently cannot calculate the motion of the body by calculating the motion of its molecules separately. But we may surmount this in the manner we overcame the difficulty just mentioned.

PROP. *To find the equations of motion.*

428. Let mX, mY, mZ be the *impressed* moving forces which act upon the particle m, not including the molecular forces which act upon this particle. Let xyz be the coordinates to m at the time t referred to three rectangular axes fixed in space: then $m\dfrac{d^2x}{dt^2}$, $m\dfrac{d^2y}{dt^2}$, $m\dfrac{d^2z}{dt^2}$ are the *effective* moving forces of m (Art. 221).

Now by the general principle enunciated in Art. 224, the forces

$$m\left(X - \frac{d^2x}{dt^2}\right),\ m\left(Y - \frac{d^2y}{dt^2}\right),\ m\left(Z - \frac{d^2z}{dt^2}\right)$$

acting on m parallel to the axes of x, y, z respectively, and similar forces acting on all the other particles ought, together with the molecular forces by which the particles of the body act upon each other, to satisfy the equations of equilibrium of forces acting on a rigid body.

But the molecular forces are of themselves in equilibrium, since the molecules retain their relative situations during the motion.

Hence the forces $m\left(X - \frac{d^2x}{dt^2}\right)$, $m\left(Y - \frac{d^2y}{dt^2}\right)$, $m\left(Z - \frac{d^2z}{dt^2}\right)$ acting on m, and similar forces acting on the other particles of the body, ought to satisfy the six equations of equilibrium of forces acting on a rigid body, given in Art. 56. This is *D'Alembert's Principle:* see Art 226. Wherefore we have the six equations of motion

$$\Sigma . m\left(X - \frac{d^2x}{dt^2}\right) = 0,\ \Sigma . m\left(Y - \frac{d^2y}{dt^2}\right) = 0,\ \Sigma . m\left(Z - \frac{d^2z}{dt^2}\right) = 0,$$

$$\Sigma . m\left\{y\left(Z - \frac{d^2z}{dt^2}\right) - z\left(Y - \frac{d^2y}{dt^2}\right)\right\} = 0,$$

$$\Sigma . m\left\{z\left(X - \frac{d^2x}{dt^2}\right) - x\left(Z - \frac{d^2z}{dt^2}\right)\right\} = 0,$$

$$\Sigma . m\left\{x\left(Y - \frac{d^2y}{dt^2}\right) - y\left(X - \frac{d^2x}{dt^2}\right)\right\} = 0.$$

These equations, in connexion with the relations which express the invariability of the parts of the body with respect to each other*, are sufficient to calculate the motion. We

* Suppose there are n particles in the body: then the conditions, necessary and sufficient for expressing the invariability of these relatively to each other, may be thus stated: select any three particles; that these may be invariable we must have three relations; and that each of the remaining $n - 3$ particles may be invariable, it must be at a constant distance from each of the chosen three: this gives for each of the $n - 3$ particles three relations. Hence, on the whole, there are $3 + 3(n - 3)$ or $3n - 6$ relations, which, with the 6 equations of motion, give $3n$ equations to calculate the $3n$ co-ordinates.

shall not write down these relations in this place; but shall introduce them when necessary in the course of our calculations: they are generally reduced to a very simple form by means of the properties proved in the Chapter of Preliminary Analysis.

We shall now transform the six equations of motion to more convenient forms, and deduce from them two Principles, one of which enables us to calculate the motion of the centre of gravity of the body in space; and the other, the motion of the other parts relatively to the centre of gravity. Hence the first Principle will enable us to calculate the motion of *translation* of the body in space; and the second the motion of *rotation*.

PROP. *The motion of the centre of gravity of a body, moving free in space and acted on by any forces, is the same, as if all the forces were applied at the centre of gravity, parallel to their former directions.*

429. By the first three equations of Art. 428,

$$\Sigma . m\left(X - \frac{d^2x}{dt^2}\right) = 0,\ \Sigma . m\left(Y - \frac{d^2y}{dt^2}\right) = 0,\ \Sigma . m\left(Z - \frac{d^2z}{dt^2}\right) = 0.$$

Let $\bar{x}\bar{y}\bar{z}$ be the co-ordinates to the centre of gravity,

$x'y'z'$ .... m from the centre of gravity;

$$\therefore\ x = \bar{x} + x',\ y = \bar{y} + y',\ z = \bar{z} + z'.$$

Now $\Sigma . mx' = 0$, $\Sigma . my' = 0$, $\Sigma . mz' = 0$ (Art. 413).

Hence, substituting for xyz, the above equations give, M being the whole mass of the body,

$$\frac{d^2\bar{x}}{dt^2} = \frac{\Sigma . mX}{M},\ \frac{d^2\bar{y}}{dt^2} = \frac{\Sigma . mY}{M},\ \frac{d^2\bar{z}}{dt^2} = \frac{\Sigma . mZ}{M}:$$

and these are the equations we should obtain for the motion of the centre of gravity supposing the forces all applied at that point. Hence the Proposition is proved.

PROP. *The motion of rotation of a body acted on by any forces and moving freely is the same as if the centre of gravity were fixed and the same forces acted.*

430. The last three of the equations of Art. 428 are

$$\Sigma . m \left\{ y \left(Z - \frac{d^2 z}{dt^2} \right) - z \left(Y - \frac{d^2 y}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ z \left(X - \frac{d^2 x}{dt^2} \right) - x \left(Z - \frac{d^2 z}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ x \left(Y - \frac{d^2 y}{dt^2} \right) - y \left(X - \frac{d^2 x}{dt^2} \right) \right\} = 0.$$

Now let $\bar{x}, \bar{y}, \bar{z}$ be the co-ordinates to the centre of gravity, and let (as before) $x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$.

Let these be put in the above equations, observing that $\Sigma . mx' = 0$, $\Sigma . my' = 0$, $\Sigma . mz' = 0$ (Art. 413), and that therefore the differential coefficients of these with respect to t vanish; also bearing in mind the equations of last Article we have, after all reductions,

$$\Sigma . m \left\{ y' \left(Z - \frac{d^2 z'}{dt^2} \right) - z' \left(Y - \frac{d^2 y'}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ z' \left(X - \frac{d^2 x'}{dt^2} \right) - x' \left(Z - \frac{d^2 z'}{dt^2} \right) \right\} = 0,$$

$$\Sigma . m \left\{ x' \left(Y - \frac{d^2 y'}{dt^2} \right) - y' \left(X - \frac{d^2 x'}{dt^2} \right) \right\} = 0.$$

These equations involve $x'y'z'$, the co-ordinates of m measured from the centre of gravity of the body; and not xyz, the co-ordinates of m measured from a fixed point in space. Hence the integrals of these equations will give the motion of the different parts of the body *relatively to the centre of gravity*, and not their absolute motions in space: and consequently they determine the rotatory motion about the centre of gravity. The motion of the centre of gravity itself was determined in the preceding Proposition.

But these are precisely the equations we should have obtained on the supposition that the centre of gravity were fixed, and that point taken as the origin of moments. Hence the Proposition is true.

From the first of the Principles demonstrated in the last two Articles we gather, that all the calculations we have made of the motion of a material particle will be true also of the centre of gravity of a rigid body. It remains then to ascertain the motion of the other parts of the body relative to the centre of gravity: and this the latter Principle enables us to accomplish. We shall consider the motion of rotation of a body first about any fixed axis, either passing through the centre of gravity or not, and lastly about a fixed point.

MOTION OF A RIGID BODY ABOUT A *FIXED AXIS.*

PROP. *To calculate the angular accelerating force of a rigid body moving about a fixed axis, and acted on by any given forces.*

431. Let the fixed axis be taken as the axis of z, and let xy be the co-ordinates to the projection of a particle m on the plane xy: also let r be the distance of m from the axis of rotation, and θ the angle r makes with the plane zx: then $x = r\cos\theta$, $y = r\sin\theta$: and r is constant with respect to the time, because the axis is *fixed*.

Now by Art. 60, we are to take only the last of the equations of Art. 428;

$$\therefore \Sigma . m\left\{x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right\} = \Sigma . m\,(xY - yX).$$

$$\text{But } \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt}, \quad \frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt};$$

$$\therefore x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{d}{dt}\left\{x\frac{dy}{dt} - y\frac{dx}{dt}\right\} = \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2}.$$

$$\text{Hence } \Sigma . mr^2\frac{d^2\theta}{dt^2} = \Sigma . m\,(xY - yX),$$

or, since $\dfrac{d^2\theta}{dt^2}$ is the same for every particle,

$$\frac{d^2\theta}{dt^2} = \frac{\Sigma . m(xY - yX)}{\Sigma . mr^2}$$

$$= \frac{\text{moment of the forces about the axis}}{\text{moment of inertia about the axis*}}.$$

By integrating this equation we shall know the angle through which the body has revolved in a given time; and shall consequently be able to determine the position of the body at any instant.

PROP. *A body moves about a fixed horizontal axis acted on by gravity only: required to determine the time of a small oscillation.*

432. Let ABC (fig. 101) be a section of the body, made by the plane of the paper, passing through the centre of gravity G, and cutting the axis of rotation perpendicularly in C; P the projection of any particle m on this plane; CX vertical; GH perpendicular to CX; $CG = h$; $CP = r$; $PCX = \theta'$; $GCX = \theta$.

Then by Art. 431, $\dfrac{d^2\theta}{dt^2} = \dfrac{\text{moment of forces}}{\text{moment of inertia}}$

$$= -\frac{\Sigma . mgr \sin\theta'}{\Sigma . mr^2} = -\frac{Mgh \sin\theta}{M(k^2+h^2)} = -\frac{gh}{k^2+h^2} \sin\theta. \text{ Arts. 413, 420.}$$

If we put $\dfrac{k^2+h^2}{h} = l$, multiply by $2\dfrac{d\theta}{dt}$ and integrate:

$$\therefore \frac{d\theta^2}{dt^2} = \frac{2g}{l}\cos\theta + \text{const.} = \frac{g}{l}(a^2 - \theta^2),$$

neglecting θ^3... and supposing $\theta = a$ at first;

$$\therefore \text{time of oscillation} = -\sqrt{\frac{l}{g}} \int_a^{-a} \frac{d\theta}{\sqrt{a^2 - \theta^2}} = \pi\sqrt{\frac{l}{g}}.$$

* This is an example of the introduction of the condition, that the particles of the body are invariable in relative position: see the latter part of Art. 428, and the note. By the last Chapter $\Sigma . mr^2$ is shewn to be equal to Mk^2, in consequence of the invariability of the system of particles.

Hence the body will move as if collected in a material point at a distance $\frac{k^2+h^2}{h}$ from the axis. Take $CO = \frac{k^2+h^2}{h}$ in the line CG produced; then O is called the *centre of oscillation*: and $\frac{k^2+h^2}{h}$ is called the length of the *isochronous simple pendulum*, the body itself being denominated, in contradistinction, a *compound pendulum*. The point C is called the *centre of suspension*.

PROP. *The centres of oscillation and suspension are reciprocal: that is, if the body be suspended on an axis through* O *parallel to that through* C, *then* C *will be the centre of oscillation.*

433. For let l' be the length of the simple pendulum in this case; then

$$l' = \frac{k^2+OG^2}{OG} = \frac{k^2}{l-h} + l - h$$

$$= \frac{lh-h^2}{l-h} + l - h \text{ (Art. 432)} = l.$$

From which the truth of the Proposition is evident.

PROP. *To determine the length of the seconds pendulum experimentally.*

434. We have already shewn (Art. 396) that if l be the length of a simple pendulum, that is, a pendulum consisting of a single particle suspended by a string without weight, t the duration of each oscillation and g the force of gravity, then

$$t = \pi \sqrt{\frac{l}{g}}.$$

But it is impossible to form a pendulum which may, with due regard to accuracy, be considered a simple pendulum. It becomes necessary, then, to use a compound pendulum, and to measure the distance between the centres of suspension and oscillation (see Art. 432). The practical difficulties in the way

of determining the latter point were considerable, and such as greatly to endanger the accuracy of the result, before Captain Kater removed the sources of difficulty by using the property of the compound pendulum proved in Art. 433, namely, that the centres of oscillation and suspension are reciprocal. We proceed to explain this.

Let AB be the pendulum (fig. 102); C the point of suspension; F a weight which may be shifted from one position to another on the pendulum: O the centre of oscillation of the pendulum including F.

The position of O is first found pretty accurately by making the pendulum oscillate about C and O till the times of oscillation are nearly the same. Knife edges are then fixed at C and O, and the weight F, which is placed near the middle point between C and O, is shifted till it is found, that the time of oscillation about C and O is exactly the same. It remains only to measure CO and observe the time of oscillation. For the details of the experiment we refer the reader to the *Philosophical Transactions* for 1818. If t be the time of oscillation in seconds and $CO = l$, then, since the length of the simple pendulum varies as the square of the time of oscillation, the length of the seconds pendulum $= l \div t^2$.

PROP. *To calculate the effect produced on the pendulum by shifting* F.

435. Let l' be the length of the simple pendulum when F is removed: $M(1+n)$ and M the masses of the pendulum with and without F, n being a small fraction: let l be the length of the simple pendulum when F is so situated, that the times of oscillation about C and O are the same: and let L and L' be the lengths when the pendulum oscillates about C and O, the weight F being then at a distance x from C: and let $\frac{1}{2}l + \delta$ be the value of x when $L = l$. Then, by Art. 432,

$$l = \frac{\text{square of rad. of gyration about axis of suspension}}{\text{dist. of centre of gravity from same axis}}$$

$$= \frac{\text{mom. of inertia}}{\text{mass} \times \text{dist. of centre of gravity}}$$

$$= \frac{Ml'h + Mn(\frac{1}{2}l + \delta)^2}{Mh + Mn(\frac{1}{2}l + \delta)} = \frac{l'h + n(\frac{1}{2}l + \delta)^2}{h + n(\frac{1}{2}l + \delta)}:$$

$$\therefore\ l' = l + \frac{n}{h}(\tfrac{1}{4}l^2 - \delta^2).$$

$$\text{Also } L = \frac{l'h + nx^2}{h + nx};$$

$$\therefore\ \frac{dL}{dx} = \frac{n^2x^2 + 2nhx - nl'h}{(h + nx)^2} = \frac{n^2(x - a)(x + \beta)}{(h + nx)^2},$$

$$\text{where } a = \frac{1}{n}\left\{\sqrt{h^2 + nl'h} - h\right\}$$

$$= \frac{h}{n}\left\{\frac{nl'}{2h} - \frac{n^2l'^2}{8h^2}\right\} \text{neg}^s.\ n^2 \ldots = \frac{l'}{2} - \frac{nl'^2}{8h} = \frac{l}{2} - \frac{n\delta^2}{2h}:$$

β is a positive quantity.

Let $CD = DO = \frac{1}{2}l$: and take $CP = a$. Then if F be below P (that is, x greater than a), the time of oscillation about C will increase or decrease according as F is shifted from or towards C, since dL and dx have the same sign: the contrary will be the case when F is placed above P.

In like manner if the pendulum be suspended from O, we have a point Q, the distance of which from O equals $\frac{1}{2}l - \frac{n\delta^2}{2h'}$ (h' being the distance of the centre of gravity from O), such that when F is beyond Q from O the time of oscillation about O is increased or diminished according as F is moved further from O or nearer to it; and *vice versâ*.

Since $DP = \frac{n\delta^2}{2h}$, and $DQ = \frac{n\delta^2}{2h'}$, and these are both less than δ (δ being by hypothesis a very small quantity), it follows that F cannot be between P and Q when the times of oscillation about C and O are the same.

PROP. *To shew that if the axes of suspension be* equal *cylinders rolling on horizontal plates, instead of knife edges, the length of the simple pendulum still equals the distance of the axes.*

436. Let AB be the pendulum (fig. 103); G its centre of gravity, O its centre of oscillation, CDE the semi-cylindrical axis of suspension, C' the point of contact with the horizontal plane of support when the pendulum hangs in its position of rest: P the point of contact at the time t, when the pendulum oscillates; $C'M = x$, $MG = y$, the co-ordinates to G, θ the angle CG makes with the vertical, R the pressure at P, F the friction on the plane of support, $CG = h$, M the mass of the pendulum, $CO = l$, k = rad. of gyration about G, a = rad. of the axis at C.

Now by Art. 429 the motion of the centre of gravity is the same as if all the forces were applied at that point;

$$\therefore \frac{d^2x}{dt^2} = -\frac{F}{M} \dots (1), \quad \frac{d^2y}{dt^2} = g - \frac{R}{M} \dots (2).$$

Also by Art. 430 the motion of rotation is the same as if G were fixed; hence by Art. 431

$$\frac{d^2\theta}{dt^2} = \frac{Fy - R(a+h)\sin\theta}{Mk^2} \dots (3)$$

we have here three equations and five unknown quantities R, F, x, y, θ: we must seek, then, two relations connecting x, y, θ: these are

$$x = PM - PC' = (a+h)\sin\theta - a\theta \dots (4)$$

$$y = (a+h)\cos\theta - a \dots (5).$$

By equations (1) (2) (3) we have

$$k^2\frac{d^2\theta}{dt^2} + y\frac{d^2x}{dt^2} + (a+h)\sin\theta\left(g - \frac{d^2y}{dt^2}\right) = 0;$$

differentiating (4) and (5) we have

$$\frac{dx}{d\theta} = (a+h)\cos\theta - a = y, \quad \frac{dy}{d\theta} = -(a+h)\sin\theta.$$

Hence our last equation becomes

$$k^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + (a+h)g\sin\theta\frac{d\theta}{dt} = 0;$$

$$\therefore\ k^2\frac{d\theta^2}{dt^2}+\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}=C+2(a+h)g\cos\theta$$

when $\theta=\alpha$, velocity $=0$;

$$\therefore\ k^2\frac{d\theta^2}{dt^2}+\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}=2(a+h)g(\cos\theta-\cos\alpha);$$

$$\therefore\ \frac{d\theta^2}{dt^2}\{k^2+(a+h)^2+a^2-2a(a+h)\cos\theta\}=2(a+h)g(\cos\theta-\cos\alpha)$$

$$\frac{d\theta^2}{dt^2}=\frac{(a+h)g(\alpha^2-\theta^2)}{k^2+h^2},$$

neglecting powers of α and θ higher than the square.

$$\therefore\ \frac{dt}{d\theta}=-\sqrt{\frac{k^2+h^2}{(a+h)g}}\frac{1}{\sqrt{\alpha^2-\theta^2}},$$

$$t=\sqrt{\frac{k^2+h^2}{(a+h)g}}\cos^{-1}\frac{\theta}{\alpha},\ \text{const.}=0;$$

$$\therefore\ \text{time of oscillation}=\pi\sqrt{\frac{k^2+h^2}{(a+h)g}}.$$

Also if b be the radius of the axis at O, and if $CO=m$,

$$\text{time of oscillation about } O=\pi\sqrt{\frac{k^2+(m-h)^2}{(b+m-h)g}},$$

and these times being equal, we have

$$\frac{k^2+h^2}{a+h}=\frac{k^2+(m-h)^2}{b+m-h}=l;$$

$$\therefore\ l(a+h)-h^2=(k^2=)\,bl+(m-h)l-(m-h)^2;$$

$$\therefore\ l=\frac{(m-h)^2-h^2}{m-2h+b-a}=\frac{m(m-2h)}{m-2h+b-a}.$$

If $b=a$, $l=m$; that is, the length of the simple pendulum equals the distance between the axes, when the cylinders are of equal radii.

437. Mr. Lubbock has calculated, in a Paper read before the Royal Society in 1830, the errors in the length of the,

simple pendulum corresponding to given deviations of the knife edges. It is there shewn, that a small deviation of one of the knife edges in azimuth is quite insensible: but that this is not the case for a small deviation in altitude: a deviation of one degree increases by 3 the vibrations of a seconds pendulum in 24 hours. A deviation from horizontality in the agate planes has a still greater influence: for a deviation in horizontality of 10′ increases by about 6 the vibrations in 24 hours.

PROP. *When a body moves about a fixed axis, required to find the pressure upon the axis at any instant.*

438. We shall suppose that the axis is fixed at two given points: let the axis of rotation be the axis of z, and let a and a' be the distances of the fixed points from the origin; let P, P' be the pressures at these points, $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ the angles which their directions make with the axes of xyz respectively: X, Y, Z the impressed accelerating forces of the particle m, the co-ordinates of which are xyz at the time t; and therefore $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ the effective accelerating forces of m: but since the *angular* accelerating force about the axis of rotation is calculated in Art. 432, we shall transform these effective forces as follows. Let f be the effective angular accelerating force, ω the angular velocity of the body at the time t, r the distance of m from the axis of rotation, θ the angle which r makes with the plane zx; then $x = r\cos\theta$, and $y = r\sin\theta$; differentiating twice with respect to t and observing that r does not vary with the time, because the axis from which it is measured is *fixed* in the body, and then replacing x and y, we have

$$\frac{d^2x}{dt^2} = -yf - x\omega^2, \quad \frac{d^2y}{dt^2} = xf - y\omega^2.$$

Then the moving forces $m(X + yf + x\omega^2)$, $m(Y - xf + y\omega^2)$, mZ acting parallel to the axes on the particle m, and similar forces acting on all the other particles, together with the pressures P, P' on the two fixed points of the axis ought to be in equilibrium at the time t, according to the first Principle of Art. 226. Hence by Art. 56,

$$P\cos\alpha + P'\cos\alpha' + \Sigma . m\,(X + yf + x\omega^2) = 0$$
$$P\cos\beta + P'\cos\beta' + \Sigma . m\,(Y - xf + y\omega^2) = 0$$
$$P\cos\gamma + P'\cos\gamma' + \Sigma . m\,Z \qquad\qquad = 0$$
$$-P\cos\beta . a - P'\cos\beta' . a' + \Sigma . m\,\{Zy - (Y - xf + y\omega^2)\,z\} = 0$$
$$P\cos\alpha . a + P'\cos\alpha' . a' \quad + \Sigma . m\,\{(X + yf + x\omega^2)z - Zx\} = 0$$
$$\Sigma . m\,\{(Y - xf + y\omega^2)x - (X + yf + x\omega^2)\,y\} = 0.$$

These equations may generally be much simplified in applying them to any particular case, as we shall see in the Chapter of Problems on this subject. The first, second, fourth, and fifth equations determine the four quantities $P\cos\alpha$, $P\cos\beta$, $P'\cos\alpha'$, $P'\cos\beta'$; from which the pressures perpendicular to the axis may be obtained. The third equation is the only equation which contains $P\cos\gamma$ and $P'\cos\gamma'$, and it shews that these quantities are indeterminate, but that their sum must $= -\Sigma . mZ$. Lastly, the sixth equation is independent of the pressures, and, in short, determines the motion as calculated in Art. 432: this is easily seen, since the equation by reduction becomes

$$f . \Sigma . m\,(x^2 + y^2) = \Sigma . m\,(Yx - Xy).$$

The following Proposition is an application of these equations.

PROP. *The principal axes through the centre of gravity are* permanent axes, *when the body is not acted on by any forces.*

439. An axis is said to be *permanent* when the body permanently revolves about it when it is not fixed.

Let us suppose the body moves about a *fixed* principal axis. Since no forces act upon the body it follows that X, Y, Z each vanish, hence the equations of last Article become (since the sixth gives $f = 0$)

$$P\cos\alpha + P'\cos\alpha' + \qquad \omega^2\,\Sigma . mx = 0$$
$$P\cos\beta + P'\cos\beta' + \qquad \omega^2\,\Sigma . my = 0$$
$$P\cos\gamma + P'\cos\gamma' \qquad\qquad\qquad = 0$$
$$-Pa\cos\beta - P'a'\cos\beta' - \omega^2\,\Sigma . myz = 0$$
$$Pa\cos\alpha + P'a'\cos\alpha' + \omega^2\,\Sigma . mxz = 0.$$

Since the axis of z passes through the centre of gravity, therefore $\Sigma . mx = 0$, $\Sigma . my = 0$ (Art. 413): also if the other two principal axes $x_, y_,$ make each an angle ϕ with the axes of xy respectively at the time t, we have

$$x = x_, \cos\phi + y_, \sin\phi, \text{ and } y = y_, \cos\phi - x_, \sin\phi, \; z = z_,;$$

$$\therefore \Sigma . mxz = \cos\phi \Sigma . mx_, z_, + \sin\phi \Sigma . my_, z_, = 0;$$

$$\text{so also } \Sigma . myz = 0.$$

Hence the equations become

$$P\cos\alpha + P'\cos\alpha' = 0, \quad P\cos\beta + P'\cos\beta' = 0, \quad P\cos\gamma + P'\cos\gamma' = 0,$$

$$Pa\cos\beta + P'a'\cos\beta' = 0, \quad Pa\cos\alpha + P'a'\cos\alpha' = 0,$$

these give $P = 0$ and $P' = 0$. Hence there is no pressure on the fixed axis, and therefore it would not move if the body were to rotate about it when it is not fixed.

MOTION OF A RIGID BODY ABOUT A *FIXED POINT*.

440. In calculating the motion of a rigid body about a fixed point it is found most convenient to transform the equations of motion so as to contain angular co-ordinates and angular velocities.

Let the axes of co-ordinates be drawn through the fixed point: and suppose that $\omega' \omega'' \omega'''$ are three angular velocities such that if they were simultaneously impressed upon the body about the axes xyz respectively at the expiration of the time t, the motion of the body shall be what it actually is; then these are called the angular velocities of the body about the axes at that instant.

We shall always estimate those angular velocities positive which make the body revolve from the axis of x to the axis of y about z; from y to z about x; and from z to x about y: and those negative which act in the opposite directions.

When the axes of co-ordinates are principal axes we shall use $\omega_1 \omega_2 \omega_3$ for $\omega' \omega'' \omega'''$. We shall see in the calculations which follow, that the equations of motion become much simplified by transforming them from being functions of $\omega' \omega'' \omega'''$ to functions of $\omega_1 \omega_2 \omega_3$.

PROP. *To find the linear velocities, parallel to the axes of co-ordinates, of any particle of the body in terms of the angular velocities about the axes.*

441. Let xyz be the co-ordinates to particle m at P (fig. 104): draw PM perpendicular to the axis of x: PN perpendicular to plane xy: then at the time t the velocity of m about the axis of $x = \omega' PM$: resolving this parallel to the axes of y and z and reckoning those linear velocities *positive* which tend *from* the origin, and *vice versâ*, we have

vel. of m arising from ω' parallel to $y = -\omega' PM \sin PMN = -\omega' z$

.... $z = \omega' PM \cos PMN = \omega' y$,

also velocity of m arising from ω'' parallel to $x = \omega'' z$

.... $z = -\omega'' x$,

velocity of m arising from ω''' parallel to $x = -\omega''' y$

.... $y = \omega''' x$.

Adding together those velocities which are parallel to the same axes, we have

velocity of m parallel to $x = \omega'' z - \omega''' y$,

.... $y = \omega''' x - \omega' z$,

.... $z = \omega' y - \omega'' x$.

If m be at rest at the instant of expiration of the time t these expressions vanish; the third is a necessary consequence of the other two.

Hence $x = \dfrac{\omega'}{\omega'''} z$, $y = \dfrac{\omega''}{\omega'''} z$ are the equations to a straight line through the fixed point, which is at rest at the instant under consideration.

This line is called the *Axis of Instantaneous Rotation.*

PROB. *To find the position of the instantaneous axis at any instant.*

442. Let $\alpha\beta\gamma$ be the angles which this line makes with the axes of xyz at the proposed instant: then by fig. 104,

$$\cos\alpha = \frac{AM}{AP} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{\omega'}{\sqrt{\omega'^2+\omega''^2+\omega'''^2}},$$

$$\cos\beta = \frac{MN}{AP} = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{\omega''}{\sqrt{\omega'^2+\omega''^2+\omega'''^2}};$$

$$\cos\gamma = \frac{PN}{AP} = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{\omega'''}{\sqrt{\omega'^2+\omega''^2+\omega'''^2}}.$$

By means of these we shall know the position at any instant when $\omega'\omega''\omega'''$ are known.

PROP. *To find the angular velocity of the body about the instantaneous axis.*

443. Let ω be the required angular velocity: r the distance of the particle m from the origin: then the distance of this particle from the instantaneous axis

$$= r\sin(\angle \text{ between } r \text{ and inst. axis}) = r\sqrt{1-\cos^2(\text{same } \angle)}$$

$$= \sqrt{x^2+y^2+z^2-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2};$$

$$\therefore \text{the velocity of } m = \omega\sqrt{x^2+y^2+z^2-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2}.$$

But by Art. 441 the whole velocity

$$= \sqrt{(\omega''z-\omega'''y)^2+(\omega'''x-\omega'z)^2+(\omega'y-\omega''x)^2}.$$

Let us substitute for $\omega'\omega''\omega'''$ in terms of $\alpha\beta\gamma$ by Art. 442, then whole velocity =

$$\sqrt{\omega'^2+\omega''^2+\omega'''^2}\sqrt{(z\cos\beta-y\cos\gamma)^2+(x\cos\gamma-z\cos\alpha)^2+(y\cos\alpha-x\cos\beta)^2}$$

$$= \sqrt{\omega'^2+\omega''^2+\omega'''^2}\sqrt{x^2+y^2+z^2-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2}.$$

Hence by equating these expressions,

$$\omega = \sqrt{\omega'^2+\omega''^2+\omega'''^2},$$

this is the angular velocity required.

444. COR. If a body revolve about an axis with an angular velocity ω, then the resolved part of this about another axis inclined to the former at an angle α

$$= (\omega' = \sqrt{\omega'^2+\omega''^2+\omega'''^2}\cos\alpha =)\,\omega\cos\alpha.$$

PROP. *To find the inclinations of the instantaneous axis to the principal axes.*

445. Let $\alpha_{,}\beta_{,}\gamma_{,}$ be the angles the instantaneous axis makes with the principal axes, and $\omega_1\omega_2\omega_3$ the angular velocities about the principal axes.

$$\therefore \cos\alpha_{,} = \cos\alpha\cos x_{,}x + \cos\beta\cos x_{,}y + \cos\gamma\cos x_{,}z$$

$$= \frac{\omega'}{\omega}\cos x_{,}x + \frac{\omega''}{\omega}\cos x_{,}y + \frac{\omega'''}{\omega}\cos x_{,}z, \text{ (Arts. 443, 444.)}$$

$= \dfrac{\omega_1}{\omega}$, since by resolving the angular velocities $\omega'\omega''\omega'''$ about the axis of $x_{,}$ we have by Art. 444,

$$\omega_1 = \omega'\cos x_{,}x + \omega''\cos x_{,}y + \omega'''\cos x_{,}z.$$

Similarly $\cos\beta_{,} = \dfrac{\omega_2}{\omega}$, $\cos\gamma_{,} = \dfrac{\omega_3}{\omega}$.

COR. Also $\omega'^2 + \omega''^2 + \omega'''^2 = \omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$.

PROP. *To obtain equations for calculating the angular velocities about the principal axes at any instant.*

446. Let Ax, Ay, Az be the axes of co-ordinates fixed in space; A being the fixed point of the body;

$Ax_{,}$, $Ay_{,}$, $Az_{,}$ the principal axes in the body.

Then the three equations of rotatory motion are, by Art. 428,

$$\Sigma . m\left\{y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right\} = \Sigma . m\{yZ - zY\} = L \text{ suppose.}$$

$$\Sigma . m\left\{z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right\} = \Sigma . m\{zX - xZ\} = M \text{ ....}$$

$$\Sigma . m\left\{x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right\} = \Sigma . m\{xY - yX\} = N \text{ ....}$$

Now by Art. 441.

$$\frac{dx}{dt} = \omega''z - \omega'''y, \quad \frac{dy}{dt} = \omega'''x - \omega'z, \quad \frac{dz}{dt} = \omega'y - \omega''x.$$

By differentiating these with respect to t,

$$\frac{d^2x}{dt^2} = \omega''\frac{dz}{dt} - \omega'''\frac{dy}{dt} + \frac{d\omega''}{dt}z - \frac{d\omega'''}{dt}y$$

$$= -(\omega''^2 + \omega'''^2)x + \omega'\omega''y + \omega'\omega'''z + \frac{d\omega''}{dt}z - \frac{d\omega'''}{dt}y,$$

so $$\frac{d^2y}{dt^2} = -(\omega'^2 + \omega'''^2)y + \omega''\omega'x + \omega''\omega'''z + \frac{d\omega'''}{dt}x - \frac{d\omega'}{dt}z;$$

$$\frac{d^2z}{dt^2} = -(\omega'^2 + \omega''^2)z + \omega'''\omega'x + \omega'''\omega''y + \frac{d\omega'}{dt}y - \frac{d\omega''}{dt}x.$$

Hence $$\Sigma.m\left\{y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right\} = (\omega'''^2 - \omega''^2)\Sigma.myz$$

$$+\left(\omega'''\omega' - \frac{d\omega''}{dt}\right)\Sigma.mxy - \left(\omega''\omega' + \frac{d\omega'''}{dt}\right)\Sigma.mxz$$

$$+\omega'''\omega''\Sigma.my^2 - \omega''\omega'''\Sigma.mz^2 + \frac{d\omega'}{dt}\Sigma.m(y^2+z^2) = L \ldots (1).$$

Now suppose the fixed axes Ax, Ay, Az were so chosen that at the instant of expiration of the time t the principal axes should coincide with them. Then *at this instant,*

$\Sigma.mxy=0$, $\Sigma.mxz=0$, $\Sigma.myz=0$: also $\omega'=\omega_1$, $\omega''=\omega_2$, $\omega'''=\omega_3$;

and likewise $\dfrac{d\omega'}{dt} = \dfrac{d\omega_1}{dt}$, for the cosines of the angles, which the instantaneous axis makes with the axes x and $x_{,}$ at a small time h after $x_{,}$ coincided with x, will differ by a small quantity varying as h^2: and therefore also ω_1 and ω' (see Art. 444) will differ by a quantity varying as h^2; and their differential coefficients with respect to t at the instant when $x_{,}$ coincides with x will be equal. Hence equation (1) becomes *at this instant*

$$\omega_2\omega_3\Sigma.m(y_{,}^2 - z_{,}^2) + \frac{d\omega_1}{dt}\Sigma.m(y_{,}^2 + z_{,}^2) = L,$$

the letters with subscript accents having reference to the principal axes.

Now this equation is independent of the epoch from which the time is measured; it is also independent of the angles which the principal axes make with the fixed axes in space. It follows, then, that this equation will hold for *every instant* of the time t; and is therefore generally true. We might have arrived at this equation by direct elimination; but the process is very tedious.

Now $\Sigma . m(y_{,}^2 + z_{,}^2) = A$; and $\Sigma . m(y_{,}^2 - z_{,}^2) = C - B$;

$$\left.\begin{aligned} \therefore A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3 &= L_{,} \\ \text{similarly } B\frac{d\omega_2}{dt} + (A - C)\,\omega_1\omega_3 &= M_{,} \\ C\frac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 &= N_{,} \end{aligned}\right\} \text{.........} (2).$$

By means of these three equations the three quantities $\omega_1\, \omega_2\, \omega_3$ must be determined.

PROP. *To determine the position of the body in space when the angular velocities about the principal axes are known.*

447. We consider, as before, those angular velocities positive which tend to turn the body from the axis of $x_{,}$ to the axis $y_{,}$ about $z_{,}$, from $y_{,}$ to $z_{,}$ about $x_{,}$, and from $z_{,}$ to $x_{,}$ about $y_{,}$.

Also by Art. 444 an angular velocity is resolved about any new axis by multiplying it by the cosine of the angle between the axes.

Now the position of the principal axes of the body at the time t, is determined by the values of θ, ϕ, ψ, these angles being measured as explained in Art. 415: it follows then, that $\omega_1\, \omega_2\, \omega_3$ must be functions of θ, ϕ, ψ, and $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, $\frac{d\psi}{dt}$.

The resolved parts of $\frac{d\theta}{dt}$ about the axes of $x_, y_, z_,$ are

$$\frac{d\theta}{dt}\cos\phi, \quad -\frac{d\theta}{dt}\sin\phi, \quad 0,$$

the resolved parts of $\frac{d\phi}{dt}$ about these axes are

$$0, \qquad 0, \qquad \frac{d\phi}{dt},$$

and the resolved parts of $\frac{d\psi}{dt}$ about these axes are

$$-\frac{d\psi}{dt}\cos zx_,, \quad -\frac{d\psi}{dt}\cos zy_,, \quad -\frac{d\psi}{dt}\cos zz_,,$$

or $-\frac{d\psi}{dt}\sin\phi\sin\theta, \quad -\frac{d\psi}{dt}\cos\phi\sin\theta, \quad -\frac{d\psi}{dt}\cos\theta$ (see Art. 415).

Hence, adding those about the same axes,

$$\omega_1 = \frac{d\theta}{dt}\cos\phi - \frac{d\psi}{dt}\sin\phi\sin\theta,$$

$$\omega_2 = -\frac{d\theta}{dt}\sin\phi - \frac{d\psi}{dt}\cos\phi\sin\theta,$$

$$\omega_3 = \frac{d\phi}{dt} - \frac{d\psi}{dt}\cos\theta.$$

In these we must substitute the values of $\omega_1\ \omega_2\ \omega_3$ obtained by integrating the equations in Art. 446, and we shall find θ, ϕ, ψ, and so determine the position of the principal axes, and consequently of the body, at any proposed instant.

COR. By the above equations we obtain

$$\frac{d\theta}{dt} = \omega_1\cos\phi - \omega_2\sin\phi$$

$$\sin\theta\frac{d\psi}{dt} = -\omega_1\sin\phi - \omega_2\cos\phi$$

$$\frac{d\phi}{dt} = \omega_3 - \frac{\cos\theta}{\sin\theta}(\omega_1\sin\phi + \omega_2\cos\phi).$$

448. We have thus obtained general equations for calculating the motion of a rigid body moving about a fixed point, and for determining its position in space. We proceed to apply these equations: first, to the case where no forces act, except such as always have their resultant passing through the fixed point; and secondly, to the case where small disturbing forces act in addition to these.

PROP. *To shew, that when the body is acted on by forces, which have a single resultant passing through the fixed point, there exists a plane, which has an invariable position in space during the motion, and which can be determined in position at any instant in terms of the co-ordinates and velocities of the different parts of the body at that instant.*

449. The equations (2) of Art. 446 become, in this case,

$$\left.\begin{array}{l} A\dfrac{d\omega_1}{dt}+(C-B)\,\omega_2\,\omega_3=0,\ \ B\dfrac{d\omega_2}{dt}+(A-C)\,\omega_1\,\omega_3=0, \\ \qquad\text{and } C\dfrac{d\omega_3}{dt}+(B-A)\,\omega_1\,\omega_2=0. \end{array}\right\}\ \ldots (1).$$

Let abc be the cos of the angles which x makes with xyz,

$a'b'c'$ .... y ....

$a''b''c''$ .... z ....

Multiply equations (1) by a, b, c, and add

$$\therefore A\left\{a\frac{d\omega_1}{dt}+\omega_1\,(b\,\omega_3-c\,\omega_2)\right\}+B\left\{b\frac{d\omega_2}{dt}+\omega_2\,(c\,\omega_1-a\,\omega_3)\right\}$$

$$+\,C\left\{c\,\frac{d\omega_3}{dt}+\omega_3\,(a\,\omega_2-b\,\omega_1)\right\}=0\ \ldots\ldots\ldots (2).$$

We shall now prove, that the coefficients of $\omega_1\,\omega_2\,\omega_3$ in this equation respectively equal $\dfrac{da}{dt}, \dfrac{db}{dt}, \dfrac{dc}{dt}$.

By Art. 416, $a=\cos\phi\cos\psi+\sin\phi\sin\psi\cos\theta$

$b=-\sin\phi\cos\psi+\cos\phi\sin\psi\cos\theta$

$c=-\sin\psi\sin\theta.$

$$\therefore \frac{da}{dt} = \{\cos\phi \sin\psi \cos\theta - \sin\phi \cos\psi\} \frac{d\phi}{dt}$$

$$+ \{\sin\phi \cos\psi \cos\theta - \cos\phi \sin\psi\} \frac{d\psi}{dt} - \sin\phi \sin\psi \sin\theta \frac{d\theta}{dt}$$

$$= \{\cos\phi \sin\psi \cos\theta - \sin\phi \cos\psi\} \left\{\frac{d\phi}{dt} - \cos\theta \frac{d\psi}{dt}\right\}$$

$$- \sin\psi \sin\theta \left\{\sin\phi \frac{d\theta}{dt} + \cos\phi \sin\theta \frac{d\psi}{dt}\right\}.$$

$$= b\omega_3 - c\omega_2 \text{ by Art. 447.}$$

In the same manner it may be shewn, that

$$\frac{db}{dt} = c\,\omega_1 - a\,\omega_3, \text{ and } \frac{dc}{dt} = a\omega_2 - b\,\omega_1.$$

Hence equation (2) becomes

$$A\left\{a\frac{d\omega_1}{dt} + \omega_1\frac{da}{dt}\right\} + B\left\{b\frac{d\omega_2}{dt} + \omega_2\frac{db}{dt}\right\} + C\left\{c\frac{d\omega_3}{dt} + \omega_3\frac{dc}{dt}\right\} = 0;$$

$$\therefore A a\omega_1 + Bb\,\omega_2 + Cc\,\omega_3 = l.$$

$$\text{Similarly,} \quad Aa'\omega_1 + Bb'\omega_2 + Cc'\omega_3 = l',$$

$$Aa''\omega_1 + Bb''\omega_2 + Cc''\omega_3 = l'';$$

where l, l', l'', are constants depending upon the configuration of the system at any given instant.

Add the squares of these three equations together; observing that since the angle between any two axes of the same system of co-ordinates equals a right angle, therefore

$$ab + a'b' + a''b'' = 0, \quad ac + a'c' + a''c'' = 0, \quad bc + b'c' + b''c'' = 0;$$

and we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = l^2 + l'^2 + l''^2 = k^2 \text{ suppose.}$$

Hence if at any instant we draw a line AI' making angles with the fixed axes of which the cosines are

$$\frac{Aa\omega_1 + Bb\omega_2 + Cc\omega_3}{\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}}, \quad \frac{Aa'\omega_1 + Bb'\omega_2 + Cc'\omega_3}{\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}},$$

$$\frac{Aa''\omega_1 + Bb''\omega_2 + Cc''\omega_3}{\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}},$$

this line, and therefore the plane perpendicular to it, will remain invariable during the whole motion: since these three expressions, though *implicitly* functions of the time, are, as has been shewn above, *explicitly* independent of the time. For this reason this plane is called the *Invariable Plane.**

450. Cor. 1.

$$\cos I'Ax_{,} = a\cos I'Ax + a'\cos I'Ay + a''\cos I'Az = \frac{A\omega_1}{k},$$

$$\text{also } \cos I'Ay_{,} = \frac{B\omega_2}{k}, \ \cos I'Az_{,} = \frac{C\omega_3}{k}.$$

451. Cor. 2. If the invariable plane be taken for the plane of xy, then

$$\frac{A\omega_1}{k} = \cos zx_{,} = \sin\phi\sin\theta: \text{ Art. 415.}$$

$$\frac{B\omega_2}{k} = \cos zy_{,} = \cos\phi\sin\theta,$$

$$\frac{C\omega_3}{k} = \cos zz_{,} = \cos\theta.$$

Prop. *A body revolves about its centre of gravity acted on by no forces but such as pass through that point: required to integrate the equations of motion.*

452. The equations (2) of Art. 446 become in this case

$$A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3 = 0, \quad B\frac{d\omega_2}{dt} + (A - C)\,\omega_1\omega_3 = 0,$$

* An indefinite number of invariable planes might be drawn: thus any plane of which the cosines of position are M, N, $\sqrt{1 - M^2 - N^2}$, where M and N are *any* functions of l, l', l'', will be invariable. But the plane spoken of in the text exclusively possesses another property, which we shall prove in another place; and for that reason it receives, in preference to the others, the distinguishing name.

$$\text{and } C\frac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 = 0,$$

the principal axes being drawn through the centre of gravity.

Multiply these equations by $\omega_1\,\omega_2\,\omega_3$ respectively and add; then

$$A\,\omega_1\frac{d\omega_1}{dt} + B\,\omega_2\frac{d\omega_2}{dt} + C\,\omega_3\frac{d\omega_3}{dt} = 0$$

$$A\,\omega_1^2 + B\,\omega_2^2 + C\,\omega_3^2 = \text{constant} = h^2.$$

Again multiply the equations by $A\omega_1$, $B\omega_2$, $C\omega_3$, and add;

$$\therefore\ A^2\omega_1\frac{d\omega_1}{dt} + B^2\omega_2\frac{d\omega_2}{dt} + C^2\omega_3\frac{d\omega_3}{dt} = 0;$$

$$\therefore\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = \text{constant} = k^2 \text{ (Art. 449)}.$$

Eliminating ω_3^2 from these two equations, we have

$$A\,(A - C)\,\omega_1^2 + B\,(B - C)\,\omega_2^2 = k^2 - Ch^2;$$

$$\therefore\ \omega_2^2 = \frac{1}{B\,(B - C)}\{k^2 - Ch^2 - A\,(A - B)\,\omega_1^2\},$$

$$\text{and } \omega_3^2 = \frac{1}{C\,(C - B)}\{k^2 - Bh^2 - A\,(A - C)\,\omega_1^2\}.$$

Hence the first of the equations of motion gives

$$\frac{d\omega_1}{dt} + \sqrt{\frac{(A-C)(A-B)}{BC}\left\{\omega_1^2 - \frac{k^2 - Ch^2}{A(A-C)}\right\}\left\{\frac{k^2 - Bh^2}{A(A-B)} - \omega_1^2\right\}} = 0;$$

the integral of this equation, which in the general case cannot be found, will give ω_1 in terms of t and then ω_2 and ω_3 will be known.

Knowing $\omega_1\omega_2\omega_3$ the position of the body at any time is determined by integrating the equations of Art. 447.

453. By means of the two integrals which we have obtained we can shew, that the angular velocity about the axis perpendicular to the invariable plane is constant.

For the cosines of the angles, which the instantaneous axis makes with the axes of $x_, y_, z_,$, are $\omega_1 \div \omega$, $\omega_2 \div \omega$, and $\omega_3 \div \omega$ (Art. 445): and therefore the cosines of the angles, which the instantaneous axis makes with the fixed axes xyz, are

$$\frac{a\omega_1 + b\omega_2 + c\omega_3}{\omega}, \quad \frac{a'\omega_1 + b'\omega_2 + c'\omega_3}{\omega}, \quad \frac{a''\omega_1 + b''\omega_2 + c''\omega_3}{\omega}.$$

From these, and the cosines of the angles, which the perpendicular to the invariable plane makes with the fixed axes, we can easily find the cosine of the angle, which this perpendicular makes with the instantaneous axis. For this end, multiply the two cosines which refer to the axis of x, then those of y, then those of z; and add the results, observing that

$$a^2 + b^2 + c^2 = 1, \quad a'^2 + b'^2 + c'^2 = 1, \quad a''^2 + b''^2 + c''^2 = 1,$$
$$ab + a'b' + a''b'' = 0, \; ac + a'c' + a''c'' = 0, \; bc + b'c' + b''c'' = 0,$$

we shall have the required cosine

$$= \frac{A\omega_1^2 + B\omega_2^2 + C\omega_3^2}{\omega\sqrt{A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2}} = \frac{h^2}{\omega k^2}.$$

Hence the angular velocity about the perpendicular to the invariable plane $= h^2 \div k$; and is therefore constant.

454. The motion of the body may be represented geometrically.

Let $x_, y_, z_,$ be the co-ordinates to any chosen point in the instantaneous axis, at the time t, referred to the principal axes: r the distance of this point from the origin: p the length of the projection of r on the axis perpendicular to the invariable plane; $\omega_0 (= h^2 \div k)$ the angular velocity of the body about this axis: then

$$\omega_1 = \frac{x_,}{r}\omega, \quad \omega_2 = \frac{y_,}{r}\omega, \quad \omega_3 = \frac{z_,}{r}\omega, \quad \omega_0 = \frac{p}{r}\omega:$$

and the integrals in the last Art. become

$$Ax_,^2 + By_,^2 + Cz_,^2 = \frac{r^2 h^2}{\omega^2} = \frac{p^2 h^2}{\omega_0^2} = \frac{p^2 k^2}{h^2}:$$

$$\text{and } A^2 x_,^2 + B^2 y_,^2 + C^2 z_,^2 = \frac{r^2 k^2}{\omega^2} = \frac{p^2 k^2}{\omega_0^2} = \frac{p^2 k^4}{h^4}.$$

The first of these equations proves, that if the point $x_{,} y_{,} z$ be so chosen at each instant on the instantaneous axis, that p be always the same, then the locus of this point in the body is an ellipsoid, having its centre at the fixed point and its axes coinciding with the principal axes. The second equation shews, that the perpendicular from the origin on the tangent plane at $x_{,} y_{,} z_{,} = p$, and therefore, that the tangent plane to the ellipsoid at $x_{,} y_{,} z_{,}$. The cosines of the angles which the perpendicular to the tangent plane makes with the principal axes $= \frac{A\omega_1}{k}, \frac{B\omega_2}{k}, \frac{C\omega_3}{k}$: and therefore (Art. 450) the tangent plane is always parallel to the invariable plane, and at a constant distance p: and is therefore itself an invariable plane.

The motion may therefore be thus conceived geometrically. Suppose an imaginary ellipsoid to be described in the body as above explained, and at the point where this ellipsoid cuts the instantaneous axis in its initial position draw a tangent plane to the ellipsoid: let this remain a fixed plane in space during the motion, and upon it make the ellipsoid roll (without sliding), in such a manner, that the body may revolve about the axis perpendicular to the invariable plane with a uniform angular velocity $= h^2 \div k$, the constants h and k being determined from the initial circumstances. This geometrical motion will exactly coincide with the actual motion of the body.

455. Cor. 1. If $A = B = C$, then the ellipsoid becomes a sphere: therefore the point of contact on the fixed tangent plane will remain stationary; and the body will revolve uniformly about the instantaneous axis, which remains fixed both in the body and in space.

456. Cor 2. If $A = B$, then the ellipsoid becomes a spheroid: and the point of contact will describe a circle on the fixed tangent plane: and the body moves uniformly about its instantaneous axis, while this axis moves uniformly in a cone about the perpendicular to the invariable plane. It will be easily seen by drawing a figure, that, since the radius-vector in an ellipse lies on the same side of the perpendicular on the tangent with the semi-major axis, the instantaneous axis will, or

will not, lie between the perpendicular to the invariable plane and the axis of C, according as the spheroid is prolate or oblate; that is, according as C is greater or less than A. Some writers represent the motion by supposing a hoop fixed in the body to revolve in contact with a fixed circle, placed inside the hoop when C is less than A, as represented by fig. 106: but placed outside when C is greater than A: AI is the instantaneous axis, AZ the perpendicular to the invariable plane, $AZ_{\prime}$ the axis of C.

The results of these Cors. may be easily obtained independently from the equations of motion by introducing the relations connecting A, B, C. In both the above cases the differential equation in ω_1 of Art. 452 can be integrated.

457. If we obssrve the apparent motion of the stars night after night we remark, that they all seem to move in parallel circles about a point nearly coinciding with the star named (on that account) the Pole Star. This proves that the axis about which the Earth revolves points towards the Pole Star, and never deviates from that direction by an angle appreciable by ordinary observation. Also geodetic measurements and other calculations for ascertaining the Figure of the Earth shew, that this axis of rotation coincides (so far as the approximation is carried) with the geometrical axis of the spheroidal form of the Earth's surface. Theory shews that there is a necessary connexion between these two facts which are apparently independent of each other. This we proceed to prove.

Prop. *Suppose the Earth revolves about an axis nearly coinciding with one of its principal axes at any given time: required to find the motion, all external forces being neglected.*

458. Let the axis of $x_{\prime}$ be that near which the instantaneous axis lies at the given time t. Now the sine of the angle which these two axes make with each other $= \sqrt{\dfrac{\omega_1^2 + \omega_2^2}{\omega_1^2 + \omega_2^2 + \omega_3^2}}$ (Art. 445), and this is small by hypothesis: hence $\omega_1^2 + \omega_2^2$ is small, and ω_1 and ω_2 are small: and the equations (2) of Art. 446 give

$$A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3=0,\quad B\frac{d\omega_2}{dt}+(A-C)\,\omega_1\omega_3=0,$$

$$\text{and}\quad C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2=0;$$

then neglecting the product of ω_1 and ω_2, the last equation gives $\omega_3 =$ constant $= n$: and the others give

$$A\frac{d\omega_1}{dt}+(C-B)\,\omega_2 n=0,\quad B\frac{d\omega_2}{dt}+(A-C)\,\omega_1 n=0;$$

$$\therefore\ \frac{d^2\omega_1}{dt^2}+\frac{(A-C)(B-C)}{AB}\,n^2\omega_1=0;$$

$$\therefore\ \omega_1=e\sin\left\{\sqrt{\frac{(A-C)(B-C)}{AB}}\,nt+f\right\},$$

e and f being constants which depend upon the circumstances at any given time:

$$\therefore\ \omega_2=\frac{A}{(B-C)\,n}\,\frac{d\omega_1}{dt}$$

$$=e\sqrt{\frac{A}{B}\frac{A-C}{B-C}}\cos\left\{\sqrt{\frac{(A-C)(B-C)}{AB}}\,nt+f\right\},$$

and since $\omega_1^2+\omega_2^2$ is small at the given time, e is small: and since e is constant it shews that ω_1 and ω_2 are *always* small so long as $(A-C)(B-C)$ is positive.

If however $(A-C)(B-C)$ be negative, then the trigonometrical expressions for $\omega_1\,\omega_2$ must be replaced by exponentials, and consequently they will not remain small.

From this we gather that if a body revolve at any time about an axis coinciding nearly with the principal axis of greatest or least moment, the axis of rotation will always nearly coincide with that principal axis. But if the axis be that of mean moment the instantaneous axis of rotation will deviate more and more from that principal axis till it approaches the principal axis of either greatest or least moment.

Cor. If the instantaneous axis actually coincide with a principal axis at first, then $e=0$, and ω_1 and ω_2 each vanish. Hence any principal axis is a permanent axis (see Art. 439).

If, however, the slightest cause tend to make the instantaneous axis of rotation deviate from the principal axis, the rotatory motion may be said to be *stable* or *unstable* according as the principal axis in question is not or is the *mean* principal axis.

This points out an admirable adaptation in the laws of nature: that the motion of rotation which causes the heavenly bodies to *bulge* at their equators, in so doing, gives them such a figure as to insure the stability of their rotation.

We shall now consider the action of the Sun and Moon on the rotatory motion of the Earth.

PROP. *To obtain equations for calculating the rotatory motion of the Earth when acted on by the Sun and Moon.*

459. The equations of motion referred to the principal axes are by Art. 446,

$$A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3 = L_{,},\quad B\frac{d\omega_2}{dt} + (A - C)\,\omega_1\omega_3 = M_{,}$$

$$\text{and } C\frac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 = N_{,}.$$

To calculate $L_{,}M_{,}N_{,}$ let S be the mass of one of the disturbing bodies: $x_{,}y_{,}z_{,}$ the co-ordinates of the centre of S; $x_{,}'y_{,}'z_{,}'$ the co-ordinates to any particle m of the Earth's mass referred to the principal axes, the origin being the centre of gravity of the Earth: $r^2 = x_{,}^2 + y_{,}^2 + z_{,}^2$.

Then the attraction of S on the particle m, resolved parallel to the axes $y_{,}$ and $z_{,}$ and estimated positive in directions *from* the origin, is

$$\frac{S\,(y_{,} - y_{,}')}{\{(x_{,} - x_{,}')^2 + (y_{,} - y_{,}')^2 + (z_{,} - z_{,}')^2\}^{\frac{3}{2}}} = Y_{,} \text{ suppose}$$

$$\frac{S\,(z_{,} - z_{,}')}{\{(x_{,} - x_{,}')^2 + (y_{,} - y_{,}')^2 + (z_{,} - z_{,}')^2\}^{\frac{3}{2}}} = Z_{,} \text{ suppose.}$$

Hence $L_{,} = \Sigma \cdot m\,(y_{,}'Z_{,} - z_{,}'Y_{,})$, see Art. 446.

$$= S\Sigma . m \left\{ \frac{z_, y_,' - y_, z_,'}{\{r^2 - 2(x_, x_,' + y_, y_,' + z_, z_,') + (x_,'^2 + y_,'^2 + z_,'^2)\}^{\frac{3}{2}}} \right\}$$

$$= \frac{S}{r^3} \Sigma . m(z_, y_,' - y_, z_,') \left(1 - \frac{2(x_, x_,' + y_, y_,' + z_, z_,') - (x_,'^2 + y_,'^2 + z_,'^2)}{r^2}\right)^{-\frac{3}{2}}$$

But $S\Sigma . m (z_, y_,' - y_, z_,') = z_, \Sigma . m y_,' - y_, \Sigma . m z_,' = 0$, because the origin of co-ordinates is at the centre of gravity.

$$\therefore\ L_, = \frac{3S}{r^5} \Sigma . m (z_, y_,' - y_, z_,') (x_, x_,' + y_, y_,' + z_, z_,'),$$

neglecting the cubes of very small quantities,

$$= \frac{3S}{r^5} \Sigma . m \{ (y_,'^2 - z_,'^2) z_, y_, - (y_,^2 - z_,^2) z_,' y_,' + z_, x_, y_,' x_,' - y_, x_, z_,' x_,' \}$$

$$= \frac{3S}{r^5} z_, y_, \Sigma . m (y_,'^2 - z_,'^2) \text{ by the property of principal axes,}$$

$$\therefore\ L_, = \frac{3S}{r^5} z_, y_, (C - B).$$

In the same manner we should find

$$M_, = \frac{3S}{r^5} x_, z_, (A - C), \quad \text{and } N_, = \frac{3S}{r^5} x_, y_, (B - A).$$

Hence the equations of motion become

$$A \frac{d\omega_1}{dt} + (C - B)\,\omega_2 \omega_3 = \frac{3S}{r^5} y_, z_, (C - B),$$

$$B \frac{d\omega_2}{dt} + (A - C)\,\omega_1 \omega_3 = \frac{3S}{r^5} x_, z_, (A - C),$$

$$C \frac{d\omega_3}{dt} + (B - A)\,\omega_1 \omega_2 = \frac{3S}{r^5} x_, y_, (B - A).$$

In these equations the disturbing body is supposed to be at a very great distance, as is the case with the Sun and Moon; but it is remarkable that they are very nearly correct even when the attracting body is very near the Earth, supposing the Earth's figure to be spheroidal. For a demonstration of this we refer the reader to the *Mécanique Céleste*. Liv. v. Chap. I. §. 3.

It will be observed that we have taken account of only one disturbing body S in these equations: but since the perturbations are small and the equations in $\omega_1\omega_2\omega_3$ linear, we may calculate the effects of the disturbing bodies singly and add them together, Art. 288.

PROP. *To prove that the velocity of rotation of the Earth, and consequently the length of the mean day, is not altered by the action of the Sun and Moon, very small quantities being neglected.*

460. If we neglect the disturbing forces and suppose the figure of the Earth to be one of revolution and not differing much from a sphere, that is, $B = A$, and each of these nearly $= C$, the difference being of the order of the ellipticity of the terrestrial spheroid; then in this case the equations of the last Article give, for a first approximation, $\omega_3 =$ const. $= n$, ω_1 and ω_2 very small quantities. These values may be put in the small terms of our equations in order to obtain a nearer approximation.

If we multiply the three equations of last Article by $\omega_1\omega_2\omega_3$ and divide them by A, B, C respectively and add them together, we have

$$\frac{d(\omega_1^2 + \omega_2^2 + \omega_3^2)}{dt} + 2\,\omega_1\omega_2\omega_3\left\{\frac{C-B}{A} + \frac{A-C}{B} + \frac{B-A}{C}\right\}$$

$$= \frac{6S}{r^3}\left\{\frac{y_,z_,}{r^2}\,\frac{C-B}{A}\,\omega_1 + \frac{x_,z_,}{r^2}\,\frac{A-C}{B}\,\omega_2 + \frac{x_,y_,}{r^2}\,\frac{B-A}{C}\,\omega_3\right\}.$$

Now $\omega_1\omega_2$ are each extremely small; $\frac{C-B}{A}$, $\frac{A-C}{B}$ are of the order of the ellipticity of the Earth; and $\frac{B-A}{C}$ is extremely small, since if we suppose the Earth a figure of revolution this expression vanishes; also $\frac{S}{r^3}$ is very small, because S varies as the cube of the radius of the body S.

Hence if we neglect extremely small quantities

$$\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \text{const.} = n,$$

since the *mean* values of $\omega_1 \omega_2 \omega_3$ are 0, 0, n.

Hence the angular velocity of the Earth is constant, and the length of the mean day is not affected by the action of the Sun and Moon, when we neglect inappreciable quantities.

A full discussion of this important question will be found in the *Méc. Céleste*, Liv. v. Chap. I. §. 8, 9: and also in the *Mémoires de l'Académie Royale des Sciences de l'Institut de France*: Vol. VII. p. 199.

Astronomical observations shew in a remarkable degree, that the length of the mean day has been invariable for a long period of time. We proceed to explain how this result is obtained from observations.

PROP. *To shew from observations made on eclipses, that the length of the mean day has been invariable for a great length of time.*

461. Let us take for the unit of time the length of any day at the present epoch: and suppose the day has been decreasing by a parts. Let n be the mean angular motion of the Moon on the day which is taken for the unit of time; then n is the number of degrees through which the Moon moves on that day: and $n(1 + a)$, $n(1 + 2a)$,...... are the angles described by the Moon during the days preceding that day in order: and the angle described during t days $= nt + \frac{1}{2} na(t - 1)t$, and if t be very large this angle $= nt + \frac{1}{2} nat^2$. Let n' be the mean motion of the Sun on the day of which the length is the unit of time, then the angle described by the Sun in the t days now elapsed $= n't + \frac{1}{2} n'at^2$, and the difference of longitude in the Sun and Moon being λ now, was $= \lambda + (n - n')t + \frac{1}{2}(n - n')at^2$ at the distance of t days from the present time.

Let δ be the error made in calculating the difference of longitudes of the Sun and Moon at a distance of t days on the supposition of the invariability of the length of the day: then $\frac{1}{2}(n - n')at^2 = \delta$.

Now the values of δ have been calculated in the *Connaissance des Tems* of 1800, from 27 eclipses observed by the Chaldees, the Greeks, and the Arabs. The greatest value of δ corresponds to an eclipse observed in the year B.C. 382: for this $\delta = -27' \, 41''$. For the most ancient eclipse $\delta = 2''$; this eclipse being observed by the Chaldees in the year B.C. 720.

Let i be the number of centuries in t days: then $t = 36525\, i$. By the mean of modern observations on the Sun and Moon it is found that $(n - n')\, 36525 = 445268^0$: for the most ancient eclipse $i = 25.56$;

$$\therefore \delta = \tfrac{1}{2}\,(36525)\,.\,(25.56)^2\, a \times 445268^0.$$

Now if the day be shorter by a ten-millionth part than at the epoch of the most ancient eclipse on record, then

$$(36525)\,(25.56)\, a = 0.0000001\,;$$

$$\therefore \delta = \tfrac{1}{2}\,(25.56)\,.\,(0.0000001)\,.\,(445268)^0 = 34',$$

a value which renders the observed eclipse impossible. The same will be true of each of the above mentioned ancient eclipses.

From this we learn, that the length of the day has not changed even by a hundred and fifteenth part of a second of time during the last twenty-five centuries. M. Poisson's *Traité de Mécanique*, Seconde Edition, Tom. II. p. 196—200.

462. By comparing the observed north-polar distances of stars made at epochs distant from each other Bradley shewed that the point in the heavens to which the Earth's axis of rotation is directed is not stationary, although for periods of time not very long this deviation, as we remarked in Art. 457, is not perceptible. It becomes an interesting question, then, to ascertain the cause of this perturbation. Since we neglected the action of the Sun and Moon in the calculation of Art. 458, we may readily conjecture that the action of these bodies is the cause required. This we proceed to demonstrate.

PROP. *To determine the position of the axis of rotation of the Earth at any given time, the action of the Sun being considered; and the figure of the Earth being taken to be one of revolution.*

463. We shall refer the disturbing body S to the ecliptic. Let the plane of the ecliptic be the plane of XY: the axis of X being drawn through the first point of Aries, which is *moveable*; the centre of the Earth the origin of co-ordinates; $x_, y_, z_,$ parallel to the principal axes; $\theta =$ the angle between the equator and ecliptic, or the angle between the axes of $z_,$ and Z; $\phi =$ the right ascension of the axis of $x_,$, or the angle between the axes of $x_,$ and X; $l =$ longitude of the Sun; $r =$ distance of the Sun from the Earth's centre; then by Spherical Trig.

$$\frac{x_,}{r} = \cos\phi\cos l + \sin\phi\sin l\cos\theta,$$

$$\frac{y_,}{r} = -\sin\phi\cos l + \cos\phi\sin l\cos\theta, \quad \frac{z_,}{r} = -\sin l\sin\theta;$$

$$\therefore \frac{y_, z_,}{r^2} = \quad \tfrac{1}{2}\sin 2l\sin\theta\sin\phi - \tfrac{1}{2}\sin^2 l\sin 2\theta\cos\phi,$$

$$\frac{x_, z_,}{r^2} = -\tfrac{1}{2}\sin 2l\sin\theta\cos\phi - \tfrac{1}{2}\sin^2 l\sin 2\theta\sin\phi;$$

then putting $P = \dfrac{3S}{2r^3}\sin 2l\sin\theta$, and $P' = \dfrac{3S}{2r^3}\sin^2 l\sin 2\theta$,

we have by the equations of Art. 459, ($B = A$),

$$\left.\begin{aligned} &\frac{d\omega_1}{dt} + \frac{C-A}{A}\omega_2\omega_3 = \frac{C-A}{A}(P\sin\phi - P'\cos\phi) \\ &\frac{d\omega_2}{dt} - \frac{C-A}{A}\omega_1\omega_3 = \frac{C-A}{A}(P\cos\phi + P'\sin\phi) \\ &\frac{d\omega_3}{dt} = 0. \end{aligned}\right\} \ldots (1).$$

The third equation gives $\omega_3 =$ constant $= n$; and therefore $\phi = nt +$ small terms (see Art. 447).

Let the time be measured from the epoch when the Sun was in Aries: then $l = n't$. Since $B = A$ any axis in the plane $x_, y_,$ is a principal axis: let the axis of $x_,$ be so chosen, that when $t = 0$ it passed through Aries: then $\phi = nt$ neglecting small terms; and

$$P = \frac{3S}{2r^3}\sin\theta\sin 2n't; \quad P' = \frac{3S}{4r^3}\sin 2\theta(1 - \cos 2n't).$$

We shall neglect the variations of the inclination (θ) of the equator and ecliptic in calculating small terms.

Since the equations (1) are linear we may take one term only of P and P' in the calculation: let $k \sin it$ and $k' \cos it$ be corresponding terms: then i admits of two values 0 and $2n'$. Considering these terms we have

$$\frac{d\omega_1}{dt} + \frac{C-A}{A} n\omega_2 = \frac{C-A}{2A}\{(k-k')\cos(n-i)t - (k+k')\cos(n+i)t\},$$

$$\frac{d\omega_2}{dt} - \frac{C-A}{A} n\omega_1 = \frac{C-A}{2A}\{-(k-k')\sin(n-i)t + (k+k')\sin(n+i)t\}.$$

To solve these differentiate the first with respect to t and eliminate $\frac{d\omega_2}{dt}$ by the second;

$$\therefore \frac{d^2\omega_1}{dt^2} + \left(\frac{C-A}{A}\right)^2 n^2\omega_1 = \frac{C-A}{2A}\left\{(k-k')\left(\frac{C-A}{A}n - n + i\right)\sin(n-i)t\right.$$
$$\left. - (k+k')\left(\frac{C-A}{A}n - n - i\right)\sin(n+i)t\right\}.$$

The integral of this is of the form

$$\omega_1 = C_1 \cos\left(\frac{C-A}{A}nt + C_2\right) + M\sin(n-i)t + N\sin(n+i)t,$$

where C_1 and C_2 are constants independent of the disturbing forces; and M and N are to be found by putting this value for ω_1 in the differential equation; we find that

$$2M = \frac{(k-k')(C-A)\{Cn - (2n-i)A\}}{(C-A)^2n^2 - A^2(n-i)^2},$$

$$2N = -\frac{(k+k')(C-A)\{Cn - (2n+i)A\}}{(C-A)^2n^2 - A^2(n+i)^2}.$$

Now the only values of $\frac{i}{n}$ are 0 and $\frac{2n'}{n}$ $(= \frac{2}{365}$, since there are 365 days in a year): hence by neglecting $\frac{i}{n}$ in the small terms, we have

$$M = \frac{k-k'}{2n}\frac{C-A}{C}, \quad N = -\frac{k+k'}{2n}\frac{C-A}{C}.$$

Now when there are no disturbing forces $\omega_1 = 0$, and consequently $C_1 = 0$;

$$\therefore\ \omega_1 = M \sin (n - i)\, t + n \sin (n + i)\, t,$$

$$\omega_2 = M \cos (n - i)\, t + N \cos (n + i)\, t.$$

Returning to the axes fixed in space and choosing the plane of the ecliptic for the plane of xy we must put the values of ω_1 and ω_2 in the equations (Art. 447. Cor. 1.)

$$\frac{d\theta}{dt} = \quad \omega_1 \cos \phi - \omega_2 \sin \phi,$$

$$\sin \theta \frac{d\psi}{dt} = - \omega_1 \sin \phi - \omega_2 \cos \phi,$$

$$\frac{d\phi}{dt} = n - \cot \theta\, (\omega_1 \sin \phi + \omega_2 \cos \phi),$$

in which θ is the obliquity of the ecliptic; ϕ the right ascension of a fixed terrestrial meridian; ψ the longitude of Aries measured in a retrograde direction: fig. 97.

Now $\phi = nt$ in the small terms:

$$\therefore\ \frac{d\theta}{dt} = \omega_1 \cos nt - \omega_2 \sin nt$$

$$= (N - M) \sin it = - \frac{C - A}{nC} . k \sin it,$$

$$\sin \theta \frac{d\psi}{dt} = - \omega_1 \sin nt - \omega_2 \cos nt,$$

$$= - (M + N) \cos it = \frac{C - A}{nC} . k' \cos it,$$

and by replacing $k \sin it$ and $k' \cos it$ by P and P', of which they have been the representatives,

$$\frac{d\theta}{dt} = - \frac{C - A}{nC} P = - \frac{3S}{2r^3} \frac{C - A}{nC} \sin \theta \sin 2n't,$$

$$\frac{d\psi}{dt} = \frac{C - A}{nC \sin \theta} P' = \frac{3S}{2r^3} \frac{C - A}{nC} \cos \theta\, (1 - \cos 2n't),$$

integrating these equations and putting $\sqrt{\frac{S}{r^3}} = n'$, the mean motion of the Sun,

$$\theta = I + \frac{3n'}{4n}\frac{C-A}{C}\sin I\cos 2n't,$$

$$\psi = \frac{3n'^2}{2n}\frac{C-A}{C}\cos I \, . \, t - \frac{3n'}{4n}\frac{C-A}{C}\cos I\sin 2n't,$$

I being the mean value of θ: and the axis of x being so chosen that $t = 0$ when Aries was in that axis.

We should obtain analogous expressions for the perturbation of the Earth's axis by the Moon.

464. The first of these expressions shews, that the obliquity of the ecliptic fluctuates; but preserves its mean value equal to the value it would have if there were no disturbing forces.

The second shews, that the first point of Aries, or the vernal equinox, has on the whole a retrograde motion on the ecliptic, though at the same time it is subject to a small oscillatory motion.

The steady retrograde motion is called *the Precession of the Equinoxes;* the solar precession (i. e. the precession caused by the Sun) equals $\frac{3n'^2}{2n}\frac{C-A}{C}\cos I$ in a unit of time. This precessional motion causes the pole of the Earth to describe a small circle about the pole of the ecliptic.

The oscillating motion of the pole, arising partly from the change of the obliquity and partly from the periodical term in ψ, is called the *Nutation of the Earth's Axis.*

465. It will be seen that the Precession and Nutation of the Earth's axis arise from the attraction of the Sun and Moon upon the protuberant parts of the Earth, i. e. upon the portion by which it exceeds a sphere touching it internally. For if the form of the Earth were spherical, then $C = A$ and the variable terms in θ and ψ would vanish.

We proceed to calculate the effect of the Moon upon the position of the Earth's Axis.

PROP. *To find the motion of the Earth's axis with respect to the plane of the Moon's orbit caused by the action of the Moon.*

466. Let θ' and ψ' be the same quantities as θ and ψ in Art. 463. with this difference, that the plane of the Moon's orbit is used instead of the ecliptic; i the inclination of the Moon's orbit to the ecliptic; this does never much exceed 5^0; we shall consider i constant, since its variations are very small, as is shewn by observation: I' the inclination of the equator to the Moon's orbit: M the mass of the Moon: a the radius of the Moon's orbit.

Now for $\dfrac{S}{r^3}$ in Art. 463. we must put $\dfrac{M}{a^3}$.

Let n'' be the mean motion of the Moon about the Earth:

$$\text{then } n'' = \sqrt{\frac{M+E}{a^3}}, \quad \therefore \frac{M}{a^3} = \frac{Mn''^2}{M+E} = \frac{n''^2}{1+\nu},$$

where E is the mass of the Earth, and ν the ratio of this mass to that of the Moon.

Hence $\dfrac{S}{r^3}$ in Art. 463. must be replaced by $\dfrac{n''^2}{1+\nu}$;

$$\therefore \frac{d\theta'}{dt} = -\frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\sin I' \sin 2n''t$$

$$\frac{d\psi'}{dt} = \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\cos I'(1-\cos 2n''t).$$

The periodical quantities $\sin 2n''t$ and $\cos 2n''t$ go through their changes in half a month; in consequence of the shortness of their period and the smallness of their coefficients they never accumulate so much as to produce a sensible effect: and are therefore omitted. Hence the inclination of the Earth's axis to the Moon's orbit suffers no sensible change from the Moon's attraction; but the line of intersection of the equator and the plane of the Moon's orbit

does change its position, which is determined by the equation

$$\psi' = \frac{3n''}{2(1+\nu)} \frac{C-A}{nC} \cos I' \,.\, (n''t + \text{const.})$$

In order to calculate the Lunar Precession and Nutation, we must refer the angle ψ' to the ecliptic. Since the oscillations of the plane of the Moon's orbit are insensible no Lunar Nutation can arise from them; but the Moon's line of Nodes continually regresses (Art. 338.) performing a revolution in 18 years and 7 months: and this is the cause of Lunar Nutation. We proceed to calculate this and Lunar Precession.

PROP. *To calculate Lunar Precession.*

467. Let K, K', P be respectively the poles of the ecliptic, Moon's orbit, and the Earth's equator (fig. 107).

Now P revolves about K' with an angular velocity $\frac{d\psi'}{dt}$:

hence the linear vel. of P about $K' = \frac{d\psi'}{dt} \sin\theta'$, rad. of sphere $= 1$

the resolved part of this about $K = \frac{d\psi'}{dt} \sin\theta' \cos KPK'$: and therefore

P revolves about K with an angular velocity $= \frac{d\psi'}{dt} \frac{\sin\theta'}{\sin\theta} \cos KPK'$

and .... ♈ .... $= \frac{d\psi'}{dt} \sin\theta' \cos ♈PK'$:

$$\therefore \frac{d\psi}{dt} = \frac{d\psi'}{dt}\frac{\sin\theta'}{\sin\theta}\cos KPK' = \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\frac{\cos\theta'\sin\theta'}{\sin\theta}\cos KPK'$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\frac{\cos\theta'}{\sin\theta}\frac{\cos i - \cos\theta\cos\theta'}{\sin\theta}$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\frac{1}{\sin^2\theta}(\cos\theta\cos i + \sin\theta\sin i\cos\Omega)$$

$$\times (\cos i - \cos^2\theta\cos i - \cos\theta\sin\theta\sin i\cos\Omega)$$

where Ω is the longitude of the Moon's node measured in a retrograde direction

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}(\cos\theta\cos i+\sin\theta\sin i\cos\Omega)(\cos i-\cot\theta\sin i\cos\Omega)$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\left(\cos\theta\cos^2 i-\frac{\cos 2\theta}{2\sin\theta}\sin 2i\cos\Omega-\cos\theta\sin^2 i\cos^2\Omega\right)$$

$$= \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\left\{\cos I\,(\cos^2 i-\tfrac{1}{2}\sin^2 i)\right.$$

$$\left.-\frac{\cos 2I\sin 2i}{2\sin I}\cos\left(\frac{2\pi t}{\tau}+\Omega_0\right)-\tfrac{1}{2}\cos I\sin^2 i\cos\left(\frac{4\pi t}{\tau}+2\,\Omega_0\right)\right\},$$

in which τ = the periodic time of the Moon's nodes: and Ω_0 is the longitude of the ascending node when $t = 0$;

$$\therefore\ \psi = \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\left\{\cos I\,(\cos^2 i-\tfrac{1}{2}\sin^2 i)\,t\right.$$

$$-\frac{\tau}{4\pi}\frac{\cos 2I\sin 2i}{\sin I}\sin\left(\frac{2\pi t}{\tau}+\Omega_0\right)$$

$$\left.-\frac{\tau}{8\pi}\cos I\sin^2 i\sin\left(\frac{4\pi t}{\tau}+2\Omega_0\right)\right\}+\text{const.}$$

Hence the Lunar Precession

$$= \frac{3n''}{2(1+\nu)}\frac{C-A}{nC}\cos I\,(\cos^2 i-\tfrac{1}{2}\sin^2 i)\,n''t.$$

The second term of ψ is periodical as well as the third; but the third is so small, in consequence of its coefficient $\sin^2 i$, that it may be neglected. They are both parts of Lunar Nutation.

PROP. *To find the effect of the Moon on the obliquity of the ecliptic.*

468. We have from the last Proposition

$$\frac{d\theta}{dt} = \frac{d\psi'}{dt}\sin\theta'\cos \Upsilon PK' = \frac{3n''^2}{2(1+\nu)}\frac{C-A}{nC}\cos\theta'\sin\theta'\cos\Upsilon PK'$$

$$= \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} (\cos\theta \cos i + \sin\theta \sin i \cos\Omega) \sin i \sin\Omega,$$

$$= \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \left\{ \tfrac{1}{2} \cos I \sin 2i \sin\left(\frac{2\pi t}{\tau} + \Omega_0\right) \right.$$

$$\left. + \tfrac{1}{2} \sin I \sin^2 i \sin\left(\frac{4\pi t}{\tau} + 2\Omega_0\right) \right\};$$

$$\therefore\ \theta = I - \frac{3n''^2}{2(1+\nu)} \frac{C-A}{nC} \left\{ \frac{\tau}{4\pi} \cos I \sin 2i \cos\left(\frac{2\pi t}{\tau} + \Omega_0\right) \right.$$

$$\left. + \frac{\tau}{8\pi} \sin I \sin^2 i \cos\left(\frac{4\pi t}{\tau} + 2\Omega_0\right) \right\}$$

the variable terms in this are periodical, and the last is so small as to be insensible. These terms and the periodical terms in the value of ψ make up the whole of Lunar Nutation.

469. Let x and y be the parts of Lunar Nutation which have been determined in Arts. 467, 468;

$$\therefore\ x^2 \left\{ \frac{8\pi n (1+\nu) C}{3\tau n''^2 (C-A) \cos 2I \sin 2i} \right\}^2 + y^2 \left\{ \frac{8\pi n (1+\nu) C}{3\tau n''^2 (C-A) \cos I \sin 2i} \right\}^2 = 1.$$

This is the equation to an ellipse of which the axes are in the ratio $\cos 2I : \cos I$. This explains the construction mentioned in works on Plane Astronomy. Woodhouse's *Plane Astronomy*, p. 857. Maddy's *Plane Astronomy*, 2nd Edition.

The whole Precession, both Solar and Lunar, equals

$$\frac{3}{2n} \frac{C-A}{C} \cos I \left\{ n'^2 + \frac{n''^2}{1+\nu} (\cos^2 i - \tfrac{1}{2} \sin^2 i) \right\} t,$$

(which agrees with *Mécanique Céleste*, Liv. v. Chap. I. §. 14.) and the Nutation is given by the equations of Arts. 467, 468.

470. Annual Precession

$$= \frac{C-A}{C} \frac{3n'}{n} \cos I \left\{ 1 + \frac{n''^2}{n'^2} \frac{1 - \frac{3}{2} \sin^2 i}{1+\nu} \right\} 180^0$$

$$I = 23^0 28' 18'',\quad i = 5^0 8' 50'',\quad \frac{n}{n'} = 365.26,\quad \frac{n''}{n'} = \frac{36526}{2732}.$$

$$\log_{10} \sin i = \bar{2}.9528656$$

$$\log_{10} (\tfrac{3}{2} \sin^2 i) = \bar{2}.0818225 = \log_{10} .0120732$$

$$\therefore \log_{10} (1 - \tfrac{3}{2} \sin^2 i) = \log_{10} .9879268 = \bar{1}.9947248$$

$$\log_{10} \frac{n''^2}{n'^2} \qquad = 2.2522428$$

$$\therefore \log_{10} \left\{ \frac{n''^2}{n'^2} (1 - \tfrac{3}{2} \sin^2 i) \right\} = 2.2469676 = \log_{10} 176.5906 ;$$

$$\frac{n''^2}{n'^2} (1 - \tfrac{3}{2} \sin^2 i) = 176.5906$$

$$\log_{10} \left\{ \frac{3n'}{n} \times 180 \times 60 \times 60 \times \cos I \right\} = 2.3856065$$

$$\begin{array}{l} 3.9030900 \\ \bar{1}.9625076 - 2.5626021 \\ \hline 6.2512041 \\ 2.5626021 \\ \hline 3.6886020 = \log_{10} 4882.05 ; \end{array}$$

$$\therefore \text{Annual Precession} = \frac{C - A}{C} \left(1 + \frac{176.5906}{1 + \nu}\right) 4882''.05.$$

The ratio of C to A depends upon the figure of the Earth: when we come to this subject we shall complete the numerical calculation of Precession, and compare it with the observed value. See Art. 551.

471. We have supposed in these calculations that the Earth is wholly solid. Laplace shews, however, that the variations of the motion of the terrestrial nucleus, covered by a fluid, are the same as if the sea formed a solid mass with it: *Mécanique Céleste*, Liv. v. §. 10—12.

CHAPTER XI.

MOTION OF A SYSTEM OF RIGID BODIES, ACTED ON BY *FINITE* FORCES.

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472. WHEN two or more bodies are in motion, and influence each other, the motion of each may be calculated by substituting unknown forces for the unknown mutual actions of the bodies, and writing down the equations for each body according to the principles of the last Chapter. We shall see this exemplified in the Chapter of Problems. In the present Chapter we intend to prove some general principles of the motion of a system of bodies.

PROP. *To find the equations of motion of a system of rigid bodies.*

473. Let mX, mY, mZ be the *impressed* moving forces acting on the particle m; xyz the co-ordinates of m at the time t: then by D'Alembert's Principle (see Arts. 226 and 428), the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right)$$

acting on m, together with similar forces acting on the other particles at the instant t ought to be in equilibrium one with another. Hence by Art. 69, we have the equations

$$\Sigma . m\left(X-\frac{d^2x}{dt^2}\right)=0,\quad \Sigma . m\left(Y-\frac{d^2y}{dt^2}\right)=0,\quad \Sigma . m\left(Z-\frac{d^2z}{dt^2}\right)=0,$$

$$\Sigma . m\left\{y\left(Z-\frac{d^2z}{dt^2}\right)-z\left(Y-\frac{d^2y}{dt^2}\right)\right\}=0,$$

$$\Sigma . m\left\{z\left(X-\frac{d^2x}{dt^2}\right)-x\left(Z-\frac{d^2z}{dt^2}\right)\right\}=0,$$

$$\Sigma . m\left\{x\left(Y-\frac{d^2y}{dt^2}\right)-y\left(X-\frac{d^2x}{dt^2}\right)\right\}=0.$$

These equations of motion are independent of the mutual actions of the bodies of the system; and are the only equations which are so, as may be seen by Art. 69. By writing down, for each body, the six equations of motion, and also the conditions which express the peculiar manner of connexion of the bodies, we may obtain other equations, which, in combination with the six written above, will give the complete motion of each body. But those six equations are the only equations, which are true for all systems, and do not involve the peculiarities of any.

Prop. *To prove, that the motion of the centre of gravity of a material system is the same as if the whole mass were collected in that point, and all the forces acted on it parallel to their real directions. Also, to deduce the Principle of the Conservation of the motion of the Centre of Gravity.*

474. By the last Art. we have the three equations

$$\Sigma . m \left(X - \frac{d^2 x}{dt^2}\right) = 0, \quad \Sigma . m \left(Y - \frac{d^2 y}{dt^2}\right) = 0, \quad \Sigma . m \left(Z - \frac{d^2 z}{dt^2}\right) = 0,$$

and the Prop. is proved exactly as in Art. 429, where this principle is proved for one rigid body.

The Principle of the *Conservation of the motion of the centre of gravity* is this, that if no external forces act on the system, the centre of gravity will either remain at rest during the motion, or move uniformly in a straight line. This is evident, for by the present Prop. the centre of gravity moves as a mass with no forces acting upon it, and will therefore, by the first law of motion, move uniformly in a straight line, if it be not at rest.

Prop. *To prove, that when a material system is in motion under the action of forces, none of which are extraneous to the system; then the sum of the products of each particle, multiplied by the projection, on any plane, of the area swept out by the radius vector of this particle measured from any fixed point varies as the time of motion. This is called the* Principle of the Conservation of Areas.

475. The last three equations of Art. 473 give

$$\Sigma . m \left\{ y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right\} = \Sigma . m (yZ - zY),$$

$$\Sigma . m \left\{ z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right\} = \Sigma . m (zX - xZ),$$

$$\Sigma . m \left\{ x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right\} = \Sigma . m (xY - yX),$$

These equations are free from the forces which arise from the mutual action of the bodies: see Art. 473.

Since then all the forces are supposed to be internal the equations of motion become

$$\Sigma . m \left(y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} \right) = 0, \; \Sigma . m \left(z \frac{d^2 x}{dt^2} - x \frac{d^2 z}{dt^2} \right) = 0,$$

$$\Sigma . m \left(x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} \right) = 0,$$

and therefore by integration

$$\Sigma . m \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) = h, \; \Sigma . m \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) = h',$$

$$\Sigma . m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = h'',$$

h, h', h'' being constants.

Let $A_x A_y A_z$ be the areas swept out by the projections of the radius vector of the particle m on the co-ordinate planes respectively perpendicular to the axes of xyz during the time t: then by the last three equations

$$\Sigma . m \frac{dA_x}{dt} = h, \; \Sigma . m \frac{dA_y}{dt} = h', \; \Sigma . m \frac{dA_z}{dt} = h'';$$

$$\therefore \; \Sigma . m A_x = ht, \; \Sigma . m A_y = h't, \; \Sigma . m A_z = h''t,$$

since the areas are measured from the epoch when $t = 0$.

Wherefore the Principle, as enunciated, is true for the three co-ordinate planes arbitrarily chosen; and consequently true for any plane, and any centre for the areas.

476. *This Principle is also true when* extraneous *forces act on the system, provided their directions all pass through the same point and the areas are estimated about that point.*

For forces which pass through a given point have no moment about that point, and therefore will not appear in the equations of the last Art. if the origin of co-ordinates be placed at the given point.

PROP. *To find the plane with respect to which the sum of the moments of the momenta of the different particles of the system, about a line perpendicular to the plane through the origin, is a maximum: and to prove, that this plane is invariable in position in space during the motion, when no external forces act, or when external forces act, but pass through a given point, and that point is taken as origin.*

This plane is called the *Principal Plane of Moments;* and also the *Invariable Plane.*

477. Let $\lambda\mu\nu$ be the angles which the required plane makes with the co-ordinate planes respectively perpendicular to the axes of xyz.

The momentum of the particle m parallel to the plane yz is $m\sqrt{\dfrac{dy^2}{dt^2}+\dfrac{dz^2}{dt^2}}$: the perpendicular from the origin upon the tangent to the projection of the curve in which m is moving on the plane $yz = \dfrac{y\dfrac{dz}{dt} - z\dfrac{dy}{dt}}{\sqrt{\dfrac{dy^2}{dt^2}+\dfrac{dz^2}{dt^2}}}$. Hence the moment of the momentum of m about the axis of $x = m\left(y\dfrac{dz}{dt} - z\dfrac{dy}{dt}\right)$ and the sum of the moments $= \Sigma . m\left(y\dfrac{dz}{dt} - z\dfrac{dy}{dt}\right) = L$ suppose; and let the sums of the moments about the axes of y and z be M and N. Hence the sum of the moments about the line perpendicular to the plane, which makes the angles $\lambda\mu\nu$ with the co-ordinate planes equals

$$L\cos\lambda + M\cos\mu + N\cos\nu;$$

and when this is a maximum

$$L\sin\lambda + M\sin\nu\frac{d\nu}{d\lambda} = 0 \text{ and } M\sin\mu + N\sin\nu\frac{d\nu}{d\mu} = 0,$$

$$\text{but } \cos^2\lambda + \cos^2\mu + \cos^2\nu = 1;$$

$$\therefore \frac{L}{N} = -\frac{\sin\nu}{\sin\lambda}\frac{d\nu}{d\lambda} = \frac{\cos\lambda}{\cos\nu} \text{ and } \frac{M}{N} = -\frac{\sin\nu}{\sin\mu}\frac{d\nu}{d\mu} = \frac{\cos\mu}{\cos\nu};$$

$$\therefore \frac{L^2 + M^2 + N^2}{N^2} = \frac{1}{\cos^2\nu}, \ \therefore \cos\nu = \frac{N}{\sqrt{L^2 + M^2 + N^2}},$$

and in like manner

$$\cos\mu = \frac{M}{\sqrt{L^2 + M^2 + N^2}} \text{ and } \cos\lambda = \frac{L}{\sqrt{L^2 + M^2 + N^2}}.$$

These determine the position of the *Principal Plane of Moments* at any instant of the motion.

The Principal Moment $= \sqrt{L^2 + M^2 + N^2}$.

If no external forces act upon the system, or only such as pass through a given point, and that point be taken as the origin of moments, then L, M, N are the constants h, h', h'' of Art. 475, 476, which depend not upon the time, but only upon the original configuration of the system. Hence in these cases the Principal Plane of Moments is called the *Invariable Plane*.

478. If the position of the invariable plane of the Solar System be calculated upon the supposition, that the heavenly bodies are intense particles without rotatory motion, it is found, that h, h', h'' are constant even in carrying the approximation to the squares and products of the masses, whatever changes the secular variations may induce in the course of ages: hence it follows that the invariable plane retains its position notwithstanding the secular variations in the elliptical elements of the planetary system. The determination of the position of the invariable plane requires a knowledge of the masses of all the bodies in the system and of the elements of their orbits. Now we know the masses of the planets only approximately; but of the masses of the comets we are in total ignorance. But from

the agreement of theory and observation, mentioned above, we learn, that, hitherto at least, the action of the comets on the planetary system is insensible. Laplace has shewn that the comet of 1770 passed through the system of Jupiter and his satellites without producing the smallest effect, though its own motion was much perturbed.

If the position of the ecliptic in the beginning of 1750 be taken as the fixed plane of xy, and the longitudes measured from the line of the equinoxes, it is found, that at the epoch 1750 the longitude of the ascending node of the invariable plane was $102^{\circ}57'30''$, and its inclination on the ecliptic was $1^{\circ}35'31''$: and if these be calculated for 1950 they are $102^{\circ}\ 57'\ 15''$ and $1^{\circ}\ 35'\ 31''$; these differ but very little from the former, and therefore shew, that the motion of the ecliptic in space is exceedingly slow.

479. It is important to remark that the terms in the equations of motion of Art. 475, which depend upon the mutual action of the parts of the system, would disappear even when the intensity of the forces varies with the time, independently of the distance: i. e. when the expression for the force is an explicit function of the time. For in this case the invariability of the principal moment and of the direction of its axis is preserved.

This shews, that the loss of heat sustained by the particles of the system by radiation, though it diminishes the intensity of their mutual action, yet has no effect on the position of the invariable plane or on the principal moment. So that if we leave out of consideration the action of the Sun Moon and planets on the Earth, and suppose, that our planet were at one time in a gaseous state, and become solid by refrigeration without losing any portion of its ponderable matter, we may feel assured that the principal moment of the system has not altered in magnitude nor has its axis changed its position during the change of condition of the globe.

If M be the whole mass: k the radius of gyration about the axis of principal moments through the centre of gravity at the time t, ω the angular velocity about this axis; then

$$Mk^2\omega = \text{principal moment} = \text{constant}.$$

This shews, that if the Earth radiate its heat into space so as to diminish its radius by contracting its dimensions, then, since k varies as the radius, ω will be increasing and the length of the day shortening.

Now it has been proved in Art. 461, by calculations of eclipses, that within the last 2556 years the length of the day is not become shorter by even a ten millionth part: and therefore, since ω varies inversely as k^2, or the length of the day varies as the square of the mean radius, the mass remaining the same, the mean radius of the Earth has not changed within the last five and twenty centuries by even a twenty-millionth part.

480. The appearance of fossil remains of tropical plants and animals in these higher latitudes has induced geologists to adopt the hypothesis, that the temperature of the Earth was in ages gone by far higher than at present. The results of the last Article shew, that no objection can be urged against this theory, at least upon mechanical principles. If this hypothesis be true we learn that the radiation goes on now very slowly, whatever its rapidity may have been at more ancient epochs.

Prop. *When a material system is in motion under the action of forces, none of which are extraneous to the system, and none of which are functions of the time explicitly; then the change of the Vis Viva of the system during a given time depends only on the co-ordinates of the particles of the system at the beginning and end of the given time, and not at all on the curves which the particles describe.*

481. This is called the *Principle of Vis Viva.*

Let XYZ be the impressed accelerating forces, which act upon the particle m, resolved parallel to the axes of co-ordinates, including pressures and reactions and neglecting only the molecular forces: xyz the co-ordinates to m at the time t: then the forces

$$m\left(X-\frac{d^2x}{dt^2}\right),\quad m\left(Y-\frac{d^2y}{dt^2}\right),\quad m\left(Z-\frac{d^2z}{dt^2}\right),$$

acting on m, and similar forces acting on the other particles of the system will satisfy the conditions of equilibrium; (Art. 224.) Wherefore by the Principle of Virtual Velocities (Art. 70.) we have

$$\Sigma . m\left\{\left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z\right\} = 0^*,$$

δx, δy, δz, being any small spaces geometrically described by m parallel to the axes, in a manner consistent with the connexion of the parts of the system one with another at the time t.

Now the spaces actually described by the particle m during the instant after the time t parallel to the axes are consistent with the connexion of the parts of the system one with another: hence we may take

$$\delta x = \frac{dx}{dt}\delta t, \quad \delta y = \frac{dy}{dt}\delta t, \quad \delta z = \frac{dz}{dt}\delta t,$$

and the above equation becomes

$$\Sigma . m\left\{\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}\right\} = \Sigma . m\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)$$

$$\therefore \Sigma . m\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\}$$

$$= 2\Sigma . m\int\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)dt + C.$$

But by the Differential Calculus

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = \frac{ds^2}{dt^2} = (\text{velocity})^2 = v^2,$$

$$\therefore \Sigma . mv^2 = 2\Sigma . m\int\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)dt + C.$$

* We might make this equation the foundation of Dynamics, and deduce from it all the equations of motion, which we have made use of in this subject. We have only to make the substitutions for δx, δy, δz which we made in Art. 83.

Now let P be the mutual pressure of two particles m and m' in contact at the point xyz: $\alpha\beta\gamma$ the angles its direction makes with the axes.

Then, the expression $m\int\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)dt$,

for the particle m becomes

$$\int P\left(\cos\alpha\frac{dx}{dt}+\cos\beta\frac{dy}{dt}+\cos\gamma\frac{dz}{dt}\right)dt,$$

and for the particle m' it becomes

$$-\int P\left(\cos\alpha\frac{dx}{dt}+\cos\beta\frac{dy}{dt}+\cos\gamma\frac{dz}{dt}\right)dt,$$

and the sum of these $=0$, and therefore P will not appear in our final equation.

Again, let xyz, $x'y'z'$ be the co-ordinates of two particles m and m' not in contact; r their distance; P their mutual action, supposed to be a function of r: then the cosines of the angles which the direction of P makes with the axes are $\frac{x-x'}{r}$, $\frac{y-y'}{r}$, $\frac{z-z'}{r}$: and the expression

$m\int\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)dt$ becomes, for the particle m

$$\int P\left\{\frac{x-x'}{r}\frac{dx}{dt}+\frac{y-y'}{r}\frac{dy}{dt}+\frac{z-z'}{r}\frac{dz}{dt}\right\}dt$$

and for the particle m'

$$-\int P\left\{\frac{x-x'}{r}\frac{dx'}{dt}+\frac{y-y'}{r}\frac{dy'}{dt}+\frac{z-z'}{r}\frac{dz'}{dt}\right\}dt,$$

Hence P will appear in $2\Sigma.m\int\left(X\frac{dx}{dt}+Y\frac{dy}{dt}+Z\frac{dz}{dt}\right)dt$ under the form

$$2\int\frac{P}{r}\left\{(x-x')\frac{d(x-x')}{dt}+(y-y')\frac{d(y-y')}{dt}+(z-z')\frac{d(z-z')}{dt}\right\}dt,$$

or $2\int P dr$; since $r^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$.

Wherefore we have finally

$$\Sigma . mv^2 = 2\Sigma . \int P dr + C,$$

and since P is a function of r the second side of this equation, when taken between limits, will be a function solely of the initial and final co-ordinates of the particles of the system. This would not be the case if P were an explicit function of t. See Prob. 32.

COR. 1. The expression for the vis viva of a system acted on by any forces (not impulsive) is

$$\Sigma . mv^2 = 2\Sigma . m\int\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)dt^*.$$

COR. 2. Any force which acts upon a fixed point of the system will not appear in the equation of vis viva, since the velocities of the point are nothing. In this way the mutual pressures of any parts of the system against immoveable obstacles will not appear. Neither will the force of friction which

* This equation may be deduced in the following manner, without assuming the Principle of Virtual Velocities.

Suppose that any rigid body of the system at the time t is moving in such a manner, that its velocities of translation parallel to the axes are a, b, c: and the angular velocities about the axes are $\omega'\ \omega''\ \omega'''$ respectively. Then if xyz be the co-ordinates of any particle m of this body at the time t, by Art. 441,

$$\frac{dx}{dt} = a + \omega''z - \omega'''y,\quad \frac{dy}{dt} = b + \omega'''x - \omega'z,\quad \frac{dz}{dt} = c + \omega'y - \omega''x.$$

Now multiply the six equations of Art. 428 by $a, b, c, \omega', \omega'', \omega'''$ respectively, and add them together, observing that these six quantities are the same for every particle of the body; and we have

$$\Sigma . m\left\{\left(X - \frac{d^2x}{dt^2}\right)\frac{dx}{dt} + \left(Y - \frac{d^2y}{dt^2}\right)\frac{dy}{dt} + \left(Z - \frac{d^2z}{dt^2}\right)\frac{dz}{dt}\right\} = 0,$$

where Σ applies to every particle in the rigid body. Similar equations may be written down for every other rigid body of the system. Then if all the equations are added together we shall have a resulting equation analogous to the above; or by transposing and putting the velocity of m equal to v, and integrating, we have the equation

$$\Sigma . mv^2 = 2\Sigma . m\int(Xdx + Ydy + Zdz),$$

in which Σ extends now to the whole system, and XYZ include the mutual actions of the bodies. This coincides with the text.

acts upon a body rolling (*not* partly rolling and partly sliding) upon a fixed obstacle appear in the vis viva; since the point of contact is for an instant at rest.

If two bodies of the system *slide* upon each other, and there is friction, the friction will enter into the expression for the vis viva: if they do not slide, the friction will not appear.

COR. 3. If forces act upon none of the particles of the system except such as remain invariably connected during the motion, then the vis viva remains the same throughout the motion. For in this case $dr = 0$; and therefore $\Sigma . mv^2 = C$.

This is called the *Principle of the Conservation of Vis Viva.*

COR. 4. By calculating the vis viva of the Solar System at distant epochs, in terms of the observed motions of the bodies which compose it, we may ascertain whether any effect is produced by the attraction of the stars, or unknown comets, or by the resistance of a medium.

COR. 5. The equation of vis viva has been applied by M. Coriolis to calculate the motion and efficiency of machines. See *Journal de l'Ecole Polytechnique*, 21ᵉ Cahier.

PROP. *The vis viva of a material system in motion is equal to the vis viva arising from the motion of translation of the centre of gravity in space added to the vis viva arising from the motion about the centre of gravity.*

482. Let xyz be co-ordinates to m at time t,

$\bar{x}\bar{y}\bar{z}$ be co-ordinates to centre of gravity of the system,

$$\text{and let } x = \bar{x} + x_{,}, \quad y = \bar{y} + y_{,}, \quad z = \bar{z} + z_{,};$$

$$\therefore v^2 = (\text{vel. of } m)^2 = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} = \frac{d\bar{x}^2}{dt^2} + \frac{d\bar{y}^2}{dt^2} + \frac{d\bar{z}^2}{dt^2}$$

$$+ 2\left\{\frac{d\bar{x}}{dt}\frac{dx_{,}}{dt} + \frac{d\bar{y}}{dt}\frac{dy_{,}}{dt} + \frac{d\bar{z}}{dt}\frac{dz_{,}}{dt}\right\} + \frac{dx_{,}^2}{dt^2} + \frac{dy_{,}^2}{dt^2} + \frac{dz_{,}^2}{dt^2},$$

and observing that $\Sigma . m \frac{dx_{\prime}}{dt} = 0$, $\Sigma . m \frac{dy_{\prime}}{dt} = 0$, $\Sigma . m \frac{dz_{\prime}}{dt} = 0$,

we have

$$\Sigma . mv^2 = \overline{V}^2 \Sigma . m + \Sigma . mv_{\prime}^2,$$

$\overline{V}$ being the velocity of the centre of gravity of the system and $v_{\prime}$ the velocity of m relative to the centre of gravity.

483. By Art. 481 we have

$$2 \Sigma . Pdr = \frac{d(\Sigma . mv^2)}{dt} dt;$$

therefore whenever during the motion the particles of the system assume such a relative position that the vis viva is a maximum or minimum $\Sigma . Pdr = 0$, and therefore (Art. 73) the system is at that instant in a position in which the forces are in equilibrium.

Also by Art. 78 we see, that when the vis viva is a *maximum*, the position which the system assumes would be a position of *stable* equilibrium, if all velocity be destroyed: and when the vis viva is a *minimum*, the position would be one of *unstable* equilibrium. Also since a function passes through its maximum and minimum values *alternately* as the variable increases continuously, the system when in motion will pass through the positions of stable and unstable equilibrium alternately.

Prop. *To prove that the variation of* $\Sigma . \mathrm{m} \int \mathrm{v\,ds}$ *taken between given limits equals zero, where* v *is the velocity and* ds *is the element of the space described in the short time* dt *by the particle* m *of a material system in free motion: if any particle move on a surface it is supposed to continue on the surface in taking the variation.*

484. This is called the *Principle of Least Action*; because, in general, $\Sigma . m \int v ds$ is a minimum.

Let δ be the symbol of variation in the Calculus of Variations: then

$$\delta (\Sigma . m \int v ds) = \Sigma . m \int \delta (v ds) = \Sigma . m \int (v \delta . ds + ds \delta v)$$
$$= \Sigma . m \int (v \delta . ds + \tfrac{1}{2} dt \delta . v^2).$$

Suppose the particle m rests on a curve surface, and that R is the normal pressure, $\alpha\beta\gamma$ the angles of its direction; X, Y, Z the accelerating forces acting on m, then (as in Art. 407)

$$\frac{d^2x}{dt^2} = X + \frac{R}{m}\cos\alpha, \quad \frac{d^2y}{dt^2} = Y + \frac{R}{m}\cos\beta, \quad \frac{d^2z}{dt^2} = Z + \frac{R}{m}\cos\gamma.$$

Let $L = 0$ be the equation to the surface; then

$$\cos\alpha = V\frac{dL}{dx}, \quad \cos\beta = V\frac{dL}{dy}, \quad \cos\gamma = V\frac{dL}{dz};$$

$$\text{where } \frac{1}{V^2} = \frac{dL^2}{dx^2} + \frac{dL^2}{dy^2} + \frac{dL^2}{dz^2}.$$

$$\text{Hence } v^2 = 2\int(Xdx + Ydy + Zdz) + 2\int\frac{R}{m}VdL:$$

if the particle do not rest on a surface, $R = 0$; and if it do, still $dL = 0$; because we suppose the motion to be such, that particles on surfaces remain on the surfaces.

$$\therefore\ v^2 = 2\int(Xdx + Ydy + Zdz) = \phi(x, y, z) + \text{const.}$$

$$\therefore\ \tfrac{1}{2}\delta\,.\,v^2 = X\delta x + Y\delta y + Z\delta z$$

$$= \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z - \frac{R}{m}V\delta L = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z.$$

Again, $ds^2 = dx^2 + dy^2 + dz^2$,

$$\therefore\ ds\delta\,.\,ds = dx\delta\,.\,dx + dy\delta\,.\,dy + dz\delta\,.\,dz;$$

$$\therefore\ v\delta\,.\,ds = \frac{dx}{dt}\delta\,.\,dx + \frac{dy}{dt}\delta\,.\,dy + \frac{dz}{dt}\delta\,.\,dz.$$

$$\text{Hence } \int(v\delta.ds + \tfrac{1}{2}dt\delta.v^2) = \frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z + \text{const.}$$

and at the limits $\delta x = 0$, $\delta y = 0$, $\delta z = 0$, because the first and last positions are given,

$$\therefore \int (v\delta \,.\, ds + \tfrac{1}{2}dt\delta \,.\, v^2) = 0,$$

$$\therefore \delta (\Sigma \,.\, m \int v ds) = 0,$$

and $\Sigma \,.\, m \int v ds$ is a maximum or minimum. It is evidently a minimum, because a path of an indefinite length can always be found for any particle of the system.

Cor. 1. Since $ds = vdt$ we learn that $\Sigma \,.\, m \int v^2 dt$ is a minimum, or the quantity of vis viva generated or expended during any given time is a minimum.

Cor. 2. If the system consist of only one particle moving on a surface, and no forces but the normal pressure act, then $\int v ds$ is a minimum: but v is a constant (Art. 407), therefore $\int ds$ is a minimum, or the particle will describe the shortest curve line that can be drawn on the surface between its positions at the beginning and end of the time t.

485. If we compare the principle of least action with the principles of the conservation of the motion of the centre of gravity, of the conservation of areas, and of vis viva, we see that this principle only serves to determine the equations of motion, and is therefore comparatively useless since these are found by much simpler means; but the other principles, which develop important properties, have the advantage of furnishing three general integrals of the equations of motion, which are in most problems the only integrals that can be found.

Prop. *To shew that the calculation of the motion of a material system may be made to depend upon the integration of a single function.*

486. We shall shew this by proving a new dynamical principle discovered by Professor Hamilton and published in the *Philosophical Transactions*, 1834.

We have seen, Art. 481, that the Principle of Virtual Velocities leads us to the dynamical equation

$$\Sigma \,.\, m \left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\} = 2\Sigma \,.\, m \int \left\{X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right\} dt.$$

Now it is easily shewn, that

$$\Sigma . m \left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right)$$

is a perfect differential coefficient with respect to t for all the forces which exist in nature; viz. forces tending to the centres of the particles of the material universe, whether fixed or moveable. Let therefore the second side $= 2(U + H)$, H being independent of t: and let $2T$ be the vis viva of the system at the time t; T_0, H_0 the values of T and H when $t = 0$;

$$\therefore\ T = U + H, \text{ and } T_0 = U_0 + H.$$

Now if the initial circumstances of the motion be varied, then H will vary, and so also will T and U: let δ be the symbol of these variations;

$$\therefore\ \delta T = \delta U + \delta H$$

$$\text{or } \Sigma . m \left\{\frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt}\right\}$$

$$= \Sigma . m \left\{\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right\} + \delta H;$$

$$\text{and therefore } 2\Sigma . m \left\{\frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt}\right\}$$

$$= \Sigma . m \frac{d}{dt}\left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\} + \delta H.$$

Now let the accumulation of the vis viva from the commencement to the termination of the time t be V;

$$\therefore\ \mathrm{V} = \int_0^t \Sigma . m \left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}\right\} dt.$$

Then V is a function of the initial and final co-ordinates of the material particles, and

$$\delta V = \Sigma . \left\{\frac{\delta V}{\delta x}\delta x + \frac{\delta V}{\delta y}\delta y + \frac{\delta V}{\delta z}\delta z + \frac{\delta V}{\delta a}\delta a + \frac{\delta V}{\delta b}\delta b + \frac{\delta V}{\delta c}\delta c\right\}$$

$$= 2\int_0^t \Sigma . m \left\{\frac{dx}{dt}\delta\frac{dx}{dt} + \frac{dy}{dt}\delta\frac{dy}{dt} + \frac{dz}{dt}\delta\frac{dz}{dt}\right\} dt$$

$$+ \Sigma . \left\{\frac{\delta V}{\delta a}\delta a + \frac{\delta V}{\delta b}\delta b + \frac{\delta V}{\delta c}\delta c\right\}$$

$$= \Sigma . m \left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\} + t\delta H + H_{,};$$

$H_{,}$ being a function of the initial co-ordinates a, b, c......

But when $t = 0$, $\delta V = 0$, hence

$$\delta V = \Sigma . m \left\{\frac{dx}{dt}\delta x + \frac{dy}{dt}\delta y + \frac{dz}{dt}\delta z\right\}$$

$$- \Sigma . m \left\{\frac{da}{dt}\delta a + \frac{db}{dt}\delta b + \frac{dc}{dt}\delta c\right\} + t\delta H \ldots\ldots (a).$$

From this equation we obtain the following groups of equations; $x_1 y_1 z_1$ being co-ordinates to m_1......

$$\left.\begin{aligned} \frac{\delta V}{\delta x_1} &= m_1\frac{dx_1}{dt}; \quad \frac{\delta V}{\delta x_2} = m_2\frac{dx_2}{dt}; \ldots\ldots \\ \frac{\delta V}{\delta y_1} &= m_1\frac{dy_1}{dt}; \quad \frac{\delta V}{\delta y_2} = m_2\frac{dy_2}{dt}; \ldots\ldots \\ \frac{\delta V}{\delta z_1} &= m_1\frac{dz_1}{dt}; \quad \frac{\delta V}{\delta z_2} = m_2\frac{dz_2}{dt}; \ldots\ldots \end{aligned}\right\} \ldots\ldots\ldots (A).$$

Second group,

$$\left.\begin{aligned} \frac{\delta V}{\delta a_1} &= -m_1\frac{da_1}{dt}; \quad \frac{\delta V}{\delta a_2} = -m_2\frac{da_2}{dt}; \ldots \\ \frac{\delta V}{\delta b_1} &= -m_1\frac{db}{dt}; \quad \frac{\delta V}{\delta b_2} = -m_2\frac{db_2}{dt}; \ldots \\ \frac{\delta V}{\delta c_1} &= -m_1\frac{dc_1}{dt}; \quad \frac{\delta V}{\delta c_2} = -m_2\frac{dc_2}{dt}; \ldots \end{aligned}\right\} \ldots\ldots\ldots (B).$$

Lastly,

$$\frac{\delta V}{\delta H} = t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (C).$$

The problem is therefore reduced to finding the function V, which Professor Hamilton denominates the *characteristic function* of the motion of a system. When V is calculated, then, by eliminating H from the equations (A) (C), we shall have the $3n$ integrals of the first order of the equations of motion by simply differentiating V. And by eliminating H from the equations (B) (C) we have the $3n$ final integrals by simple differentiation.

It may be observed that V must satisfy the two following partial differential equations,

$$\tfrac{1}{2}\Sigma \cdot \frac{1}{m}\left\{\frac{\delta V^2}{\delta x^2} + \frac{\delta V^2}{\delta y^2} + \frac{\delta V^2}{\delta z^2}\right\} = U + H,$$

$$\text{and } \tfrac{1}{2}\Sigma \cdot \frac{1}{m}\left\{\frac{\delta V^2}{\delta a^2} + \frac{\delta V^2}{\delta b^2} + \frac{\delta V^2}{\delta c^2}\right\} = U_0 + H.$$

These equations furnish the principal means of discovering the form of the function V, and are of essential importance in Professor Hamilton's Theory.

The equation (a) is denominated the *law of varying action.*

487. "It has been shewn by Lagrange and others, in treating of the motion of a system, that the variation δV vanishes when the extreme co-ordinates and constant H are given (Art. 489): and they appear to have deduced from this result only the principle which is called the law of *least action:* namely, that if the particles of a system be imagined to move from a given set of initial to a given set of final positions, not as they do, nor even as they could move consistently with the general dynamical laws, or differential equations of motion, but so as not to violate any supposed geometrical connexions, nor that one dynamical relation between velocities and configuration which constitutes the law of vis viva: and if, moreover, this geometrically imaginable, but dynamically impossible motion,

be made to differ infinitely little from the actual manner of motion of the system, between the given extreme positions, then the varied value of the definite integral called *action*, or the accumulated vis viva of the system in the motion thus imagined, will differ infinitely less from the actual value of that integral.

But when this principle of least action, or," as Professor Hamilton proposes to call it, "of *stationary action*, is applied to the determination of the actual motion of a system, it serves only to form, by the rules of the Calculus of Variations, the differential equations of motion of the second order, which can always be otherwise found."

In this, then, appears the excellence of this new principle called the *law of varying action*, that we pass from an actual motion to another motion dynamically possible, by varying the extreme positions of the system and (in general) the quantity H: but more especially that it serves to express, by means of a single function, not the mere differential equations of motion, but their intermediate and their final integrals.

We hope that the slight sketch we have given of this new principle will tempt our readers to consult the original Memoirs in the Transactions of the Royal Society of London for the years 1834, 1835, from which this notice has been gathered.

PROP. *To prove the general laws of the very small oscillations of a vibrating system of particles.*

488. If the oscillations of the particles be extremely small we may always reduce the equations of motion to linear equations and obtain approximately the co-ordinates in terms of the time. Very many and various phenomena depend upon the principles of small oscillations.

Let i be the number of particles, and m the number of equations $L=0$, $L'=0,\ldots$ connecting their co-ordinates: let $3i-m=n$, then these equations determine m of the variable co-ordinates in terms of the other n, or, more generally, all the co-ordinates may be determined by means of these equations in

functions of n independent variables. Let $\alpha, \beta \ldots$ be the initial values of these variables, and $\alpha + u$, $\beta + v \ldots$ their values at the time t; in which we suppose that $u, v \ldots$ are very small during the whole motion: hence the co-ordinates $x, y, z, x', \ldots$ can be expanded in very converging series of $u, v \ldots$

$$\begin{aligned}
\text{Let } x &= p + a\,u + b\,v + \tfrac{1}{2}c\,u^2 + \tfrac{1}{2}e\,v^2 + f\,uv + \ldots\ldots \\
y &= p_1 + a_1 u + b_1 v + \tfrac{1}{2}c_1 u^2 + \tfrac{1}{2}e_1 v^2 + f_1 uv + \ldots\ldots \\
z &= p_2 + a_2 u + b_2 v + \tfrac{1}{2}c_2 u^2 + \tfrac{1}{2}e_2 v^2 + f_2 uv + \ldots\ldots \\
x' &= p' + a' u + b' v + \tfrac{1}{2}c' u^2 + \tfrac{1}{2}e' v^2 + f' uv + \ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

Also since the forces $X, Y, Z, X', \ldots\ldots$ are supposed to be functions of the co-ordinates, these may be expanded in converging series: let

$$X = P + Au + Bv + \ldots, \quad Y = P_1 + A_1 u + B_1 v + \ldots,$$
$$Z = P_2 + A_2 u + B_2 v + \ldots, \ \&c.$$

$P, A, B \ldots$ being functions of $p, a, b, c \ldots$

Now, by Art. 481, we have

$$\Sigma . m \left\{ \left(\frac{d^2x}{dt^2} - X\right) \delta x + \left(\frac{d^2y}{dt^2} - Y\right) \delta y + \left(\frac{d^2z}{dt^2} - Z\right) \delta z \right\} = 0,$$

and $\delta x = (a + cu + fv + \ldots)\,\delta u + (b + ev + fu + \ldots)\,\delta v + \ldots$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If we substitute these and put the coefficients of the n arbitrary quantities $\delta u, \delta v \ldots$ equal to zero, we have n equations

$$\Sigma . m \left\{ \left(\frac{d^2x}{dt^2} - X\right)(a + cu + fv + \ldots) + \left(\frac{d^2y}{dt^2} - Y\right)(a_1 + c_1 u + f_1 v + \ldots) \right.$$
$$\left. + \left(\frac{d^2z}{dt^2} - Z\right)(a_2 + c_2 u + f_2 v + \ldots) \right\} = 0.$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It remains to substitute for $X, Y, Z \dots x, y, z \dots$; this substitution being made we shall neglect the squares and products of $u, v \dots$ and of their second differential coefficients with respect to t: we shall thus have n linear equations of the form

$$\left.\begin{array}{l} D\dfrac{d^2u}{dt^2} + E\dfrac{d^2v}{dt^2} + \dots\dots + Fu + Gv + \dots\dots = Q \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots (1),$$

where $D, E, F, G, Q \dots$ are given functions of the constants which enter the formulæ for $x, y, z \dots X, Y, Z \dots$ We may suppose $Q = 0$, since we can always add to $u, v \dots$ such constant values as to strike Q out: this amounts to supposing that $\alpha, \beta, \gamma \dots$ are the values of the n independent variables which correspond to a state of equilibrium of the system; since when $u = 0$, $v = 0 \dots$ the accelerating forces vanish.

We may satisfy the equations (1), putting $Q=0$, by $u = RN \sin(t\sqrt{\rho} - r)$, $v = RN' \sin(t\sqrt{\rho} - r) \dots\dots R$ and r being arbitrary constants of which the second may be considered positive and less than π, and $\rho, N, N', \dots$ are constants to be determined. By putting these values in (1), we have n equations,

$$\left.\begin{array}{l} (DN + EN' + \dots.)\,\rho = FN + GN' + \dots. \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array}\right\} \dots (2).$$

In eliminating from these $n - 1$ of the quantities $N, N' \dots$ the n^{th} equation will be of the n^{th} degree in ρ and will be free from all the quantities $N, N' \dots.$ in consequence of the form of equations (2). And the values of $n - 1$ of $N, N', \dots. viz.$ $N' \dots.$ suppose, obtained from (2) will be rational fractions of the n^{th} degree with respect to ρ, having a common denominator, and being each multiplied by N, which remains indeterminate; we may therefore choose N equal to the common denominator, and $N, N' \dots.$ will be expressed in terms of symmetrical functions of ρ of the n^{th} degree.

In consequence of the linear form of equations (1) they are satisfied not only by the values of $u, v \dots.$ corresponding to each of the n values of ρ, but also by taking for $u, v \dots.$

the sums of these particular values, in which we may change the values of R and r as ρ changes.

If then, we call $\rho\ \rho_1\ \rho_2\ \rho_3 \ldots$ the values of ρ, and use corresponding subscript figures for the other letters, we have the following general solutions of equations (1),

$$\left.\begin{aligned} u &= RN \sin (t\sqrt{\rho} - r) + R_1 N_1 \sin (t\sqrt{\rho_1} - r_1) + \ldots \\ v &= RN' \sin (t\sqrt{\rho} - r) + R_1 N'_1 \sin (t\sqrt{\rho_1} - r_1) + \ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots (3),$$

R, $R_1 \ldots r$, $r_1 \ldots$ being the $2n$ arbitrary constants in these complete integrals. The constants must be determined in terms of the initial values of u, $v \ldots$ and their differential coefficients: they are small because the original displacements are small. If the values $\rho\ \rho_1\ \rho_2 \ldots$ be all real, then the motions of the particles will be periodical and will always be very small. If, however, one or more of $\rho\ \rho_1\ \rho_2 \ldots$ be imaginary, we must replace the circular functions by exponentials, and therefore as the time increases u, $v \ldots$ will increase indefinitely and the above formulæ will cease to be true. In the first case the state of equilibrium of the system is stable; in the second unstable.

489. Suppose, for example, that all of R, $R_1 \ldots$ except the first vanish: then

$$\left.\begin{aligned} x &= p + (a N + b N' + \ldots) R \sin (t\sqrt{\rho} - r), \\ y &= p_1 + (a_1 N + b_1 N' + \ldots) R \sin (t\sqrt{\rho} - r), \\ z &= p_2 + (a_2 N + b_2 N' + \ldots) R \sin (t\sqrt{\rho} - r), \\ x' &= p' + (a' N + b' N' + \ldots) R \sin (t\sqrt{\rho} - r), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \ldots (4).$$

Hence the particles all perform their oscillations in the same period, *viz.* $\dfrac{2\pi}{\sqrt{\rho}}$: and all the particles return to their places of equilibrium at the same instant.

490. A system of material particles, in which the relations connecting the co-ordinates are of such a number as to leave n of them independent variables, will, when slightly disturbed from the position of rest, assume a number (n) of oscillatory motions, each analogous to that described in the last Article, corresponding to the n values $\rho\, \rho_1\, \rho_2 \ldots.$ And in virtue of equations (3) and the corresponding values of $x, y, z \ldots.$ all the oscillations, or only some of them may exist at the same time in the system: and conversely, whatever be the initial derangement we may always resolve the motion of each particle parallel to each co-ordinate axis into n or less than n simple oscillations analogous to that represented by equations (4), the periods being $\frac{2\pi}{\sqrt{\rho}}, \frac{2\pi}{\sqrt{\rho_1}} \ldots.$: when these are commensurable the whole system will return to the same state in a period equal to the least common multiple of these periods: this is the case in vibrating cords, and vibrating surfaces. The principle proved in this article is called the *Principle of the Co-existence of Small Vibrations.*

491. Suppose that $U, V \ldots.$ are values of $u, v \ldots.$ when the system is in vibration under the action of one set of forces, the initial values of $u, v \ldots. \frac{du}{dt}, \frac{dv}{dt} \ldots.$ being $u_0, v_0 \ldots.$ $u_1, v_1 \ldots.$ Again suppose that $U', V' \ldots.$ are the values of $u, v \ldots.$ when the system is under the action of a second set of forces and $u_0', v_0', \ldots. u_1', v_1' \ldots.$ the initial values of $u, v \ldots. \frac{du}{dt}, \frac{dv}{dt} \ldots.$ and so on: then, if the initial values of $u, v \ldots. \frac{du}{dt}, \frac{dv}{dt} \ldots..$ be $u_0 + u_0' + \ldots.., \quad v_0 + v_0' + \ldots..,$ $\ldots. u_1 + u_1' + \ldots., \quad v_1 + v_1' + \ldots.$ the general values of $u, v \ldots.$ are

$$u = U + U' + \ldots., \quad v = V + V' + \ldots.$$

This principle, the truth of which arises from equations (1) being linear, is called the *Principle of the Superposition of Small Motions:* see Art. 288.

CHAPTER XII.

THE MOTION OF A FLEXIBLE BODY.

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492. In the present Chapter we shall calculate and explain some of the simpler cases of the motion of vibrating strings: for more information on this subject and on the motion of elastic springs we refer the reader to M. Poisson's *Traité de Mécanique*, Tom. II. Seconde Edition; to the *Journal de l'Ecole Polytechnique*, Cahier XVIII, p. 442; and lastly, to M. Poisson's Memoir on the Equilibrium and Motion of Elastic Bodies in the *Mémoires de l'Académie des Sciences*, Tom. VIII.

Prop. *To determine equations for calculating the motion of a perfectly flexible cord, very slightly extensible, of the same thickness and density throughout, fixed at its two extremities, and very little disturbed from its position of rest.*

493. Let A and B be the fixed extremities of the cord (fig. 109), we shall suppose that the cord is straight when in equilibrium: Let P be the position of a particle of the cord in motion at the time t, which is at Q when the cord is at rest: $AQ = x$, $AM = x + u$, $MN = y$, $NP = z$, $AB = l$: M the mass of the cord, T the tension at the point P: the resolved parts of T parallel to the axes are

$$T\frac{d(x+u)}{ds}, \quad T\frac{dy}{ds}, \quad T\frac{dz}{ds}; \text{ where } ds = PP';$$

the excesses of the corresponding tensions at P' over those at P are therefore, by Taylor's Theorem,

$$\frac{d}{dx}\left(T\frac{d(x+u)}{ds}\right)dx, \quad \frac{d}{dx}\left(T\frac{dy}{ds}\right)dx, \quad \frac{d}{dx}\left(T\frac{dz}{ds}\right)dx:$$

these are the *impressed* forces acting on PP': the mass of $PP' = M\frac{ds}{l}$, hence the *effective* forces acting on PP' are

$$M\frac{ds}{l}\frac{d^2u}{dt^2}, \quad M\frac{ds}{l}\frac{d^2y}{dt^2}, \quad M\frac{ds}{l}\frac{d^2z}{dt^2}:$$

hence by the Principle of Art. 224 we have

$$\frac{d}{dx}\left\{T\frac{d(x+u)}{ds}\right\} = \frac{M}{l}\frac{d^2u}{dt^2}, \quad \frac{d}{dx}\left(T\frac{dy}{ds}\right) = \frac{M}{l}\frac{d^2y}{dt^2},$$

$$\frac{d}{dx}\left(T\frac{dz}{ds}\right) = \frac{M}{l}\frac{d^2z}{dt^2}.$$

Let W be the tension of the cord when at rest; then it is found by experiment that the change in extension (as $ds - dx$,) of a piece of cord dx varies as the change in tension $T - W$: suppose that $T - W = Q\frac{ds - dx}{dx}$.

$$\text{Now } \frac{ds^2}{dx^2} = \left(1 + \frac{du}{dx}\right)^2 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2} = \left(1 + \frac{du}{dx}\right)^2,$$

neglecting small quantities of the second order. Hence $T - W = Q\frac{du}{dx}$, and our equations of motion become

$$\frac{d^2u}{dt^2} = b^2\frac{d^2u}{dx^2}, \quad \frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2}, \quad \frac{d^2z}{dt^2} = a^2\frac{d^2z}{dx^2},$$

if we neglect small quantities of the second order, and put $Ql = Mb^2$, and $Wl = Ma^2$: hence a^2 and b^2 are in the ratio $W : Q$.

The variables u, y, z are separated in these equations; from which we conclude that the vibrations of the cord parallel to the axes of x, y, z are independent of each other, and

co-exist without any interference. The transversal vibrations are the same in the directions of y and z. We shall calculate the motion parallel to y.

PROP. *To integrate the equations of motion, and to interpret the integrals.*

494. For the transversal vibrations,

$$\frac{d^2y}{dt^2} = a^2 \frac{d^2y}{dx^2}.$$

To integrate this, add to each side $a\dfrac{d^2y}{dx\,dt}$;

$$\therefore \frac{d}{dt}\left\{\frac{dy}{dt} + a\frac{dy}{dx}\right\} = a\frac{d}{dx}\left\{\frac{dy}{dt} + a\frac{dy}{dx}\right\},$$

$$\text{or } \frac{dv}{dt} = a\frac{dv}{dx}, \text{ if we put } \frac{dy}{dt} + a\frac{dy}{dx} = v:$$

$$\therefore dv = \frac{dv}{dt}dt + \frac{dv}{dx}dx = \frac{dv}{dx}d(x + at),$$

$$\therefore v, \text{ or } \frac{dy}{dt} + a\frac{dy}{dx}, = \phi(x + at);$$

ϕ representing any arbitrary function of $x + at$.

In like manner by subtracting $a\dfrac{d^2y}{dx\,dt}$ from each side we have

$$\frac{dy}{dt} - a\frac{dy}{dx} = \psi(x - at),$$

ψ representing another arbitrary function.

$$\text{Hence } \frac{dy}{dt} = \tfrac{1}{2}\phi(x + at) + \tfrac{1}{2}\psi(x - at),$$

$$\frac{dy}{dx} = \frac{1}{2a}\phi(x + at) - \frac{1}{2a}\psi(x - at);$$

$$\therefore\ dy = \frac{dy}{dt}\,dt + \frac{dy}{dx}\,dx$$

$$= \frac{1}{2a}\,\phi\,(x + at)\,d\,(x + at) - \frac{1}{2a}\,\psi\,(x - at)\,d\,(x - at)\,;$$

$$\therefore\ y = F\,(x + at) + f\,(x - at),$$

F and f being arbitrary functions depending on ϕ and ψ; but we shall cease to use ϕ and ψ.

We proceed to explain how to determine the values of these functions. We are supposed to know the initial circumstances of the motion, namely the values of y and $\dfrac{dy}{dt}$ for all values of x between $x = 0$ and $x = l$ when $t = 0$: hence $F(x) + f(x)$ is known for all values of x between 0 and l: and also $\dfrac{dF(x)}{dx} - \dfrac{df(x)}{dx}$ is known, and consequently $F(x)$ and $f(x)$ are known between these limits; but the values of these quantities for values of x greater than l and less than 0 are not given, nor are they necessarily known since the functions $F(x)$ and $f(x)$ may be discontinuous; that is, the original form of the curve need not be such as can be expressed in analysis, but may be a series of pieces of curve so long as they have the same tangent at their points of junction.

The condition that the extremities of the cord are always stationary enables us to determine the values of $F\,(x + at)$ and $f\,(x - at)$ for all values of x and t. For by this condition $y = 0$ and $\dfrac{dy}{dt} = 0$ when $x = 0$ and l whatever t be: hence by putting $at = v$ we have

$$F\,(v) + f\,(-v) = 0\ldots(1),\quad F\,(l + v) + f\,(l - v) = 0\ldots(2)$$

for all positive values of v.

Put $l + v$ for v in (2), then by (2) and (1)

$$F\,(2l + v) = F\,(v)\ \ldots\ldots\ldots\ldots\ (3).$$

The initial circumstances make known $F(v)$ and $f(v)$ from $v = 0$ to $v = l$: then (2) gives $F(v)$ from $v = l$ to $v = 2l$,

and then (3) gives $F(v)$ from $v = 2l$ to $4l$, then to $6l$ and so on to ∞: hence $F(x + at)$ is known for all positive values of t. Hence, by (1) $f(-v)$ is known from $v = 0$ to $v = \infty$: but also $f(v)$ is known from $v = 0$ to $v = l$ by the initial circumstances, hence $f(x - at)$ is known for every point of the cord during the whole motion. Hence the value of y, and therefore the form of the string, is known at every instant. There is nothing to make known $F(v)$ for negative values of v, or $f(v)$ for values of v between l and ∞.

In (1) put $v + 2l$ for v, then

$$f(-2l - v) = -F(2l + v) = -F(v) \text{ by (3)}$$
$$= f(-v) \text{ by (1)}\ldots\ldots\ldots(4).$$

Hence
$$y = F(x + at) + f(x - at)$$
$$= F(x + at + 2l) + f(x - at - 2l)$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$= F(x + at + 2nl) + f(x - at - 2nl) \text{ by (3) (4)}$$

n a positive integer: hence the cord repeatedly assumes the same form relatively to the plane xz, performing a vibration in the time $\frac{2l}{a}$; substituting for a its value,

$$\text{time of vibration} = 2\sqrt{\frac{Ml}{W}}:$$

the same is true of the motion parallel to z: and also parallel to x the oscillations take place in the time $2\sqrt{\frac{Ml}{Q}}$.

Prop. *A portion only of the cord is set in motion at first, as a piano-forte wire by the sudden blow of the hammer of a key: required to determine the motion.*

495. To simplify the calculation we shall at first suppose that one end of the string is at an indefinitely great distance. Let the original displacement extend over a small space $2a$; and let the origin of x be at the mid-point of this space: h the

distance of the nearest extremity. Then when $t = 0$ we have $y = 0$ and $\frac{dy}{dt} = 0$ from $x = -\infty$ to $x = -a$ and from $x = a$ to $x = h$;

$$\therefore\ F(v) = 0 \text{ and } f(v) = 0 \ldots\ldots\ldots\ldots (1),$$

from $v = -\infty$ to $v = -a$, and from $v = a$ to $v = h$.

Because of the fixed extremity

$$F(h + v) + f(h - v) = 0,$$

for positive values of v. Hence when v is greater than h,

$$F(v) = -f(2h - v) \ldots\ldots\ldots\ldots\ldots (2).$$

Since $x - at$ is always negative for negative values of x, it follows by (1) that beyond the limits of disturbance, that is when x is $< -a$, we have $f(x - at) = 0$: also for negative values of x we have $F(x + at) = 0$ unless t lie between $\frac{-a - x}{a}$ and $\frac{a - x}{a}$.

Hence the initial disturbance is propagated to the left and each particle of the cord begins to move after a time $\frac{-a - x}{a}$, vibrates for a time $\frac{2a}{a}$ and then returns to rest.

In the same way the motion is propagated to the right: but in consequence of the fixed extremity this will require a little further examination.

Let us consider the motion of a particle at a distance x; $y = F(x + at) + f(x - at)$ is its general displacement.

When $t = 0$ this particle is at rest, for $y = 0$ by (1): and it remains so till $t = \frac{x - a}{a}$, for, then $y = f(a)$, and the particle moves till $t = \frac{x + a}{a}$, and after this is at rest again for a time. Ever after this $x - at$ is negative and $< -a$, and therefore $f(x - at) = 0$ by (1). But when t becomes

$$\frac{x-a}{a}+\frac{2h-2x}{a} \text{ or } \frac{2h-x-a}{a};$$

$$y = F(2h-a) = -f(a) \text{ by } (2),$$

and as t increases and becomes $\frac{x+a}{a}+\frac{2h-2x}{a}=\frac{2h-x+a}{a}$;

$$y = F(2h+a) = -f(-a) \text{ by } (2).$$

Hence at a time $\frac{2h-2x}{a}$ after the particle began to move, it again begins to move: and ceases to move at the same time after it ceased before. Likewise the displacements of the particle are exactly the same that they were before, but on the *opposite side* of the line of rest.

When t is $> \frac{2h-x+a}{a}$, $x+at$ is $> 2h+a$;

$$\therefore F(x+at) = -f\{2h-(x+at)\} = 0 \text{ always.}$$

Hence the particle oscillates for a period $\frac{2a}{a}$ commencing at the time $\frac{x-a}{a}$: it then rests: and after a time $\frac{2(h-x)}{a}$, it oscillates for the same period in a manner precisely similar to the former; except on the opposite side of the line of rest: after this the particle remains permanently at rest. This is true whatever x be: and it is to be remarked that $\frac{2(h-x)}{a}$ = the time the bent portion of the cord, or the *pulse*, would take to move from the particle to the fixed point and *back again*. Hence, since this second motion arises solely in consequence of the fixity of one extremity of the string, it follows that when the right hand pulse reaches the fixed point *it is reflected*, but to the opposite side of the string.

Hence the original disturbance divides itself into two pulses, one moving continually to the left; the other to the right till it reaches the fixed point, after which it moves back, towards the left, on the other side of the line and with the same velocity as before.

Let the string be of definite length. Then the pulse after reflexion will be reflected again at the other extremity and move on the upper part of the string to the right.

Let C, D (fig. 110) be the fixed extremities of the string and A the origin of disturbance. The initial disturbance divides into two a and b: b is reflected at D and moves along to b' at the same time that a moves to a' having been reflected at C: a' and b' meet at B and confirm each other, forming a disturbance exactly similar to the original disturbance.

Evidently B is such a point that

$$AD + DB = AC + CB, \text{ or } AD = CB.$$

Now the interval between two maximum disturbances at A = time of describing $AC + CD + DA = \dfrac{2l}{a}$. Hence the velocity of the pulses $= a$.

CHAPTER XIII.

MOTION OF ONE OR MORE RIGID BODIES ACTED ON BY *IMPULSIVE FORCES.*

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496. IN the preceding Chapters we have obtained differential equations for calculating the motion of a body acted on by any forces of finite intensity. Since these differential equations are of the second order, their first integrals will be of the first order, and will therefore be functions of the velocities and co-ordinates of position of the various parts of the body. The values of the arbitrary constants introduced by the process of integration are determined by knowing the velocity and position of the parts of the body at any given instant of the motion: the instant generally chosen is the epoch from which the time is measured. In calculating the motion of a heavenly body, the values of the arbitrary constants are found by observations, made on the position and velocity of the body at any given time; because we are altogether unacquainted with the initial circumstances of the motion. But we may wish to calculate the motion of a material system, when the original circumstances of projection are known. It becomes necessary, then, to calculate the motion, which results from the action of Impulsive Forces. We have explained the nature of these forces in Chap. I.

497. In the following Articles, whenever we speak of the *Impressed Forces*, we mean the momenta, which the different particles have at the instant the impulsive action *begins;* and also the forces which are put in play *during* the action, by the connexion of the parts of the system one with another; *neglecting only the molecular forces.* And by the *Effective Forces* we mean the momenta which the different particles have at the

instant the impulsive action *ceases*. These forces act at different instants: but the whole duration of the action is so extremely short, that the spaces described by the parts of the system during the action are insensible, and the accumulated effect of the forces is the same as if they acted simultaneously.

498. We have laid down a Principle in Art. 225, by means of which we can obtain equations for calculating the effect of impulsive forces upon any material system. This principle coincides with *D'Alembert's Principle* applied to impulsive action, (as is remarked in Art. 226,) in those cases in which the only molecular forces are those arising from the action of the molecules of *rigid* bodies: because then these forces will of themselves be in equilibrium, and may therefore be neglected in the enunciation of our Principle.

MOTION OF ONE RIGID BODY.

499. In the following calculations V is the velocity, which measures the dynamical effect of the *impressed* force acting on the particle m, not including molecular forces: V_1, V_2, V_3 are the resolved parts of V parallel to the axes: and v, v_1, v_2, v_3 are similar quantities in reference to the *effective* force of m.

PROP. *To obtain the equations of motion of a rigid body acted on by impulsive forces.*

500. By the Principle of Art. 225 applied to rigid bodies, (or by D'Alembert's Principle, see Art. 226), the impulsive forces

$$m(V_1 - v_1),\ m(V_2 - v_2),\ m(V_3 - v_3)$$

acting on m parallel to the axes of co-ordinates, together with similar forces acting on the other particles of the system, ought to be in equilibrium.

Hence we have from Art. 56 the equations

$$\Sigma . m(V_1 - v_1) = 0,\ \Sigma . m(V_2 - v_2) = 0,\ \Sigma . m(V_3 - v_3) = 0,$$

$$\Sigma . m\{y(V_3-v_3)-z(V_2-v_2)\}=0,\ \Sigma . m\{z(V_1-v_1)-x(V_3-v_3)\}=0,$$

$$\Sigma . m\ \{x(V_2-v_2)-y(V_1-v_1)\}=0.$$

By means of these six equations we shall be able to calculate the motion of a rigid body acted on by any impulsive forces.

They lead immediately to two fundamental principles, analogous to those of Arts. 429, 430, for finite forces.

Prop. *The velocity of the centre of gravity is the same, as if the forces, which act upon the various particles of the body, were all transferred to that point, their directions being parallel to their former directions.*

501. For the first three of the equations of last Article give

$$\Sigma . m(V_1-v_1)=0,\ \Sigma . m(V_2-v_2)=0,\ \Sigma . m(V_3-v_3)=0.$$

Now let $\overline{V}$ be the velocity of the centre of gravity when the impulse is over: and let $\overline{V}_1$, $\overline{V}_2$, $\overline{V}_3$, be the resolved parts of $\overline{V}$ parallel to the axes. But if we differentiate the formulæ of Art. 413, with respect to t,

$$\Sigma . mv_1=M\overline{V}_1,\ \Sigma . mv_2=M\overline{V}_2,\ \Sigma . mv_3=M\overline{V}_3.$$

Therefore by the equations above, we have

$$M\overline{V}_1=\Sigma . mV_1,\ M\overline{V}_2=\Sigma . mV_2,\ M\overline{V}_3=\Sigma . mV_3.$$

But these are the equations we should have obtained by supposing the forces transferred to the centre of gravity, their directions being preserved.

Hence the Proposition, as enunciated, is true.

Prop. *The velocity of rotation of the body will be the same as if the centre of gravity were fixed.*

502. Let $x'y'z'$ be the co-ordinates of m measured from the centre of gravity parallel to the original axes: then $x=\bar{x}+x'$, $y=\bar{y}+y'$, $z=\bar{z}+z'$; also

$$v_1 = \bar{V}_1 + v_1', \quad v_2 = \bar{V}_2 + v_2', \quad v_3 = \bar{V}_3 + v_3'$$

the accented velocities being the velocities relative to the centre of gravity: these formulæ are obtained by differentiating the above values of xyz.

If we substitute these quantities in the last three equations of Art. 500, the first of them becomes

$$\Sigma . m \{(\bar{y} + y')(V_3 - \bar{V}_3 - v_3') - (\bar{z} + z')(V_2 - \bar{V}_2 - v_2')\} = 0.$$

The quantities which refer to the centre of gravity may be written on the left hand of Σ: also by Art. 413 we have $\Sigma . mx' = 0$, $\Sigma . my' = 0$, $\Sigma . mz' = 0$. If we introduce these conditions, and make use of the equations of the last Art., the above equation becomes

$$\Sigma . m \{y'(V_3 - v_3') - z'(V_2 - v_2')\} = 0.$$

$$\text{Similarly, } \Sigma . m \{z'(V_1 - v_1') - x'(V_3 - v_3')\} = 0,$$

$$\Sigma . m \{x'(V_2 - v_2') - y'(V_1 - v_1')\} = 0.$$

But these are exactly the equations we should arrive at by supposing the centre of gravity fixed, during the action of the forces.

Hence the Proposition, as enunciated, is true.

The Principles proved in the last two Propositions reduce the calculation of the motion of a rigid body moving freely and acted on by impulsive forces to the calculation of the motion of a single particle, and of a rigid body moving about a fixed point. We shall now determine more convenient equations for calculating the rotatory motion of a body about its centre of gravity when acted on by impulsive forces.

PROP. *To calculate the rotatory motion of a body moving about a fixed axis and acted on by impulsive forces.*

503. The equation for determining the rotatory motion is (Art. 500.)

$$\Sigma . m \{x(V_2 - v_2) - y(V_1 - v_1)\} = 0,$$

the axis of z being the axis of revolution; let r be the distance of any particle m from this axis: θ the angle which r makes with the plane zx; and ω the angular velocity at the time t;

$$\therefore\ x = r\cos\theta,\quad y = r\sin\theta;$$

$$\therefore\ \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} = -y\frac{d\theta}{dt},\quad \frac{dy}{dt} = r\cos\theta\frac{d\theta}{dt} = x\frac{d\theta}{dt};$$

$$\therefore\ v_1 = -y\omega,\quad v_2 = x\omega:$$

r is constant with respect to t, because the axis is fixed in the body. Our equation now becomes

$$\Sigma.m(x^2+y^2)\,\omega = \Sigma.m(xV_2 - yV_1);$$

$$\therefore\ \omega = \frac{\Sigma.m(xV_2 - yV_1)}{\Sigma.m(x^2+y^2)}$$

$$= \frac{\text{moment of the impressed forces about the axis}}{\text{moment of inertia about the axis}}.$$

Prop. *A body in which one point is fixed is acted on by impulsive forces: required to determine the motion.*

504. Let the fixed point be the origin: and let $r_1\,r_2\,r_3$ be the distances of the particle m from the axes of x, y, z; and let $\theta_1\,\theta_2\,\theta_3$ be the angles which $r_1 r_2 r_3$ make respectively with the planes xy, zy, zx; and $\omega_1\,\omega_2\,\omega_3$ the angular velocities about the axes:

$$\therefore\ y = r_1\cos\theta_1,\quad z = r_2\cos\theta_2,\quad x = r_3\cos\theta_3,$$

$$z = r_1\sin\theta_1,\quad x = r_2\sin\theta_2,\quad y = r_3\sin\theta_3;$$

$$\therefore\ v_2 = -z\omega_1,\quad v_3 = -x\omega_2,\quad v_1 = -y\omega_3,$$

$$v_3 = y\omega_1,\quad v_1 = z\omega_2,\quad v_2 = x\omega_3.$$

Hence the three last equations of Art. 500 become

$$\omega_1\Sigma.m(y^2+z^2) = \Sigma.m(yV_3 - zV_2),$$

$$\omega_2\Sigma.m(x^2+z^2) = \Sigma.m(zV_1 - xV_3),$$

$$\omega_3\Sigma.m(x^2+y^2) = \Sigma.m(xV_2 - yV_1),$$

or

$$\omega_1 = \frac{\text{sum of moments of the impressed forces about axis of } x}{\text{moment of inertia about axis of } x},$$

$$\omega_2 = \frac{\text{sum of moments of the impressed forces about axis of } y}{\text{moment of inertia about axis of } y},$$

$$\omega_3 = \frac{\text{sum of moments of the impressed forces about axis of } z}{\text{moment of inertia about axis of } z}.$$

We shall apply the principles proved in the last Articles to the solution of a few questions.

PROP. *A smooth and imperfectly elastic sphere impinges on a fixed plane at a given angle, and with a given direct velocity, but no rotation: required to find its velocity and direction of motion immediately after impact.*

505. Questions of this description, where the bodies are more or less elastic, always divide themselves into two parts; first, the action of the forces during the compression of figure of the bodies; secondly, the action during the restitution of figure. Bodies differ in their elasticity owing to their physical constitution: but the law to which we are led by experiment is this, that when two bodies of given materials (the same or different), and of any masses, come in contact, the momentum gained by each body during the restitution bears a constant ratio to the momentum lost during the compression of figure: this ratio we write e, and is called the *elasticity* of the material of which the bodies are made. This is not strictly true for extremely great or extremely small velocities. See Art. 203 and the Note. Hence if P be the force which measures the mutual action of two bodies during compression, $e \cdot P$ will be the mutual action of the bodies during restitution of figure, e being the elasticity appertaining to the material or materials of which the bodies are made, and chosen from a Table of experimental results.

Let M be the mass of the sphere, V its velocity, and α the angle the direction of its motion makes with a normal to the plane at the instant it comes in contact with the plane: P the

impulsive pressure between the sphere and plane during compression: and therefore $e.P$ the pressure during restitution: u and θ the velocity and direction of motion the instant compression ceases: v and β the velocity and direction the instant the sphere leaves the plane.

Then by Art. 501, resolving the forces parallel and perpendicular to the plane, *at the instant compression ceases* we have

$$MV \sin a - Mu \sin \theta = 0 \ldots\ldots\ldots\ldots (1),$$

$$MV \cos a - P - Mu \cos \theta = 0 \ldots\ldots (2),$$

and because the plane is immoveable

$$u \cos \theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again, *at the instant restitution ceases,*

$$Mu \sin \theta - Mv \sin \beta = 0 \ldots\ldots\ldots\ldots\ldots (4),$$

$$Mu \cos \theta - e.P + Mv \cos \beta = 0 \ldots\ldots (5).$$

By (3) $\theta = 90^0$; hence from (1) (2) we have

$$u = V \sin a, \quad \text{and} \quad P = MV \cos a.$$

Hence by (4) and (5),

$$v \sin \beta = V \sin a, \quad v \cos \beta = Ve \cos a;$$

$$\therefore \cot \beta = e \cot a,$$

$$\text{and} \quad v = V\sqrt{\sin^2 a + e^2 \cos^2 a}, \quad \text{or} = V \sin a \div \sin \beta,$$

which determine the velocity and direction of motion after impact.

Cor. If we could find substances perfectly devoid of elasticity, then, in that case, $e = 0$; and $\beta = 90^0$, and $v = V \sin a$, or the body would move along the plane in its compressed form incapable of recovering its figure, and therefore not receding from the plane. If the elasticity were perfect, then $e = 1$, $\beta = a$, $v = V$, and the motion after impact is exactly the same as before impact.

PROP. *Two spheres moving in the same straight line with given velocities without rotation, come in contact: required their velocities after impact.*

506. Let M and M' be the masses of the spheres: e their relative elasticity: V, u, v the velocities of M, at the commencement of the action, at the instant compression ceases, and at the instant restitution of figure ceases. Also let the same letters accented apply to M'.

Then by Art. 501, *at the instant compression ceases*,

$$MV - P - Mu = 0 \ldots\ldots (1),$$

$$M'V' + P - M'u' = 0 \ldots\ldots (2),$$

but at the instant compression ceases the bodies move with the same velocity; hence

$$u = u' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again, *at the instant restitution ceases*,

$$Mu - e\,.\,P - Mv = 0 \ldots. (4),$$

$$M'u' + e\,.\,P - M'v' = 0 \ldots. (5).$$

Eliminating u and u' from (1) (2) (3), we have

$$P = \frac{MM'}{M + M'}(V - V'), \text{ and } u = u' = \frac{MV + M'V'}{M + M'}.$$

Then by (4) and (5), we have

$$v = \frac{MV + M'V'}{M + M'} - e\frac{M'(V - V')}{M + M'}$$

$$v' = \frac{MV + M'V'}{M + M'} + e\frac{M(V - V')}{M + M'}.$$

COR. If the bodies were devoid of elasticity, then $e = 0$, and

$$v = v' = \frac{MV + M'V'}{M + M'}.$$

If the elasticity were perfect, $e = 1$, and

$$v = V - \frac{2M'}{M+M'}(V - V'), \quad v' = V' + \frac{2M}{M+M'}(V - V').$$

Some questions of this description will be given in the Chapter of Problems.

When a body at rest is acted on by any forces there is a line about which it *begins* to revolve. This line is called the *Axis of Spontaneous Rotation.*

PROP. *To find the position of the axis of spontaneous rotation of a body, when it is acted on by an impulsive force.*

507. Let P be the momentum which measures the impulsive force: M the mass of the body.

Then by Art. 501 the centre of gravity moves with the velocity $\frac{P}{M}$ in a direction parallel to that of the impulse.

Let the line of the impulse be taken for the axis of x: and the plane through this and the centre of gravity for the plane of xy: h and r the distances of the centre of gravity from the line of impulse and from the projection on the plane xy of any particle m: θ the angle which r makes with the axis of x: ω the angular velocity of the body. Hence, by Arts. 502 and 503,

$$\omega = \frac{\text{moment of } P}{\text{moment of inertia}} = \frac{Ph}{Mk^2},$$

k being the radius of gyration about the centre of gravity.

Let xy be the co-ordinates of the projection of m: then by compounding the velocities of translation and rotation, we have (fig. 108),

$$\text{vel. of } m \text{ parallel to } x = \frac{P}{M} - \omega r \sin\theta = \frac{P}{M} - \frac{Phr\sin\theta}{Mk^2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; y = \omega r \cos\theta = \frac{Phr\cos\theta}{Mk^2}.$$

To find the point in the plane xy which is at rest at the beginning of the motion we must equate these two velocities to zero;

$$\therefore\ k^2 - hr\sin\theta = 0, \quad r\cos\theta = 0;$$

$$\therefore\ \theta = 90^0, \text{ and } r = \frac{k^2}{h} = GO \text{ in the figure,}$$

and therefore the axis of spontaneous rotation is at right angles to the direction of the impulse; and also cuts at right angles the perpendicular from the centre of gravity upon the direction of the impulse at a distance $\frac{k^2}{h}$, the centre of gravity lying *between* the axis of spontaneous rotation and the impulse.

The point O coincides with the centre of oscillation, if H be the projection of the axis of suspension: see Art. 432.

Prop. *A body revolves about a fixed axis and impinges upon a fixed point, so that the direction of the impulse is perpendicular to the plane passing through the axis and the centre of gravity: required to find the position of the fixed point so that the pressure on the fixed axis at the instant of impact may be wholly in the plane perpendicular to the direction of the impulse. The fixed point so found is called the* Centre of Percussion.

508. Let the fixed axis be the axis of z: and the plane passing through the centre of gravity at the instant of the impulse the plane yz. P the momentum which measures the impulse on the fixed point: $y_, z_,$ the co-ordinates to the point in which the direction of the impulse cuts the plane yz: ω the angular velocity of the body at the instant the impulsive action commences: then, $-y\omega$ and $x\omega$ are the velocities of a particle m of which the co-ordinates are x and y, and $-my\omega$ and $mx\omega$ the momenta parallel to the axes of x and y, where the action commences.

We shall suppose the axis to be fixed at two points (since if fixed at more they can always be reduced to two) of which the distances from the origin are a and a'; let $R\cos\alpha$,

$R\cos\beta$, $R\cos\gamma$, $R'\cos\alpha'$, $R'\cos\beta'$, $R'\cos\gamma'$ be the momenta which measure the impulsive pressures parallel to the axes of x and y on these points at the instant of impact: h the distance of the centre of gravity from the axis of z.

The forces, then, which act upon the body at the instant of impact are

$R\cos\alpha$, $R'\cos\alpha'$, $-my\omega$ on each particle m, and P parallel to x,

$R\cos\beta$, $R'\cos\beta'$, $mx\omega$ on each particle m, ... y,

$R\cos\gamma$, $R'\cos\gamma'$, ... z.

But the body is reduced to rest, by hypothesis; and consequently by the six equations of Art. 56 we have

$$R\cos\alpha + R'\cos\alpha' - \omega\Sigma . my + P = 0,$$

$$R\cos\beta + R'\cos\beta' + \omega\Sigma . mx = 0, \quad R\cos\gamma + R'\cos\gamma' = 0,$$

$$-R\cos\beta . a - R'\cos\beta' . a' - \omega\Sigma . mxz = 0,$$

$$R\cos\alpha . a + R'\cos\alpha' . a' - \omega\Sigma . myz + Pz_{,} = 0,$$

$$\omega\Sigma . m(x^2 + y^2) - Py_{,} = 0,$$

in these $\Sigma . mx = 0$, and $\Sigma . my = Mh$, as the axes have been chosen. From these equations the pressures may be found.

Now for the centre of percussion we must have $R\cos\alpha = 0$ and $R'\cos\alpha' = 0$: hence

$$-Mh\omega + P = 0, \quad -\omega\Sigma . myz + Pz_{,} = 0,$$

$$\omega\Sigma . m(x^2 + y^2) - Py_{,} = 0;$$

therefore $P = Mh\omega$ is known, since the motion previous to the impact, and consequently ω, may be calculated by the principles of Chapter XI: and the co-ordinates to the centre of percussion are,

$$y_{,} = \frac{\Sigma . m(x^2 + y^2)}{Mh} = \frac{k^2 + h^2}{h}; \quad z_{,} = \frac{\Sigma . myz}{Mh}.$$

If the body be symmetrical about a plane through the centre of gravity and perpendicular to the axis of z, then, if $\bar{z}$ be

the distance of the centre of gravity from the plane xy, and if $z = \bar{z} + z'$ we have

$$\Sigma . myz = \bar{z}\Sigma . my + \Sigma . myz' = \bar{z}\Sigma . my = M\bar{z}h.$$

In this case $y_{\prime} = \dfrac{k^2 + h^2}{h}$, $z_{\prime} = \bar{z}$: and the centre of percussion will then coincide with the centre of oscillation: see Art. 432.

MOTION OF A SYSTEM OF BODIES.

509. When two or more bodies are so connected, that when acted on by impulsive forces they influence each other's motion, the motion of each may be calculated by substituting unknown forces for the unknown mutual actions, as mentioned in Art. 472. In the remaining Articles of this Chapter we shall prove some general principles of the motion of a system acted on by impulsive forces. We shall use the same notation as in the previous Articles (see Art. 499), except that V &c.... will not include the mutual actions of the bodies on one another.

Prop. *To find equations of motion.*

510. As in Art. 500 the forces

$$m(V_1 - v_1),\quad m(V_2 - v_2),\quad m(V_3 - v_3)$$

acting on m, and all similar forces acting on the other particles, ought, together with the mutual actions of the bodies, and the molecular forces, to be in equilibrium at the time t.

But the molecular forces are of themselves in equilibrium, because we suppose the bodies rigid: and therefore by Art. 69 we have

$$\Sigma . m(V_1 - v_1) = 0,\quad \Sigma . m(V_2 - v_2) = 0,\quad \Sigma . m(V_3 - v_3) = 0,$$

$$\Sigma . m\,\{y(V_3 - v_3) - z(V_2 - v_2)\} = 0,$$

$$\Sigma . m\,\{z(V_1 - v_1) - x(V_3 - v_3)\} = 0,$$

$$\Sigma . m\,\{x(V_2 - v_2) - y(V_1 - v_1)\} = 0,$$

in which the mutual pressures do not appear, as we learn from Art. 69. The same remarks may be made here that we made in Art. 473.

PROP. *To prove, that the motion of the centre of gravity of the system is the same as if the whole mass were collected in that point, and all the forces acted on it parallel to their real directions. Also to prove that the Principle of the Conservation of the Motion of the Centre of Gravity* (Art. 474) *is true when internal impulsive forces act.*

511. The first part of this Prop. may be proved exactly as in Art. 501: and the second part is a very simple deduction from the first.

PROP. *To prove that the Principle of the Conservation of Areas* (Art. 475) *is true when internal impulsive forces act:* i.e. that, though the forces may change the velocities and directions of the parts of the system: yet, they are so changed, as not to make the sum of the areas multiplied by the masses after the impulse different to that sum before the impulse.

512. This Prop. is an immediate deduction from the equations of motion. For the first of the last three of the equations of Art. 510 gives

$$\Sigma . m (y V_3 - x V_2) = \Sigma . m (y v_3 - x v_2),$$

and the internal impulsive forces do not enter this and the other two equations, as is observed in Art. 510.

Now the first side of this equation is the sum of the products of the masses and the areas described in the unit of time immediately *before* the impact (see Art. 475), and the second side is the same sum *after* the impact. These sums are therefore equal, and the Prop. is true.

It is also true in the peculiar case mentioned in Art. 476.

513. Hence the Invariable Plane and the Principal Moment of a System (Art. 477) are not affected by the action of internal impulsive forces: and therefore, combining the results just deduced and those of Art. 477, we learn from the

Principle of the Conservation of Areas, or rather the Principle of the Conservation of the Principal Moments which springs from it, that earthquakes, volcanic explosions, the action of winds upon the surface of the Earth, the friction and pressure of the Ocean upon the solid nucleus of the terrestrial spheroid, produce no variation in the principal moment on the direction of its axis: since these forces all arise from the mutual action of the parts of the system. And since the displacements produced by these causes in any portions composing the Earth's mass are too inconsiderable sensibly to alter the value of k, it follows, that their effect on the angular velocity and upon the length of the day will be inappreciable.

PROP. *When a material system in motion is acted on by impulsive forces, none of which are supposed external to the system, vis viva is lost or gained according as the impulse is of the nature of collision or explosion. When the system is perfectly elastic the vis viva is the same before and after the impulse.*

514. Let P be the resultant of the internal forces acting on m, and $\alpha\beta\gamma$ the angles its direction makes with the axes. Then the forces

$$mV_1 + P\cos\alpha - mv_1, \quad mV_2 + P\cos\beta - mv_2,$$
$$mV_3 + P\cos\gamma - mv_3$$

acting on m parallel to the axes and similar forces acting on all the other particles of the system will satisfy the conditions of equilibrium (Art. 224, 226), supposing the bodies to be *rigid*.

Hence by the Principle of Virtual Velocities (Art. 70.)

$$\Sigma . m\left\{\left(V_1 + \frac{P}{m}\cos\alpha - v_1\right)\delta x\right.$$
$$\left. + \left(V_2 + \frac{P}{m}\cos\beta - v_2\right)\delta y + \left(V_3 + \frac{P}{m}\cos\gamma - v_3\right)\delta z\right\} = 0,$$

δx, δy, δz being any small spaces geometrically described by m parallel to the axes in a manner consistent with the connexion of the parts of the system one with another at the time t.

We shall first observe, that P will disappear from the equation above. For if P be the action between two bodies of the system which touch each other in the point xyz, then δx, δy, δz will be the virtual velocities of the point of contact with respect to P acting on one, and $-\delta x$, $-\delta y$, $-\delta z$ those with respect to the other body; and consequently in the above expression when a term of the form $P \cos a \delta x$ occurs we find also $-P\cos a\delta x$; and therefore P disappears, and the equation becomes

$$\Sigma . m \{(V_1 - v_1)\delta x + (V_2 - v_2)\delta y + (V_3 - v_3)\delta z\} = 0 \ldots (1).$$

In applying this equation to calculate the motion of a system suddenly acted on by impulsive forces we must make a few important remarks. When a body yields or expands, the centres of its particles approach or recede from each other; but, during the action of the impulsive forces, the spaces through which they yield or recede are so extremely small, that we wholly neglect them; but this is not the case with their velocities, for although the change of distance of the centres of the particles during the impulse is indefinitely small, yet this change divided by the time elapsed during the impulse will give a difference of velocities which is not necessarily insensible. In consequence of this, when two bodies come into collision the particles in contact do not move with the same velocity at the first instant of the contact, but after all compression ceases and the restitution of figure has not begun to take place, at this instant and at this instant alone, do the particles in contact move with the same velocity. Again, when two bodies are acted upon by impulsive forces of the nature of internal explosion, the particles in contact move with the same velocity at the first instant of the action of the forces, but at every other instant of the action they move with different velocities.

Now δx, δy, δz may be any small spaces provided they be consistent with the connexion of the parts of the system one with another at the time of the impulse; this connexion remains the same during the impulse, because all small spaces

described in that time are insensible. Wherefore we must not give to these quantities such arbitrary values as will imply, that the particles in contact at the point (xyz) separate, or penetrate each other, or (in other words) move with opposite or unequal velocities.

If, then, the impulsive forces be of the nature of *collision* and xyz be co-ordinates of the point of contact, the initial velocities of the particles in contact will not be the same, but after the collision ceases they will have the same effective velocities v_1, v_2, v_3. Hence, in this case we may put

$$\delta x = v_1 \delta t, \quad \delta y = v_2 \delta t, \quad \delta z = v_3 \delta t,$$

since these virtual velocities are consistent with the connexion of the parts of the system one with another, and they imply that the particles in contact remain in contact when the principle of virtual velocities is applied to the system in its imaginary state of equilibrium.

If the impulsive forces be of the nature of internal *explosion*, then it will easily be seen, after what has been said, that we may put

$$\delta x = V_1 \delta t, \quad \delta y = V_2 \delta t, \quad \delta z = V_3 \delta t,$$

but we must not put the other values for δx, δy, δz.

I. Suppose the impulse is of the nature of collision, the bodies being inelastic.

Then substituting for δx, δy, δz in equation (1), and putting v for the resulting velocity of m

$$\Sigma . mv^2 = \Sigma . m\{V_1 v_1 + V_2 v_2 + V_3 v_3\};$$

$$\therefore \Sigma . mv^2 = \Sigma . mV^2 - \Sigma . m\{(V_1 - v_1)^2 + (V_2 - v_2)^2 + (V_3 - v_3)^2\},$$

and, since the last term of this is essentially negative, we see that vis viva is lost during the collision.

II. Suppose the impulse is of the nature of internal explosion.

By substituting in (1) the values of δx, δy, δz above specified we have

$$\Sigma . mV^2 = \Sigma . m\{V_1 v_1 + V_2 v_2 + V_3 v_3\};$$

$$\therefore \Sigma . mv^2 = \Sigma . mV^2 + \Sigma . m\{(V_1 - v_1)^2 + (V_2 - v_2)^2 + (V_3 - v_3)^2\},$$

and consequently vis viva is gained during the separation.

III. Suppose that the impulse is in part of the nature of collision, and in part of the nature of explosion.

In this case we must combine the cases already mentioned. When, for instance, the bodies are perfectly elastic the impulsive forces which act during the collision are the same exactly as those which act during the separation of the bodies; it follows, by examining the above expressions, that the vis viva lost during the collision is exactly regained during the separation, and that the state of the system is consequently unaffected by the whole impulse.

515. The degradation of rocks and the consequent action of collision which is incessantly taking place in large portions of matter on the surface of the Earth, the unceasing action of waves on the sea-shore and the collision of the waters of the ocean upon the solid nucleus of the Earth, and other like causes are continually causing a loss of vis viva in the Earth's mass, and if allowed to act without any compensating phenomena would in the course of time produce a sensible effect in the length of the day: but on the other hand the explosions of volcanoes are compensating causes. Also the downward motion of rivers, the descent of vapour and cloud in the form of rain, the descent of boulders and avalanches, and various other causes, all tend to remove large portions of matter nearer to the Earth's centre, and would in the course of time produce a sensible decrease in the length of the day, since we have seen (Art. 499.) that the vis viva of the Earth is constant, if we neglect the attraction of the Sun, Moon, and planets and consider only the action of finite forces. But the ascent of vapour by evaporation, and the effect of earthquakes and volcanoes in removing masses of matter to a greater distance from the centre have an opposite effect. On the whole all these causes balance each other, since observations have shewn that the length of the day has been invariable for many ages, Art. 461.

CHAPTER XIV.

PROBLEMS ON THE MOTION OF RIGID BODIES, AND ANY MATERIAL SYSTEM.

516. We shall commence this Chapter with some observations upon the best methods of solving dynamical problems and the application of the general principles proved in Chapters XI. and XIII. in facilitating their solution.

To determine the motion of a rigid body in space we have six differential equations of the second order: these contain the three co-ordinates to the centre of gravity and the three angles of position of the principal axes of the body; see Arts. 429, 446, 447; and 501, 504. These are the only relations that can exist among the *mechanical quantities* (Art. 144).

If all the forces and other quantities involved in these equations be known, then we have sufficient equations for solving the problem, and determining the position of the body at every instant.

If, however, the equations involve unknown forces, or unknown geometrical quantities (as angular and linear measures), or both, then there must exist as many more equations as there are of these unknown quantities; and, moreover, these relations must be among the geometrical quantities, since the six equations of motion, as we have mentioned, are the only mechanical relations that can exist.

Suppose that from the nature of the problem we have, involved in the six mechanical equations, one unknown force, and n unknown geometrical quantities besides those necessarily contained in the six equations: then we must have $n + 1$ additional equations among the $n + 6$ geometrical quantities: when we have obtained these we have enough equations for

the solution of the problem. To determine then, these $n+6$ geometrical quantities, and therefore to determine the position of the body, we have already $n+1$ equations free from unknown mechanical quantities, and must therefore obtain five more such equations; these are found by eliminating the unknown force from the six equations of motion. In the same way we should proceed if there were two, three, or more unknown forces. The equations which we obtain among the unknown geometrical quantities must be integrated, that we may have these quantities in terms of the time.

Now the principles of the conservation of motion of the centre of gravity, and the conservation of areas, and the principle of vis viva demonstrated in Chapters XI. and XIII. are the first integrals of the equations of motion, under peculiar suppositions as to the nature of the forces which act upon the system. If, then, in any proposed problem, one or more of these principles apply, we may write them down as the integrals of our equations, and so diminish the labour of elimination and integration. If the integrals involved in these principles cannot be obtained in consequence of their involving unknown forces, the principles, though they may be true in these cases, will nevertheless not answer our purpose.

To find the unknown forces we must obtain their values from the equations of motion in terms of the geometrical quantities and their differential coefficients; and since these are supposed to be found the forces will be known also; see Problem 16.

If we find, after all the equations are written down, that there are more unknown quantities than equations, then the general solution of the problem is indeterminate; though it does not necessarily follow that all the unknown quantities are indeterminate (as in Art. 438). If we find more equations than unknown quantities it follows, that the general solution of the problem is impossible unless certain relations among the known quantities are fulfilled, the number of these relations being equal to the number by which the equations exceed the unknown quantities. Nevertheless, as in the last, some of the unknown quantities may be independent of these conditions.

We shall illustrate the remarks which we have made upon the solution of problems by referring to Art. 436. Here we have a case of motion in parallel planes, and therefore only three equations of motion: but these contain the unknown forces F and R, besides the three necessary geometrical measures of position x, y, θ: hence two more equations must exist, and these among x, y, θ; these are equations (4) (5) in that Article; and we require only one more relation connecting x, y, θ; this we have by elimination from the equations of motion. But since the point of application of the forces F and R has no velocity, (for the body at each instant is revolving about that point as an instantaneous centre of rotation), F and R will not appear in the equation of vis viva of Art. 481. Cor. 1. Hence this equation gives the integral we require; and we have (by Art. 482),

$$M\left\{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right\}+Mk^2\frac{d\theta^2}{dt^2}=2\Sigma\,.\,m\int g\,dy',$$

y' being the vertical ordinate of m,

$$=2g\Sigma\,.\,m\,(y'+\text{constant})=2Mg\,(y+\text{constant}),\ \text{(Art. 413)}.$$

This is the equation obtained in Art. 436, by elimination.

We shall now give some Problems; we shall solve a few, or give hints to guide to their solution.

PROB. 1. A sphere rolls down an inclined plane; required to determine the motion: (fig. 111.)

Since the motion of the centre of gravity is evidently parallel to the fixed inclined plane we shall measure its distance (x) from the point C, which it occupies at the commencement of the motion, E the point which was then in contact at B with the plane, $\angle EOD=\theta$, P the pressure of the plane, F the friction acting upwards, a the radius of the sphere: α the angle the plane makes with the horizon. Then for the motion of the centre of gravity (Art. 429.) and the motion of rotation about the centre of gravity (Art. 430, 431),

$$\frac{d^2x}{dt^2}=g\sin\alpha-\frac{F}{M}\ldots\ldots(1),\quad \frac{d^2\theta}{dt^2}=\frac{Fa}{Mk^2}\ldots\ldots(2);$$

three unknown quantities; we want another equation, this is

$$x = a\theta \;\ldots\ldots\; (3).$$

Since the object is to determine the position of the body at a given time we must obtain an equation between x and θ in addition to (3); this is obtained by eliminating F from (1) and (2): we thus have

$$\frac{d^2 x}{dt^2} = g \sin \alpha - \frac{k^2}{a}\frac{d^2\theta}{dt^2} = g \sin \alpha - \frac{k^2}{a^2}\frac{d^2 x}{dt^2} \text{ by (3)};$$

$$\therefore \frac{d^2 x}{dt^2} = \frac{a^2 g \sin \alpha}{a^2 + k^2}, \quad \frac{dx}{dt} = \frac{a^2 g \sin \alpha \,.\, t}{a^2 + k^2}, \text{ constant} = 0,$$

$$x = \frac{a^2 g \sin \alpha}{a^2 + k^2}\,\frac{t^2}{2}, \quad \theta = \frac{a g \sin \alpha}{a^2 + k^2}\,\frac{t^2}{2}.$$

We might have used the principle of vis viva to obtain the second equation between x and θ, since F does not occur in the equation of vis viva, because the velocity of its point of application equals zero: but the elimination was so simple that we preferred this method.

Cor. If the body partly roll and partly slide, then F is constant, and must be determined by experiment. Hence equation (3) does not hold, and in short, (1) (2) are sufficient for determining the motion in this case.

Prob. 2. Suppose the inclined plane or wedge, on which the cylinder rests is capable of moving on a smooth horizontal plane: to determine the motion of the sphere and wedge: (fig. 111.)

The quantities as before, except that x and y are the horizontal and vertical co-ordinates of O measured from A in the horizontal plane: x' the horizontal co-ordinate to the point K of the wedge, M and M' the masses of the sphere and wedge. Then for the sphere we have the three equations (Art. 429, 430, 431)

$$\frac{d^2 x}{dt^2} = \frac{F \cos \alpha - P \sin \alpha}{M} \;\ldots\ldots\; (1),$$

$$\frac{d^2y}{dt^2} = -g + \frac{F \sin \alpha + P \cos \alpha}{M} \text{............} (2),$$

$$\frac{d^2\theta}{dt^2} = \frac{Fa}{Mk^2} \text{.....................} (3).$$

For the wedge $\dfrac{d^2x'}{dt^2} = \dfrac{P \sin \alpha - F \cos \alpha}{M'}$ ...... (4).

Here are six unknown quantities; there must therefore be two relations connecting x, y, θ, x': these are

$x' - x - a \sin \alpha = a\theta \cos \alpha$ ...... (5), $y = h - a\theta \sin \alpha$ ...... (6),

h the initial value of y.

We must obtain two relations connecting x, y, θ, x' from (1) (2) (3) (4). But since there are no forces acting externally to the system of the sphere and wedge parallel to the horizon, there is a conservation of the horizontal motion of the centre of gravity (Art. 474): hence

$$M\frac{dx}{dt} + M'\frac{dx'}{dt} = \text{constant} = 0,$$

in our case, since there is supposed to be no initial velocity;

$$\therefore\ Mx + M'x' = \text{constant} = 0 \text{......} (7),$$

if we properly choose the origin A.

Again, the principle of vis viva gives us an integral; for although the point of application of P and F does move in this case, yet the velocity of this point will have exactly opposite signs relatively to P and F acting on the sphere, and P and F acting on the wedge, and therefore P and F will not occur in the equation of vis viva: in short, they are *internal* forces;

$$\therefore\ M\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right\} + M'\frac{dx'^2}{dt^2} = 2\Sigma . m\int -g\,dy'$$
$$= 2gM(h-y) \text{......} (8).$$

The equations (5) (6) (7) (8) will determine the position.

By (5) (7) (6) $\dfrac{dx'}{dt} = \dfrac{Ma\cos\alpha}{M+M'}\dfrac{d\theta}{dt} = -\dfrac{M}{M'}\dfrac{dx}{dt}$, $\dfrac{dy}{dt} = -a\sin\alpha\,\dfrac{d\theta}{dt}$;

$$\therefore \text{ by (8) } \frac{d\theta^2}{dt^2}\left\{a^2+k^2-\frac{M}{M+M'}a^2\cos^2\alpha\right\}=2ag\theta\sin\alpha;$$

$$\therefore \theta=\left\{a^2+k^2-\frac{M}{M+M'}a^2\cos^2\alpha\right\}^{-1}.\tfrac{1}{4}ag\sin\alpha\, t^2,$$

this coincides with the result of Prob. 1. if we put $M'=\infty$.

The equation to the path of O is, by (5) (6) (7)

$$y=h+a\sin\alpha\tan\alpha+\frac{M+M'}{M'}x\tan\alpha;$$

therefore the path of the centre of the sphere is a straight line.

Prob. 3. A groove in the form of a cycloid with its vertex downwards and base horizontal is cut in a solid vertical board: determine the motion of a ball moving along it, while the board itself is capable of moving freely along a smooth horizontal plane; and find the curve which the ball describes in space.

Let x, y be the horizontal and vertical co-ordinates to the ball at time t: x' the co-ordinate to the vertex of the cycloid supposed to be in the horizontal plane: s the distance of the ball from the vertex measured along the groove, R the mutual pressure of the ball and groove, M and m the masses of the board and ball: then the equations of the problem are

$$\frac{d^2x}{dt^2}=-\frac{R}{m}\frac{dy}{ds}\ \ldots\ldots (1),\qquad \frac{d^2y}{dt^2}=-g+\frac{R}{m}\frac{d(x-x')}{ds}\ \ldots\ldots (2),$$

$$\frac{d^2x'}{dt^2}=\frac{R}{M}\frac{dy}{ds}\ \ldots\ldots (3),$$

$$x-x'=a\operatorname{vers}^{-1}\frac{y}{a}+\sqrt{2ay-y^2}\ \ldots\ldots\ldots\ldots\ldots (4).$$

The principle of conservation of the horizontal motion of the centre of gravity and the principle of vis viva both apply: they give

$$Mx'+mx=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

by choosing the origin under the initial position of the centre of gravity, and also

$$M\frac{dx'^2}{dt^2}+m\left\{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right\}=2mg(h-y)\ldots\ldots(6),$$

h the initial value of y.

By (4) (6) the equation to the path of the ball in space is

$$\frac{M+m}{M}x=a\,\text{vers}^{-1}\frac{y}{a}+\sqrt{2ay-y^2}\ \ldots\ldots\ldots\ (7).$$

By (7) (5) $\dfrac{dx}{dt}=\dfrac{M}{M+m}\sqrt{\dfrac{2a-y}{y}}\dfrac{dy}{dt}=-\dfrac{M}{m}\dfrac{dx'}{dt}$;

$\therefore$ by (6) $m\dfrac{dy^2}{dt^2}\left\{\dfrac{M}{M+m}\,\dfrac{2a-y}{y}+1\right\}=2mg(h-y)$,

from which the motion must be calculated.

Prob. 4. Two equal balls are fixed to the end of a rod without weight; the rod is connected at its middle point with a fixed vertical axis, so as to allow the rod to move in a vertical plane passing through the axis, and to revolve with the axis in a horizontal direction: required the motion of the balls.

Let M be the mass of each ball: $2a$ the length of the rod: ω the angular velocity of projection about the fixed point: β the angle the direction of projection makes with the plane in which a is measured: ϕ the angle the body has described in the plane of original motion at the time t; θ the angle described in a plane perpendicular to that plane. We shall not write down the equations of motion in this case, but resort immediately to the principles of the conservation of areas (which applies since the resultant of the weights of the balls always passes through the fixed point,) and the conservation of vis viva. The principle of areas gives for the plane in which ϕ is measured

$$2Ma^2\frac{d\phi}{dt}=\text{const.}\quad\therefore\frac{d\phi}{dt}=\omega.$$

The conservation of vis viva gives

$$2M\left\{a^2\frac{d\theta^2}{dt^2}+a^2\frac{d\phi^2}{dt^2}\right\}=\text{const.}=2Ma^2\omega^2,\quad\therefore\frac{d\theta}{dt}=0,$$

and therefore the bodies move uniformly in the original plane of projection.

PROB. 5. A rod acts by one extremity with a uniform force in the direction of its length on the fly-wheel-crank of a steam-engine, the other extremity moving in a straight line passing through the centre of the fly-wheel, and a uniform resistance is to be overcome by the fly-wheel. Find the velocity of the wheel at any time: and find the relation between the forces when they are so adjusted that after half a revolution the velocity may be unaltered.

PROB. 6. A uniform lever ACB, of which the arms AC and BC are at right angles to each other, rests in equilibrium when AC is inclined at a^0 to the horizontal: shew that if AC be raised to a horizontal position (C being fixed) it will fall through an angle $= 2a$.

PROB. 7. Given the radii and masses of the wheels in Atwood's Machine (Art. 212.) and the constant friction on the fixed axles of the wheels A and B (fig. 75); shew that the accelerating force of P and Q when in motion is much less affected by the friction at A and B, than if the wheel C turned about a fixed axle.

PROB. 8. A horizontal wheel moves freely about a vertical axis through its centre; a string of definite length is wrapt round its circumference, and passing through a ring has fixed to it a weight which falls by gravity; determine the whole motion.

PROB. 9. A hemisphere rests on a horizontal plane with a string fastened to its edge, which, passing over a pully, supports a weight: when the string is cut find the motion of the hemisphere.

PROB. 10. A beam is drawn from a horizontal to a vertical position about one extremity, which is fixed, by means of a string, which is attached to the other extremity of the beam, and, after passing over a pully placed above the fixed extremity at a height equal to the length of the

beam, is attached to a falling body; determine the velocity of the beam when vertical.

PROB. 11. A beam is projected perpendicularly upwards, and has a rotatory motion round its centre of gravity in a vertical plane; it is observed at a given altitude to be in one of its horizontal positions, and to be then ascending with a given velocity; after this it performs a given number of revolutions and strikes the ground at a given angle: find the angular velocity.

PROB. 12. An inflexible straight rod is set in motion round a vertical axis passing through one extremity, about which it is capable of revolving freely in an horizontal plane: determine the motion of a ring sliding freely along it; and prove that the whole vis viva of the system is constant.

PROB. 13. A body is placed on a smooth wedge, which rests upon a smooth horizontal plane, and the wedge is acted on by a horizontal and constant force f in a vertical plane perpendicular to the inclined plane of the wedge: determine the motion: and find f when the body is at rest on the plane.

PROB. 14. A semi-cylinder rests with its plane surface on the ground, on which it is capable of moving freely; shew that a sphere rolling down its curved surface will describe an ellipse.

PROB. 15. Determine the equations of motion of two heavy particles connected by an inflexible rod without weight, one of which moves on a surface of revolution and the other is constrained to move in the axis of the surface, this axis being vertical. Find the velocity of the particle on the surface when the other continues stationary.

PROB. 16. A cylinder rolls down a fixed quadrant; find where the cylinder will leave the quadrant.

The pressure must be calculated; the body leaves at the instant that this is zero.

PROB. 17. A sphere revolves round an axis touching its surface, find the length of the simple isochronous pendulum.

PROB. 18. A sector of a circle revolves round an axis perpendicular to its plane, and passing through the centre of the circle; find the angle of the sector when the length

of the isochronous simple pendulum equals three fourths of the length of the arc.

Prob. 19. For what axes of suspension is the time of a small oscillation of a solid body an absolute minimum? Take the case of an ellipsoid.

Prob. 20. A rough vertical cylinder, capable of revolving about a concentric but smooth and smaller cylinder fixed as an axis, rests upon a rough horizontal plane, on every point of which the pressure is the same: determine the force applied by a string wrapt round the cylinder which will just make it move. If the force be greater than this, determine the motion.

Prob. 21. A cylinder is made to rotate about its axis, and is then suddenly placed in contact with a rough horizontal plane with its axis parallel to the plane; the force of friction is of *finite* intensity, and is not sufficiently great to prevent the line of contact of the cylinder from *sliding* on the plane at the *beginning* of the motion: required to determine the motion, and to shew how long the cylinder will continue to combine a sliding motion with its rolling motion.

Let ω be the angular velocity communicated to the cylinder before the contact: the friction does not affect this velocity at the first instant of the contact because the force of friction by hypothesis is of finite intensity: θ the angle described in the time t by that radius of the cylinder that was in contact with the plane at first: a the radius: x the distance of the axis of the cylinder at time t from its initial position: F the friction. Then F is constant and has its greatest value so long as the cylinder slides as well as rolls; in which case the equations of motion are

$$\frac{d^2\theta}{dt^2} = -\frac{Fa}{Mk^2} \ldots\ldots (1), \quad \frac{d^2x}{dt^2} = \frac{F}{M} \ldots\ldots (2),$$

but when the sliding motion ceases, if F' be the friction, which is then not necessarily constant, we must put F' for F in (1) and (2), and add the equation

$$x = a\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

I. So long as the sliding motion continues we have, then,

$$\frac{d\theta}{dt} = \omega - \frac{Fat}{Mk^2}, \quad \frac{dx}{dt} = \frac{Ft}{M},$$

$$\theta = \omega t - \frac{Fat^2}{2Mk^2}, \quad x = \frac{Ft^2}{2M}.$$

The sliding motion ceases when the motion of translation and the motion of rotation give exactly equal and opposite motions to the point of contact, or when

$$\frac{dx}{dt} = a\frac{d\theta}{dt}, \text{ and } \therefore t = \frac{M}{F}\frac{k^2 a\omega}{a^2+k^2}, \text{ and } \frac{d\theta}{dt} = \frac{k^2\omega}{a^2+k^2}.$$

II. After the sliding motion ceases, the equations of motion are

$$\frac{d^2\theta}{dt^2} = -\frac{F'a}{Mk^2}\ldots(1), \quad \frac{d^2x}{dt^2} = \frac{F'}{M}\ldots(2), \quad x = a\theta\ldots(3).$$

$$\text{By (1) (2) } \frac{k^2}{a}\frac{d^2\theta}{dt^2} + \frac{d^2x}{dt^2} = 0, \quad \therefore \text{by (3) } \frac{d^2\theta}{dt^2} = 0;$$

$$\therefore F' = 0 \text{ and } \frac{d\theta}{dt} = \text{constant} = \frac{k^2\omega}{a^2+k^2}.$$

From this we learn that the friction has gradually reduced the angular motion of the body till the velocity of the point of contact is zero, and after that the body proceeds to move uniformly, and to rotate uniformly, and no friction is called into play.

Prob. 22. A rough body lies upon a rough board, and this lies upon a smooth horizontal plane, the friction between the body and board is of finite intensity (as in the last Problem): the board is projected with a given velocity, determine the motion of the body and board.

Prob. 23. A sphere is fastened by an inflexible rod to a horizontal axis fixed at two points: when the sphere revolves about the axis required the pressure on the two fixed points.

If we use the notation of Art. 438, and put $\gamma = 90^0$, $\gamma' = 90^0$ and therefore $\beta = 90^0 - a$, then, the axis of rotation being the axis of z and the plane in which the centre of the sphere moves the plane of xy and the axis of x drawn vertically downwards, the moving forces $m(g + yf + x\omega^2)$, $m(y\omega^2 - xf)$, 0 acting on m parallel to the axes of x, y, z and similar forces acting on all the other particles of the system, together with the pressures of the fixed points ought to be in equilibrium at the time t. Hence

$$P\cos a + P'\cos a' + \Sigma \,.\, m\,(g + yf + x\omega^2) = 0,$$

$$P\sin a + P'\sin a' + \Sigma \,.\, m\,(y\omega^2 - xf) = 0,$$

$$-P\sin a \,.\, a - P'\sin a' \,.\, a' - \Sigma \,.\, m\,(gz + yzf + xz\omega^2) = 0;$$

$$P\cos a \,.\, a + P'\cos a' \,.\, a' + \Sigma \,.\, m\,(yz\omega^2 - xzf) = 0,$$

$$\Sigma \,.\, m\,(xy\omega^2 - x^2f - gy - y^2f - xy\omega^2) = 0.$$

Let $\bar{x}\bar{y}0$ be the co-ordinates to the centre of gravity; then, since every axis through the centre of a sphere is a principal axis, we have

$$\Sigma \,.\, m\,(y - \bar{y})z = 0,\ \Sigma \,.\, m\,(x - \bar{x})\,z = 0,\ \Sigma \,.\, m(y - \bar{y})\,(x - \bar{x}) = 0:$$

$$\therefore \Sigma \,.\, myz = \bar{y}\Sigma \,.\, mz = 0,\quad \Sigma \,.\, mxz = 0,\quad \Sigma \,.\, myx = M\bar{x}\bar{y}.$$

Hence the equations become

$$P\cos a + P'\cos a' + M\,(g + f\bar{y} + \omega^2\bar{x}) = 0,$$

$$P\sin a + P'\sin a' + M\,(\omega^2\bar{y} - f\bar{x}) = 0;$$

$$Pa\sin a + P'a'\sin a' = 0,\ Pa\cos a + P'a'\cos a' = 0,\ Mfk_{,}^2 + Mg\bar{y} = 0.$$

From which P, P', a, a' may be found.

PROB. 24. If a body revolve round an axis by the action of a force in a direction always perpendicular to the plane passing through the axis and the centre of gravity of the body, determine the point of application of this force, so that there may be no pressure on the axis, except in the plane to which the direction of the force is perpendicular: and shew whether this point varies in position.

PROB. 25. A hemisphere oscillates about a horizontal axis, which coincides with a diameter of the base; shew that if the

base be at first vertical, the ratio of the greatest pressure on the axis to the weight of the hemisphere $= 109 \div 64$.

PROB. 26. A sphere, when acted on separately by three forces, revolves round three diameters inclined at the same angle to each other, and with the same angular velocity, determine the angular velocity and the new axis of rotation when the three forces are applied at the same instant.

PROB. 27. A sphere, attracted to a given centre of force varying as the distance, is projected with a given velocity along a plane passing through that centre, friction being such as to destroy all sliding: prove that the path will be an ellipse, and find the velocity that the ellipse may be a circle.

PROB. 28. A cone of given form, and supported at G its centre of gravity, has a motion communicated to it round an axis through G perpendicular to the line joining G with a point in the circumference of the base, and in a plane passing through this point and the axis of the cone: determine the position of the *invariable plane*.

PROB. 29. Explain how the rotation of a hoop preserves it from falling.

PROB. 30. A solid of revolution moveable about its centre of gravity G, which is fixed and is the origin, and having its axis inclined to the axis of x at an angle ϕ, has an angular motion impressed upon it about a line between these two axes, and inclined to the axis of the figure at an angle θ, such that $k^2 \tan\phi = k'^2 \tan\theta$, where k and k' are the radii of gyration about its axis and a line perpendicular to the axis through G: prove that the axis of the solid will constantly preserve the same inclination to the axis of x, and will revolve uniformly about it; and the solid will at the same time revolve uniformly about its own axis, which is in motion.

PROB. 31. If the Moon moved with one of its principal axes always perpendicular to its orbit, shew that the angular force of the Earth to produce rotation about that axis would nearly $= \dfrac{3\mu \sin 2\theta}{2r^3} \dfrac{A-B}{C}$; μ, r being the Earth's mass and distance from the Moon, A, B, C the principal moments of inertia of the lunar spheroid, and θ the angular distance, at the

Moon's centre, of the Earth from one of the principal axes in the plane of the orbit.

PROB. 32. Two spheres with their centres connected by a hot wire are projected into space: required to find their motion, supposing that we know the law of contraction of the wire, which arises from the radiation of the heat.

To simplify the calculation we shall suppose, that one sphere M' is originally at rest, and the other M projected with a vel. V at right angles to the wire: r the length of the wire at the time t; $r = a$ when $t = 0$; the origin at the original position of the centre of gravity: the original position of the wire the axis of x: T the tension of the wire at the time t. Then the equations of motion are,

$$\frac{d^2 x}{dt^2} = -\frac{T}{M}\frac{x - x'}{r} \ldots\ldots (1), \quad \frac{d^2 x'}{dt^2} = \frac{T}{M'}\frac{x - x'}{r} \ldots\ldots (3),$$

$$\frac{d^2 y}{dt^2} = -\frac{T}{M}\frac{y - y'}{r} \ldots\ldots (2), \quad \frac{d^2 y'}{dt^2} = \frac{T}{M'}\frac{y - y'}{r} \ldots\ldots (4),$$

$$r^2 = (x' - x)^2 + (y' - y)^2 \ldots\ldots (5).$$

The conservation of the motion of the centre of gravity gives

$$Mx + M'x' = 0, \quad My + M'y' = MV.t.$$

The Conservation of Areas (Art. 479) gives

$$M\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) + M'\left(x'\frac{dy'}{dt} - y'\frac{dx'}{dt}\right) = \frac{MM'}{M + M'}aV.$$

These three equations with (5) will determine the motion. The Principle of Vis Viva does not apply in this case: Art. 481, because T is an explicit function of t. The equations give

$$x = \frac{M'}{M+M'}r\cos\phi, \quad y = \frac{M'}{M+M'}r\sin\phi - \frac{MVt}{M+M'} \; : \; x' = \&c.$$

$$\text{where } \phi = \int \frac{Va\,dt}{r^2}.$$

$$\text{Also } T = \frac{MM'}{M + M'}\left(\frac{V^2a^2}{r^3} - \frac{d^2 r}{dt^2}\right).$$

PROB. 33. Find the centre of spontaneous rotation when a body is acted on by a *finite* force: see Art. 507.

PROB. 34. In what point and in what direction could a single blow be given to a body like the Earth, so as instantaneously to give it the velocities of progression and rotation, which the Earth has.

PROB. 35. Two imperfectly elastic and smooth balls impinge upon each other, the motion of their centres taking place in the same plane: required their velocities after impact.

Since the balls are perfectly smooth there will be no rotatory motion produced by the impulse. We must first consider the motion till the compression ceases. Let P be the mutual pressure acting in the common normal at the points in contact; V, V' the velocities at the commencement of the contact: $\alpha\alpha'$ the angles their directions make with the line passing through their centres when the contact takes place; v, v' the velocities of the balls at the instant the compression ceases: θ, θ' the angles their directions make with the axes.

Therefore

$$MV\cos\alpha - P - Mv\cos\theta = 0\ldots(1),\quad MV\sin\alpha - Mv\sin\theta = 0\ldots(2),$$

$$M'V'\cos\alpha' + P - M'v'\cos\theta' = 0\ldots(3),\quad M'V'\sin\alpha' - M'v'\sin\theta' = 0\ldots(4);$$

and since the points in contact move with the same velocities in the direction of the normal at the instant the compression ceases, then

$$v\cos\theta - v'\cos\theta' = 0 \ldots\ldots\ldots\ldots\ldots (5).$$

Again, during the restitution of figure the mutual pressure $= Pe$: and if u and u' be the velocities after the restitution of figure is complete, and ϕ and ϕ' the angles of the directions of motion, the equations of motion are

$$Mv\cos\theta - Pe - Mu\cos\phi = 0\ldots(6),\quad Mv\sin\theta - Mu\sin\phi = 0\ldots(7)$$

$$M'v'\cos\theta' + Pe - M'u'\cos\phi' = 0\ldots(8),\quad M'v'\sin\theta' - M'u'\sin\phi' = 0\ldots(9)$$

In these nine equations are involved nine unknown quantities, $P, v, v', \theta, \theta', u, u', \phi, \phi'$: we have to determine u, u', ϕ, ϕ'.

By (1) (2) (5) $(M + M')v\cos\theta = MV\cos\alpha + M'V'\cos\alpha'$;

$$\text{eliminating } e \text{ by (1) (6) } (M + M')\, u \cos\phi$$
$$= (M + M')(1 + e)\, v \cos\theta - (M + M')\, V \cos\alpha$$
$$= M'V' \cos\alpha' - M'V\cos\alpha + e\,(MV\cos\alpha + M'V'\cos\alpha'),$$
$$\text{by (2) (7) } u \sin\phi = V \sin\alpha\,;$$

from which u and ϕ may easily be determined: in the same way u' and ϕ' may be determined.

PROB. 36. A rough ball A is placed on a rough horizontal table, and another rough ball B lying on the table is struck in a direction not passing through the centre of gravity, but so as to cause B to strike A: find the motion after impact, the bodies being inelastic.

PROB. 37. Supposing, in the last Problem, that the friction of the Table is so slight as not altogether to prevent sliding, find the conditions that B may move through its original place of rest.

The four following Problems are intended to illustrate the action of springs in removing the shock arising from the sudden collision of bodies.

PROB. 38. A ball A moves along a smooth horizontal plane with a velocity V, and sets in motion another ball B, equal to A and originally at rest, by impinging upon a spring CD (fig. 112), which is fastened to B at the point D: the inertia of the spring is neglected, and we suppose the force of the spring to vary as the space through which it is compressed: required to determine the motion of the balls.

Let O be the place of A, the centre of the first ball, at the time of first contact with the spring: $OA = x$, $CD = z$ ($= b$ when the spring is not compressed), $OB = x'$: then the force exerted by the spring on the balls at the time t varies as $b - z$; let it $= c^2(b - z)$. The equations of motion are

$$\frac{d^2x}{dt^2} = -c^2(b - z)\ldots\ldots(1), \quad \frac{d^2x'}{dt^2} = c^2(b - z)\ldots\ldots(2);$$

$$\text{also } x' - x = 2a + z\ldots\ldots(3), \quad a \text{ the radius of the balls,}$$

three equations and three unknown quantities x, x', z.

Differentiating (3) and subtracting (1) (2) we have

$$\frac{d^2x}{dt^2} = 2c^2(b - x);$$

$$\therefore \frac{dx^2}{dt^2} = \text{constant} - 2c^2(b - x)^2 = V^2 - 2c^2(b - x)^2,$$

since the point C of the spring (having no inertia) instantly acquires the velocity (V) of the body A at the first contact;

$$\therefore x = b - \frac{V}{c\sqrt{2}} \sin(c\sqrt{2}t + C') = b - \frac{V}{c\sqrt{2}} \sin c\sqrt{2}t \ldots (4).$$

This shews that the greatest compression of the spring is equal to $\frac{V}{c\sqrt{2}}$, and that the time of compression $= \frac{\pi}{2c\sqrt{2}}$; after an equal duration of time the spring is restored to its original form, since x equals b when $c\sqrt{2}t = \pi$.

$$\text{By (1) (4)} \ \frac{d^2x}{dt^2} = -\frac{Vc}{\sqrt{2}} \sin c\sqrt{2}t;$$

$$\therefore \frac{dx}{dt} = \text{const.} + \tfrac{1}{2}V \cos c\sqrt{2}t = \tfrac{1}{2}V(1 + \cos c\sqrt{2}t);$$

$$\therefore x = \frac{V}{2}\left(t + \frac{1}{c\sqrt{2}} \sin c\sqrt{2}t\right);$$

$$\therefore \text{by (3)} \ x' = 2a + b + \frac{V}{2}\left(t - \frac{1}{c\sqrt{2}} \sin c\sqrt{2}t\right);$$

$$\therefore \frac{dx'}{dt} = \tfrac{1}{2}V(1 - \cos c\sqrt{2}t).$$

From these equations we readily gather the following results. The ball A stops when $\frac{dx}{dt} = 0$, or $t = \frac{\pi}{c\sqrt{2}}$; but at this instant (as we have shewn) the spring has returned to its natural form, consequently the contact between A and the spring at this instant ceases, and A remains permanently at rest: the

space through which A has moved during the action of the spring $= \dfrac{\pi V}{2c\sqrt{2}}$. The velocity of B is zero when the spring begins to act and is V when its action ceases, and with this velocity B henceforth moves uniformly along the plane. Hence A *gradually* imparts all its velocity to B: and the duration of time which this communication of velocity occupies is $\dfrac{\pi}{c\sqrt{2}}$. If the elastic force of the spring be of very great intensity, as is the case with the forces put into play by the impact of hard balls of ivory, c is very great, and the duration of collision is exceedingly short.

PROB. 39. Suppose that A and B (in the last Problem) are of the same size, but of different masses M and M', and that they move with the velocities V and V' before they come in contact: required to determine the motion.

PROB. 40. Suppose, in the last Problem, that the force exerted by the spring during the restitution of its figure is less than the force exerted during the compression in the ratio $e : 1$, but that a complete restitution of figure takes place: required to determine the motion.

PROB. 41. A heavy carriage (represented in fig. 113.) rests upon a spring B, and is also held in its place by two springs pressing at C and C': the carriage moves uniformly along a horizontal plane with a velocity V, and its four wheels (two only of which are seen in the figure) which are all of the same size suddenly impinge at the same instant on four very small and equal pointed obstacles, and move over them; the force exerted by each spring is supposed to vary as the extent of displacement of its point of contact with the carriage, and the springs are supposed to be bent into such a form that for all small displacements of the body of the carriage the resultant of their pressures always passes through the centre of gravity of the body and so prevents rotatory motion: required the motion of the centre of gravity.

We shall merely give the results with a few of the steps of the calculation.

Let the dotted lines in the figure represent the state of things at the instant of the impact: and the dark lines the state of things at a time t after the impact: a the radius of each wheel. In consequence of the elasticity of the springs (which is supposed perfect) the body of the carriage is not rigidly connected with the wheels and axle-tree, and therefore the body can produce no instantaneous effect upon the velocity when the impulse takes place. By the impact of the wheels on the obstacles the parts of the springs which are connected with the axes of the wheels and the axle-tree have their motion suddenly changed, this causes the springs to assume new forms and in that way the forces are brought into action which gradually change the motion of the body.

Let $x'y'$ be the horizontal and vertical spaces described by the point B of the axle-tree in the time t; x and y the spaces described by the centre of gravity of the body: let c^2 and e^2 be constants which depend upon the elasticity of the springs at C and C' and that at B; we neglect the downward effect of the spring B on the axle-tree but consider only the dead weight to act at B: let θ be the angle which the spoke of each wheel, which passes through the obstacle, makes with the vertical at the time t; $\theta = a$ when $t = 0$; $M' =$ mass of each wheel: ω the angular velocity of each wheel after impact.

The equations of motion are, for each wheel,

$$\frac{d^2\theta}{dt^2} = \frac{(M' + \frac{1}{4}M)\,ga\sin\theta}{M'k^2} = \frac{\sin\theta}{n^2}, \text{ suppose} \ldots\ldots (1),$$

for the motion of G

$$\frac{d^2x}{dt^2} = -c^2(x - x') \ldots\ldots (2), \quad \frac{d^2y}{dt^2} = -g + e^2(y' - y) \ldots\ldots (3),$$

and x', y', θ are connected by the equations

$$x' = a(\sin\alpha - \sin\theta) \ldots\ldots (4), \quad y' = a(\cos\theta - \cos\alpha) \ldots\ldots (5).$$

These equations are sufficient to solve the problem: but they cannot be integrated unless α and θ (and therefore the obstacles) be supposed small: we shall neglect powers higher

than the second. After reducing the equations their integrals will be found to be of the forms

$$x = A \sin (ct + B) + aa + h\varepsilon^{-\frac{t}{n}} + k\varepsilon^{\frac{t}{n}},$$

$$y = C \sin (et + D) - l - m\varepsilon^{-\frac{2t}{n}} - p\varepsilon^{\frac{2t}{n}},$$

A, B, C, D, being arbitrary constants to be determined by the initial circumstances, and h, k, l, m, p being written for known quantities. After determining these it will be found that when $t = 0$, $\dfrac{dy}{dx} = 0$ and therefore the original rectilinear path of G is a tangent at the first point of the curve described by G; also the values of the constants will shew that the velocity is V at first, and *gradually* decreases: hence there is no *jerk* in the body of the carriage.

We might in the same way obtain the circumstances after the wheels again come to the horizontal plane.

Prob. 42. A rectangular parallelopiped slides down a smooth inclined plane and meets a fixed obstacle: determine the impulse and the subsequent motion.

Prob. 43. A beam is projected in any manner along a smooth horizontal plane and impinges upon a fixed obstacle: determine the impulse.

Prob. 44. A beam is fixed at one extremity, what vertical force applied instantaneously at the other will throw it exactly vertical?

Prob. 45. A rectangular parallelopiped revolves about one of its edges, which rests in a horizontal groove, and impinges on a fixed line parallel to the groove and in the same horizontal plane with it: find the angle through which the parallelopiped must fall so that it may be just on the point of revolving about the fixed line as a new axis, all *sliding* being prevented by friction.

Prob. 46. A beam is placed with one end against a smooth vertical wall and the other on a smooth horizontal plane so as to move in a vertical plane when left to the action of gravity; the horizontal plane does not extend to the wall, but is terminated by a straight edge parallel to the wall: find the distance of this edge from the wall that

the beam may just be prevented from revolving about the edge and finally falling beneath the horizontal plane.

PROB. 47. At what point must a given uniform circular body be struck by a force perpendicular to its plane, that in the first instant of the body's motion one extremity of a given diameter may remain at rest?

PROB. 48. A beam falls from a vertical position by revolving about one extremity which rests on a rough horizontal plane, and impinges on a vertical post: determine the magnitude and direction of the impulse on the post and on the horizontal plane at the immoveable extremity of the beam.

PROB. 49. In the last Problem determine the initial circumstances that the beam may just fall over the post.

PROB. 50. If a rough ball be projected against a rough beam on a smooth horizontal plane, determine the centre of spontaneous rotation.

PROB. 51. An elastic beam falls upon a horizontal fixed line: determine the motion.

PROB. 52. A beam, moveable about a fixed horizontal axis at a given altitude above a horizontal plane, falls through a given angle: determine the point at which a given sphere should be opposed to its impact, that it may be projected to the greatest possible distance on the horizontal plane, the beam being in its vertical position at the instant of impact.

PROB. 53. A hoop rolling down an inclined plane suddenly comes in contact with a horizontal plane; find the change in angular velocity.

PROB. 54. In lowering a bale of goods from the higher story of a warehouse by means of a given crane, the whole weight of the bale is allowed to wind off the rope freely from the axle, and when the bale is half way down, the handle of the crane suddenly flies off; determine the motion.

PROB. 55. Explain the use of fly-wheels in machinery, and if a fly-wheel of given dimensions and weight move with a given angular velocity what force applied perpendicularly at a given point of one of the spokes of the wheel will instantaneously destroy the motion.

Prob. 56. A perfectly flexible chain has one end fixed to a peg, which is at the extremity and highest point of a quadrant of a circle of which the plane is vertical, and all the chain is collected at that point; it will just cover the quadrant, and being suffered to descend freely, it is required to find the stress upon the peg at the end of the motion.

Prob. 57. A groove is cut in a horizontal table in the form of a regular hexagon and an inelastic ball is projected with a given velocity along one of its sides, find the velocity with which it will successively describe each of the other sides of the figure.

Prob. 58. A perfectly elastic solid of revolution, turning about its axis at a given rate, impinges on a hard smooth plane: if before impact the centre of gravity move perpendicular to the plane with a velocity V, determine the motion of rotation after impact, and prove that the centre of gravity will move in the same direction with a velocity $\frac{p^2 - k^2}{p^2 + k^2} V$, where p is the perpendicular from the centre of gravity on the normal at the point of impact, and k is the radius of gyration round an axis through the centre of gravity perpendicular to the axis of the solid.

Prob. 59. A solid sphere is placed in a hollow sphere, which rests on a smooth horizontal plane; determine the small oscillations, when they are slightly disturbed from the state of rest.

Prob. 60. Prove, by means of the principle of least action, that the orbit a body describes about a centre of force varying inversely as the square of the distance is a conic section.

Prob. 61. Prove the laws of reflexion and refraction of light by the principle of least action, on the supposition that light consists of luminous particles moving uniformly in the same homogeneous medium, but with different velocities in different media.

Prob. 62. A bullet is fired into a thick board hanging from a fixed horizontal axis about which it is capable of revolving; the board has a sheet of iron on its back to prevent the bullet from passing through: a ribbon is fastened to the bottom of the board and runs through a ring touching

the bottom of the board in its position of rest: shew how to compare the velocities of bullets by observing the lengths of ribbon drawn out by the motion of the board.

This is Robins' Ballistic Pendulum.

Prob. 63. The weights suspended from a wheel and axle are in motion, the wheel and axle move about a fixed axis very nearly fitting into the cylindrical aperture concentric with the axle, so as to suffer only one point to be in contact: determine the position of this point when friction is considered and when it is neglected.

HYDROSTATICS.

CHAPTER I.

DEFINITIONS AND PRINCIPLES.

517. HYDROSTATICS is the science, which treats of the equilibrium of fluid bodies.

By a fluid body we mean an assemblage of material molecules, which yield without resistance to the slightest effort, which we can make to separate them. No fluids with which we meet in nature are exactly of this character; but they approach more or less to *perfect fluidity*, as it is termed. The adherence that exists among the molecules of most fluids, known by the term *viscosity*, prevents the separation of its parts by the slightest forces; but, in this work, we shall suppose the fluids to be perfect; for, if we except certain fluids the viscosity of which is considerable and of which we do not treat, the laws of equilibrium which we deduce are true, without sensible error.

Fluids are divided into *liquids* and *aeriform fluids:* they are also said to be *incompressible* and *elastic*. In truth, all fluids are more or less elastic: but some, as water, are compressible, and that but slightly, only when subjected to enormous pressure. Aeriform fluids may be divided into *vapours*, and *permanent fluids*, such as air and gasses. A given space will not contain above a determinate quantity of vapour under a given temperature: so that if the vapour attain the limit of temperature, and we diminish ever so little the space or the temperature, a portion of the vapour becomes liquified. Experiment proves that the maximum quantity of vapour is the

same, when the temperature is the same, in a space void of air, and in a space filled with air more or less dilated or compressed. On the other hand, air and the gasses do not under any circumstances become liquids; though it is the opinion of some that this would be the case if sufficient force of compression could be exerted, or if the temperature could be reduced to a sufficient degree of cold.

518. In order that a fluid mass may be in equilibrium the forces acting upon each molecule must satisfy the equations of equilibrium of a particle deduced in Art. 56. But we are ignorant of the forms of the particles of fluids, and of the forces by which they influence each other: and consequently we fall upon the same difficulty as in the Article cited, and must therefore seek for some principle to serve us the office, which that of the transmission of force did, when we considered the equilibrium of a rigid body.

The following Principle is proved by experiment: *that any pressure communicated to a fluid mass in equilibrium is equally transmitted through the whole fluid in every direction.*

This is perfectly independent of the form of the molecules: we may therefore suppose the fluid to consist of an indefinite number of small parallelopipeds formed by planes drawn very near to each other and parallel to the co-ordinate planes.

The following experiment will illustrate the principle of the uniform distribution of pressure: we take the case of an incompressible fluid. Let fig. 114. represent a closed box full of water; A, B two vertical pistons of equal transverse section, neatly fitted into the upper face of the box, and made to move as freely as possible. It is found by experiment, that if a weight be placed on A an equal weight must be placed on B to preserve the equilibrium, shewing that the pressure of the weight A is propagated through the fluid to the under surface of B, and equably too, since it requires an equal weight on B to balance this pressure. Again if a piston equal to A or B be fitted at C, it is found that to preserve equilibrium, a pressure must be exerted at C: and when the equilibrium exists if additional pressure act at C, or a weight be placed on A or B, an

equal force must act on the other two pistons to preserve equilibrium: which shews, that the pressure upon any portion of the fluid is transmitted through the fluid, and acts equally upon every equal area upon which it presses.

When a fluid is placed in a vessel, since the only direction, in which any small portion of the surface of the vessel can sustain pressure, is that of the normal to the surface, it follows, that the pressure of a fluid on any small element of a surface, containing the fluid or immersed in it, is perpendicular to the surface. The magnitude of the pressure at any point in the fluid is, as yet, an unknown quantity: it depends upon the position of the point and upon the forces which act upon the fluid; and therefore, in the general case, varies as the position of the point varies, the forces being given. We measure the pressure at any point in terms of the force exerted on a plane of a unit of area, and acted on at every point by a pressure equal to the pressure to be measured. Thus let p be the pressure upon a unit of area acted on uniformly by a pressure equal to that at the point (xyz), then $p\omega$ is the pressure sustained by an indefinitely small portion (ω) of this surface. The coefficient p is a function of the three co-ordinates (xyz), and is termed *the pressure referred to a unit of surface.*

519. The pressure of the same elastic fluid against the sides of a containing vessel is proportional to the density of the fluid, when the temperature is constant: $p = k\rho$, where ρ is the density at the point (xyz) and k a constant when the temperature is constant: when this is not the case k is a function of the temperature.

PROP. *To find the pressure at any point in the interior of a fluid mass in equilibrium.*

520. Let the fluid be referred to three rectangular co-ordinate planes, and let xyz be the co-ordinates to the angular point, nearest the origin, of a small parallelopiped of which the sides are δx, δy, δz, drawn parallel to the co-ordinates. Let ρ be the density of the fluid at the point (xyz), X, Y, Z the accelerating forces acting upon the fluid, and p the pressure at the same point estimated as explained in Art. 518. Now the

parallelopiped is held in equilibrium by the forces acting on its particles and the pressure of the surrounding fluid on its sides. Let us suppose that X, Y, Z and ρ are the same for each particle of the parallelopiped. Now the pressure at a point $(x, y, z + \delta z)$ is $p + \dfrac{dp}{dz}\delta z$, and therefore if we suppose the pressure to act uniformly over the faces of the parallelopiped parallel to the plane of xy, then the forces acting parallel to the axis of z are a pressure $\dfrac{dp}{dz}\delta z\delta x\delta y$ acting from the origin and $\rho Z\delta x\delta y\delta z$ acting towards the origin: and since the parallelopiped is in equilibrium the sum of the forces parallel to each axis must vanish. Hence

$$\frac{dp}{dz}\delta z = \rho Z\delta z, \text{ also } \frac{dp}{dy}\delta y = \rho Y\delta y, \text{ and } \frac{dp}{dx}\delta x = \rho X\delta x;$$

$$\therefore \delta p, \text{ or } \frac{dp}{dx}\delta x + \frac{dp}{dy}\delta y + \frac{dp}{dz}\delta z, = \rho(X\delta x + Y\delta y + Z\delta z.)$$

Now let us diminish the parallelopiped indefinitely, in which case the supposition we have made respecting the uniformity of the density of the parallelopiped and of the action of the forces will be true; and

$$dp = \rho(Xdx + Ydy + Zdz).$$

From this equation we shall obtain the conditions of equilibrium of a fluid mass.

PROP. *To find the conditions of equilibrium of a mass of fluid acted on by any forces.*

521. The first member of the equation of last Article is a perfect differential, and therefore, when the equilibrium is possible, the second member must be so too: hence the forces must satisfy the condition, that $\rho(Xdx + Ydy + Zdz)$ shall be a perfect differential.

If this condition be fulfilled, then equilibrium will subsist in the interior of the fluid provided the surface be of a proper form: for since at the surface $p = 0$ it is easily seen, that

XYZ must satisfy the additional condition, that for all points at the surface $Xdx + Ydy + Zdz = 0$; or, in other words, this must be the differential equation to the surface.

This latter condition amounts to the same as saying, that the resultant of the forces acting on any particle of fluid at the surface must be in the direction of the normal. For the cosines of the angles which the normal at a point (xyz) of a surface makes with the axes are

$$-V\frac{dz}{dx}, \quad -V\frac{dz}{dy}, \quad V; \text{ where } \frac{1}{V} = \sqrt{1 + \frac{dz^2}{dx^2} + \frac{dz^2}{dy^2}}:$$

or in this case,

$$\frac{X}{\sqrt{X^2+Y^2+Z^2}}, \quad \frac{Y}{\sqrt{X^2+Y^2+Z^2}}, \quad \frac{Z}{\sqrt{X^2+Y^2+Z^2}}.$$

But these are the cosines of the angles which the direction of the resultant of X, Y, Z makes with the axes. Hence our remark is correct.

When $\rho\,(Xdx + Ydy + Zdz)$ is a perfect differential, then

$$\frac{d.\rho X}{dy} = \frac{d.\rho Y}{dx}, \quad \frac{d.\rho Z}{dx} = \frac{d.\rho X}{dz}, \quad \frac{d.\rho Y}{dz} = \frac{d.\rho Z}{dy}.$$

Performing the differentiations we have

$$\rho\left\{\frac{dX}{dy} - \frac{dY}{dx}\right\} = Y\frac{d\rho}{dx} - X\frac{d\rho}{dy},$$

$$\rho\left\{\frac{dZ}{dx} - \frac{dX}{dz}\right\} = X\frac{d\rho}{dz} - Z\frac{d\rho}{dx},$$

$$\rho\left\{\frac{dY}{dz} - \frac{dZ}{dy}\right\} = Z\frac{d\rho}{dy} - Y\frac{d\rho}{dz}.$$

Multiply these respectively by Z, Y, X and add, then

$$Z\left(\frac{dX}{dy} - \frac{dY}{dx}\right) + Y\left(\frac{dZ}{dx} - \frac{dX}{dz}\right) + X\left(\frac{dY}{dz} - \frac{dZ}{dy}\right) = 0.$$

This is independent of the density and furnishes a partial criterion before we make use of the others.

PROP. *To prove that for central forces tending to fixed centres* $X dx + Y dy + Z dz$ *is a perfect differential.*

522. Let a, b, c be the co-ordinates to any one of the centres of force, P the corresponding force: then the resolved parts of its action on the particle (xyz) are

$$P\frac{x-a}{r}, \quad P\frac{y-b}{r}, \quad P\frac{z-c}{r},$$

$$\text{where } r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2.$$

Hence, by resolving all the forces in this manner and adding together those parallel to the same axes,

$$X = \Sigma . P\frac{x-a}{r}, \quad Y = \Sigma . P\frac{y-b}{r}, \quad Z = \Sigma . P\frac{z-c}{r},$$

$$\therefore Xdx + Ydy + Zdz =$$

$$\Sigma . \frac{P}{r}\{(x-a)\,dx + (y-b)\,dy + (z-c)\,dz\} = \Sigma . Pdr.$$

But P is a function of r, and therefore Pdr and the similar expressions for the other forces are perfect differentials. Hence $Xdx + Ydy + Zdz$ is a perfect differential.

All the forces in nature are central forces. It follows then that equilibrium will always be possible provided the surface of the fluid be of the proper form.

PROP. *To prove that the particles of a mass of fluid when in equilibrium are so arranged, that the same surfaces are surfaces of equal pressure, of equal density, and of equal temperature.*

523. By Art. 520 we have

$$dp = \rho\,(Xdx + Ydy + Zdz), = \rho d\phi, \text{ suppose},$$

where ϕ is a certain function of xyz, (Art. 522.)

But dp is a perfect differential, therefore $\rho d\phi$ is so also: hence ρ is a function of ϕ, and then p is a function of ϕ and therefore of ρ; consequently any two of p, ρ, ϕ can be expressed in terms of the third: hence all values of xyz, which make one of them constant, make all the others constant also.

If the fluid be elastic the density varies directly as the pressure, as was first shewn by the experiments of Mariotte. Let $p = k\rho$; then it is found, that if the temperature vary, k is a function of the temperature.

$$\text{Now } dp = \rho d\phi, \quad \therefore \frac{dp}{p} = \frac{d\phi}{k}, \text{ and } p = \Pi \epsilon^{\int \frac{d\phi}{k}},$$

where Π is an arbitrary constant: this expression shews, that if there be equilibrium, k must be a function of ϕ, and therefore p and ρ and the temperature must be functions of ϕ: and consequently any three of these quantities can be expressed in terms of the fourth, and therefore when one is constant the others are also. Hence the truth of the Proposition is manifest.

COR. 1. It is evident, then, that the atmosphere can never be in equilibrium; for the sun heats unequally masses of air which are equally pressed by the superincumbent air; consequently the layers of equal pressure, density, and temperature do not coincide; a condition necessary when there is equilibrium.

COR. 2. If, then, we integrate the equation $Xdx + Ydy + Zdz = 0$ and give to the arbitrary constant introduced by the integration as many particular values as we please, the determinate equations which we thus obtain belong to surfaces, each of which has the equation $Xdx + Ydy + Zdz = 0$ for its differential equation, and, in consequence, possesses the property, that it is equally pressed on every part, and cuts at right angles in every point the direction of the resultant of the forces X, Y, Z. These internal surfaces are called *surfaces de niveau*. If we make the constant vary by indefinitely small degrees we divide the mass of

the fluid into an indefinitely great number of thin shells; these are called *couches de niveau.*

The value of the constant which corresponds to the external surface is determined from knowing the volume of the fluid.

CHAPTER II.

FIGURE OF THE EARTH.

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524. We have already remarked that the heavenly bodies are nearly spherical in their figure. It is found, however, by measuring degrees of latitude in places near the poles and equator, that the figure of the Earth approaches more nearly to a spheroid than a sphere. Also experiments made with pendulums lead to the same conclusion. The polar radius is found to be between 13 and 14 miles shorter than the equatorial radius. Now the altitude of the highest mountains is not greater than five miles above the level of the ocean. It follows from this that the form of the sea cannot be spherical, but must partake more or less of the spheroidal form of the land. It becomes, then, a matter of especial interest to ascertain whether the ocean covering the solid nucleus of the Earth and the nucleus itself would according to the theory of gravitation assume a spheroidal form. The calculation is one of great difficulty, and indeed would be impracticable did we not know that the figure does not differ greatly from a sphere. As a first approximation we shall enquire whether a homogeneous fluid mass revolving about a fixed axis with a uniform angular velocity will assume a spheroidal figure. It is nevertheless *à priori* highly improbable that the density of the Earth should be homogeneous, since the weight of the superincumbent mass must be sufficient of itself to produce a great condensation of the matter in the interior of the Earth.

Prop. *A homogeneous mass of fluid in the form of a spheroid is revolving with a uniform angular velocity about its axis: required to determine whether the equilibrium of the fluid is possible.*

525. Let a and c be the axes of the spheroid, c being that about which it revolves: also let $c^2 = a^2(1-e^2)$. Now the forces which act upon the particle (xyz) are the centrifugal force and the attractions of the spheroid parallel to the axes, and these latter are given in Art. 158, as follows:

$$\frac{2\pi\rho}{e^3}\left\{\sqrt{1-e^2}\sin^{-1}e - e(1-e^2)\right\}x,$$

$$\frac{2\pi\rho}{e^3}\left\{\sqrt{1-e^2}\sin^{-1}e - e(1-e^2)\right\}y,$$

$$\frac{4\pi\rho}{e^3}\left\{e - \sqrt{1-e^2}\sin^{-1}e\right\}z.$$

Let these be represented by Ax, Ay, Cz. Let w be the angular velocity of rotation, then $w^2\sqrt{x^2+y^2}$ is the centrifugal force of the particle (xyz), and the resolved parts of it parallel to the axes are w^2x, w^2y, 0.

Hence $X = -(A-w^2)x$, $Y = -(A-w^2)y$, $Z = -Cz$.

Now these satisfy the criterions given in Art. 521; hence, so far, the equilibrium is possible. Then

$$\frac{1}{\rho}dp = Xdx + Ydy + Zdz$$

$$= -(A-w^2)(xdx+ydy) + Czdz;$$

$$\therefore \frac{2p}{\rho} = \text{const.} - (A-w^2)(x^2+y^2) + Cz^2,$$

and at the surface $p = 0$;

$$\text{and } \therefore \frac{A-w^2}{C}(x^2+y^2) + z^2 = \text{const.}$$

hence the surface is a spheroid, and therefore the equilibrium is possible, and the eccentricity is given by the equation

$$\frac{A-w^2}{C} = \frac{c^2}{a^2} = 1-e^2,$$

$$\text{or } \frac{w^2}{2\pi\rho} = \frac{\sqrt{1-e^2}}{e^3}\sin^{-1}e - 3\,\frac{1-e^2}{e^2} + \frac{2}{e^3}(1-e^2)^{\frac{3}{2}}\sin^{-1}e,$$

$$\text{or } \frac{w^2}{2\pi\rho} + \frac{3(1-e^2)}{e^2} - \frac{(3-2e^2)\sqrt{1-e^2}}{e^3}\sin^{-1}e = 0.$$

$$\text{Now } \frac{\text{centrifugal force at equator}}{\text{gravity at equator}} = \frac{w^2 a}{\frac{4}{3}\pi\rho a - w^2 a},$$

and observation proves that this equals $\frac{1}{289}$;

$$\therefore\ \frac{4\pi\rho}{3w^2} = 290;\quad \therefore\ \frac{w^2}{2\pi\rho} = \tfrac{1}{435}.$$

Now by expanding in powers of e, and neglecting powers higher than the second (since we know that the figure of the Earth is not far from spherical), we have

$$\sin^{-1}e = \int_0^e \frac{de}{\sqrt{1-e^2}} = \int_0^e \left\{1 + \tfrac{1}{2}e^2 + \frac{1.3}{2.4}e^4 + \ldots\right\}de$$

$$= e + \tfrac{1}{2}\frac{e^3}{3} + \frac{1.3}{2.4}\frac{e^5}{5} + \ldots\ldots$$

$$\sqrt{1-e^2} = 1 - \tfrac{1}{2}e^2 - \frac{1.1}{2.4}e^4 + \ldots\ldots;$$

$$\therefore\ \tfrac{1}{435} = \left(\frac{3}{e^2} - 2\right)\left(1 - \tfrac{1}{2}e^2 - \frac{1.1}{2.4}e^4\right)\left(1 + \tfrac{1}{2}\frac{e^2}{3} + \frac{1.3}{2.4}\frac{e^4}{5}\right) - \frac{3}{e^2} + 3$$

$$= \left(\frac{3}{e^2} - 2\right)\left(1 - \tfrac{1}{3}e^2 - \tfrac{2}{15}e^4\right) - \frac{3}{e^2} + 3 = \tfrac{4}{15}e^2;$$

$$\therefore\ e^2 = \tfrac{1}{116}.$$

If ϵ be the ellipticity, then

$$\epsilon = \frac{a-c}{a} = 1 - \sqrt{1-e^2} = \frac{e^2}{2} = \tfrac{1}{232}.$$

This result is so much greater than that obtained by measuring the arcs of a meridian, which gives $\epsilon = \frac{1}{306}$, and which agrees very nearly with the results arrived at by other

independent processes, that we are led to the conclusion that the mass of the Earth is not homogeneous.

526. Another value of e, nearly $= 1$, also satisfies the equation: but this evidently does not give the figure of any of the heavenly bodies, for none of them are very elliptical.

There are no other possible values of e (except -1 which of course we reject): this may be shewn by putting the first side of the equation for calculating e equal to y, and tracing the curve considering e the variable abscissa: then those values of e which make the curve cut the axis of abscissas are the values we are seeking. It is found that the curve cuts the axis in only two points on the positive side of the origin.

Prop. *To shew that there is a limiting angular velocity beyond which the equilibrium is impossible.*

527. The equation of Art. 525 shews, that as we alter the value of w the value of e will also vary. To find the greatest value of w we must put $\frac{dw}{de} = 0$: this gives after some long numerical calculations,

$$\text{time of rotation} = 0.10090 \text{ day, and } \epsilon = \tfrac{17197}{27197}.$$

528. Since there are two forms of equilibrium it might perhaps be imagined, that the form of the equilibrium of a homogeneous fluid mass is unstable; and that when a fluid mass is set in rotation it will be indifferent which form it will ultimately assume. Supposing that there is a cohesive force among the particles, which is the case with all known fluids, the mass would, after revolving for a greater or less time, attain a rotatory motion comprised within the limits of equilibrium and maintain itself in that state, which of the two states being apparently doubtful. But Laplace has shewn (*Méc. Céles.* Liv. III. §. 21), that for a given primitive impulse there is but one form of equilibrium. In fact, it will easily be seen, that for a given value of the angular velocity w the vis viva of two equal masses, so different in their form as to have e small and $e = 1$ nearly, must be very different.

529. Since the ellipticity of the Earth, deduced on the supposition of its being a homogeneous mass, is greater than that given by geodetic measurements and experiments made with pendulums, we are constrained to reject the hypothesis of the Earth's homogeneity, and shall proceed to calculate its ellipticity on the supposition of its being heterogeneous. But before proceeding to this we shall investigate the following Proposition, since we shall find it of use hereafter.

PROP. *To calculate the ellipticity of a mass of fluid revolving about a fixed axis and attracted by a force wholly residing in the centre of the fluid and varying inversely as the square of the distance.*

530. Let M be the mass of the fluid, the other quantities as before;

$$\therefore\ X = -\frac{Mx}{r^3} + w^2 x, \quad Y = -\frac{My}{r^3} + w^2 y, \quad Z = -\frac{Mz}{r^3}.$$

Then the equation $Xdx + Ydy + Zdz = 0$ becomes

$$\frac{M}{r^3}(xdx + ydy + zdz) - w^2(xdx + ydy) = 0;$$

$$\therefore\ \frac{M}{r^2}dr - \frac{w^2}{2}d(x^2 + y^2) = 0,$$

$$\therefore\ \frac{M}{r} + \frac{w^2}{2}(x^2 + y^2) = \text{constant} = C;$$

let the ratio of the centrifugal force at the equator to the gravity at the equator be represented by am, where a is a very small numerical quantity, and $am = \frac{1}{289}$, then

$$am = \frac{w^2 a}{\dfrac{M}{a^2} - w^2 a}, \quad \therefore\ w^2 = \frac{M}{a^3}\frac{am}{1 + am};$$

$$\therefore\ \frac{1}{r} \text{ or } \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{C}{M} - \frac{1}{2a^3}\frac{am}{1 + am}(x^2 + y^2),$$

when $x = 0$ and $y = 0$ then $z = c$: when $z = 0$ then $x^2 + y^2 = a^2$,

$$\therefore \frac{1}{c} = \frac{C}{M}; \quad \frac{1}{a}\left(1 + \frac{am}{2 + 2am}\right) = \frac{C}{M};$$

$$\therefore \frac{c}{a} = \frac{2 + 2am}{2 + 3am}, \quad \epsilon = 1 - \frac{c}{a} = \frac{am}{2 + 3am}.$$

$$\text{Now } am = \tfrac{1}{289}, \quad \therefore \epsilon = \tfrac{1}{581}.$$

We shall find this result of use in a following part of this Chapter. The value of ϵ is far too small on this hypothesis, as the value of ϵ was too large on the hypothesis of every particle attracting. We shall, therefore, pass on to consider the form of a heterogeneous mass.

PROP. *To find the equation of equilibrium of a heterogeneous mass of fluid consisting of strata each nearly spherical, and revolving about a fixed axis passing through the centre of gravity with a uniform angular velocity.*

531. The general equation of equilibrium of fluids deduced in Art. 520, gives

$$\int \frac{dp}{\rho'} = \int (X dx + Y dy + Z dz),$$

ρ' being the density of the mass at the point (xyz); p the pressure at that point; and X, Y, Z the sums of the resolved parts parallel to the axes of the forces acting on the particle at (xyz). These forces are the attraction of the mass itself, and the centrifugal force.

Let $V =$ the sum of the quotients formed by dividing each particle of the body by its distance from the attracted point: then $-\frac{dV}{dx}$, $-\frac{dV}{dy}$, $-\frac{dV}{dz}$ are the attractions parallel to the axes of the body on the particle at (xyz).

Let w be the angular velocity of the mass, r the distance of (xyz) from the origin of co-ordinates, μ the sine of the latitude of this particle. Then the resolved parts parallel to the axes of x and y of the centrifugal force of this particle are $w^2 x$, $w^2 y$: the axis of revolution being the axis of z.

It is usual to express the angular velocity in terms of the ratio of the centrifugal force at the equator to the equatorial gravity, a fraction which observation shews to be $\frac{1}{289}$: we shall call it am, a being used throughout the calculation as a very small numerical quantity, of which the square may be neglected.

Then $am = \dfrac{w^2 \mathrm{a}^3}{Mass}$, a being the equatorial radius: to calculate the mass in this fraction we suppose its strata spherical, because of the smallness of the numerator. Hence mass $= 4\pi \int_0^{\mathrm{a}} \rho' a'^2 da'$, a' being the radius of any stratum. Let $3\int_0^{\mathrm{a}} \rho' a'^2 da' = \phi(\mathrm{a})$;

$$\therefore \; am = \frac{3 w^2 \mathrm{a}^3}{4\pi\phi(\mathrm{a})}, \quad \therefore \; w^2 = \frac{4\pi}{3} am \frac{\phi(\mathrm{a})}{\mathrm{a}^3}.$$

Hence, then,
$$X = \frac{dV}{dx} + \frac{4\pi}{3} am \frac{\phi(\mathrm{a})}{\mathrm{a}^3} x,$$

$$Y = \frac{dV}{dy} + \frac{4\pi}{3} am \frac{\phi(\mathrm{a})}{\mathrm{a}^3} y,$$

$$Z = \frac{dV}{dz};$$

$$\therefore \int \frac{dp}{\rho'} = V + \frac{2\pi}{3} am \frac{\phi(\mathrm{a})}{\mathrm{a}^3} (1 - \mu^2) r^2.$$

If we suppose the mass partly solid this equation determines the figure of the strata of the fluid part. We shall, however, consider the whole fluid, or else suppose that the solid and fluid parts follow the same law of density in passing from the circumference to the centre.

Now p is a function of ρ': and $\int \frac{dp}{\rho'}$ is a function of ρ' and is therefore constant for the outer surface and for every *level* surface, (or *surface de niveau*, Art. 523, Cor. 2) or stratum.

Hence the general equation to the strata is

$$\text{constant} = V + \frac{2\pi}{3}\alpha m \frac{\phi(\text{a})}{\text{a}^3}(1-\mu^2) r^2$$

$$= V + \frac{4\pi}{9}\alpha m \frac{\phi(\text{a})}{\text{a}^3} r^2 + \frac{2\pi}{3}\alpha m \frac{\phi(\text{a})}{\text{a}^3}(\tfrac{1}{3}-\mu^2) r^2$$

this arrangement being made, because each of the last terms, as they now stand, satisfies the partial differential equation of Laplace's coefficients; they are of the orders 0 and 2 respectively. (See Art. 170).

Now by Art. 186, we have

$$V = \frac{4\pi}{r}\int_0^a \rho'\left\{a'^2 + \alpha\frac{d}{da'}\left(\frac{a'^4}{3r}Y_1' + \ldots + \frac{a'^{i+3}}{(2i+1)r^i}Y_i' + \ldots\right)\right\}da'$$

$$+ 4\pi\int_a^{\text{a}} \rho'\left\{a' + \alpha\frac{d}{da'}\left(\frac{a'r}{3}Y_1' + \ldots + \frac{r^i}{(2i+1)a'^{i-2}}Y_i' + \ldots\right)\right\}da'.$$

In this put $r = a(1 + \alpha Y_1 + \ldots + \alpha Y_i + \ldots)$ and $\int_0^a 3\rho' a'^2 da' = \phi(a)$ as before. Then substitute this value of V in the equation to the strata, and equate terms of the order i.

The constant parts when equated give

$$\int\frac{dp}{\rho'} = \frac{4\pi}{3}\frac{\phi(a)}{a} + 4\pi\int_a^{\text{a}}\rho' a' da' + \frac{4\pi}{9}\alpha m a^2\frac{\phi(\text{a})}{\text{a}^3},$$

and the terms of the order i (See Art. 181. Cor.) give, after dividing by $-4\pi\alpha$,

$$\frac{\phi(a)}{3a}Y_i - \frac{1}{(2i+1)a^{i+1}}\int_0^a \rho'\frac{d}{da'}(a'^{i+3}Y_i')\,da'$$

$$-\frac{a^i}{2i+1}\int_a^{\text{a}}\rho'\frac{d}{da'}\left(\frac{Y_i'}{a'^{i-2}}\right)da' = 0,$$

except when $i = 2$, in which case the second side is

$$\frac{m}{6}\frac{a^2\phi(\text{a})}{\text{a}^3}(\tfrac{1}{3}-\mu^2).$$

By this equation Y_i is to be calculated, and then the form of the stratum of which the mean radius is a is known by the equation $r = a(1 + \alpha Y_1 + \ldots + \alpha Y_i + \ldots)$.

PROP. *To prove that* $Y_i = 0$, *excepting the case of* $i = 0$.

532. Since Y_i and ρ are functions of a, they may be expanded into ascending series of the form

$$Y_i = Wa^s + \ldots\ldots, \qquad \rho = D + D'a^n + \ldots\ldots$$

where D is the density at the centre of the Earth, and is, as well as W, D', independent of a: $s, n, \ldots\ldots$ must not be negative, otherwise Y_i and ρ would be infinite at the centre. Now when these series, and the corresponding series obtained by putting a' for a, are substituted in the equation of last Article, and the first side arranged in powers of a, the various coefficients ought to vanish; excepting the case of $i = 2$, because then the second side is not zero We shall therefore substitute these series, and search for values of W and s, which satisfy this condition.

$$\text{Now } \phi(a) = 3\int_0^a \rho' a'^2 da' = Da^3 + \frac{3D'}{n+3} a^{n+3} + \ldots\ldots$$

and after two easy integrations the equation of last Art. becomes

$$WD\left\{\tfrac{1}{3}a^{s+2} - \frac{a^{s-i+2}}{2i+1}a^i\right\} + \ldots\ldots = 0.$$

No value of s will cause these terms to vanish separately, nor together. The only apparent case is when $i = 1$, for then by putting $s = i - 2$, the part in the brackets vanishes: but in this particular case $s = -1$, and is negative, and therefore inadmissible.

Hence the only way of satisfying the condition is by putting $W = 0$: this shews that Y_i has no first term, that is, that it has no term at all, and is therefore zero.

PROP. *To find the value of* Y_2, *and to prove that the strata are all spheroidal, with a common axis.*

533. The equation for calculating Y_2 is found in Art. 531, and is as follows:

$$\frac{\phi(a)}{3a} Y_2 - \frac{1}{5a^3} \int_0^a \rho' \frac{d}{da'} (a'^5 Y_2') \, da' - \frac{a^2}{5} \int_a^a \rho' \frac{dY_2'}{da'} da'$$

$$= \frac{m}{6} \frac{a^2 \phi(a)}{a^3} (\tfrac{1}{3} - \mu^2).$$

Suppose Y_2 (and consequently Y_2') is expanded in a series of powers of $\frac{1}{3} - \mu^2$ with indeterminate coefficients to be ascertained by the condition, that they shall satisfy the above equation: these coefficients will be functions of a only: it is evident that Y_2 is not a function of ω. It is evident from the form of the equation, that Y_2 consists of only one term, that involving the simple power of $\frac{1}{3} - \mu^2$. Let then $a Y_2 = \epsilon (\frac{1}{3} - \mu^2)$; and the value of the radius vector, in consequence of this and the last Article, becomes

$$r = a \{1 + \epsilon (\tfrac{1}{3} - \mu^2)\}, \quad \mu = \sin (\text{latitude}) = \sin l$$

$$= a (1 - \tfrac{2}{3}\epsilon) \{1 + \epsilon \cos^2 l\}, \quad \text{since } \epsilon \text{ is small.}$$

This is the equation to a spheroid from the centre, ϵ being the ellipticity. The axis-minor coincides with the axis of revolution of the whole mass. Hence *the strata are concentric spheroids, the axes of which coincide with the axis of revolution of the whole mass.*

PROP. *To obtain an equation for calculating the ellipticity of the strata, when the law of density is known.*

534. Substitute $\epsilon (\frac{1}{3} - \mu^2)$ for $a Y_2$ in the equation of last Art. and we have, after dividing by $\frac{1}{3} - \mu^2$,

$$\frac{\phi(a)}{3a} \epsilon - \frac{1}{5a^3} \int_0^a \rho' \frac{d}{da'} (a'^5 \epsilon') \, da' - \frac{a^2}{5} \int_a^a \rho' \frac{d\epsilon'}{da'} da'$$

$$= \frac{am}{6} \frac{a^2 \phi(a)}{a^3} \quad \ldots\ldots\ldots\ldots (1),$$

this is the equation for calculating ϵ.

We may reduce it to a differential equation thus: divide both sides by a^2, and differentiate with respect to a, observing the remark in the note;* then multiply by a^6 and differentiate again; and divide by the coefficient of $\frac{d^2\epsilon}{da^2}$; and we have

$$\frac{d^2\epsilon}{da^2} + \frac{6\rho a^2}{\phi(a)}\frac{d\epsilon}{da} - \left\{1 - \frac{\rho a^3}{\phi(a)}\right\}\frac{6\epsilon}{a^2} = 0 \ldots (2).$$

This may be put into another form. Multiply by $\phi(a)$; then

$$\frac{d}{da}\left\{\phi(a)\frac{d\epsilon}{da}\right\} + \frac{d}{da}\{3\rho a^2\epsilon\} = \frac{6\phi(a)\epsilon}{a^2} + 3a^2\epsilon\frac{d\rho}{da};$$

$$\text{or } \frac{d^2}{da^2}\{\phi(a)\,\epsilon\} = \frac{6}{a^2}\phi(a)\,\epsilon + 3a^2\epsilon\frac{d\rho}{da} \ldots (3).$$

535. Cor. In equation (1) given above put $a = \mathrm{a}$, then the second integral vanishes and

$$\int_0^{\mathrm{a}} \rho'\frac{d}{da'}(a'^5\epsilon')\,da' = \frac{5\,\mathrm{a}^2}{3}\phi(\mathrm{a})\left\{\epsilon - \tfrac{1}{2}\mathrm{a}m\right\},$$

we shall call this $\psi(\mathrm{a})$: hereafter we shall see the use of this.

Prop. *To prove that the ellipticity of the strata increases from the centre to the surface.*

* In order to explain how to differentiate a definite integral with respect to a quantity involved in the limit, suppose that

$$\int f(x)\,dx = F(x) + C;$$

$$\therefore \int_b^a f(x)\,dx = F(a) - F(b);$$

$$\therefore \frac{d}{da}\int_b^a f(x)\,dx = \frac{dF(a)}{da} = f(a);$$

$$\text{and } \frac{d}{db}\int_b^a f(x)\,dx = -\frac{dF(b)}{db} = -f(b).$$

536. We assume that the density of the Earth increases from the surface to the centre. Let then $\rho = D - E a^n + \ldots$, where E is positive: and $\varepsilon = A + B a^m \ldots$: put these in equation (2) of Art. 534.

$$\frac{\rho a^3}{\phi(a)} = 1 - \frac{n}{n+3} \frac{E}{D} a^n + \ldots = 1 - H a^n + \ldots\ldots$$

where H is a positive constant.

Then equation (1) gives

$$B(m^2 + 5m) a^{m-2} - 6 A H a^{n-2} + \ldots\ldots = 0.$$

Neither m nor B can equal zero, because then the second term of ε only merges into the first; nor can $m = -5$, a negative quantity: hence the first term will not vanish of itself: but we may make the first and second vanish together by making $n = m$, and $B(m^2 + 5m) = 6 A H$: hence B must be positive; and therefore, *near the centre*, ε increases from the centre towards the surface.

In thus increasing suppose it attains a *maximum*, and then decreases. At this point $\frac{d\varepsilon}{da} = 0$, and equation (2) of Art. 534, gives $\frac{d^2\varepsilon}{da^2} = \left(1 - \frac{\rho a^3}{\phi(a)}\right) \frac{6\varepsilon}{a^2}$ a *positive quantity*, which corresponds to a *minimum*. Hence ε does not become a maximum; and therefore it continually increases to the surface. $\phi(a)$ is always greater than ρa^3, because

$$\phi(a) = 3 \int_0^a \rho' a'^2 da' = \rho a^3 - \int_0^{a'} a'^3 \frac{d\rho'}{da'} da',$$

and $\frac{d\rho'}{da'}$, is negative by hypothesis.

Cor. The expression for the attraction of the Earth considered as a spheroid upon an external object (as the Moon, see Art. 556.) admits of very great simplification in consequence of the results of Arts. 534, 535, 536.

The function V of Art. 185 in this case

$$= \frac{4\pi}{r}\int_0 \rho' \left\{a'^2 + a\frac{d}{da'}\left(\frac{a'^5}{5r^2}Y_2'\right)\right\} da'$$

$$= \frac{4\pi\phi(\mathrm{a})}{3r} + \frac{4\pi\psi(\mathrm{a})}{5r^3}\left(\tfrac{1}{3} - \mu^2\right)$$

$$= \frac{4\pi\phi(\mathrm{a})}{3r} + \frac{4\pi\mathrm{a}^2\phi(\mathrm{a})}{3r^3}\left(\epsilon - \tfrac{1}{2}am\right)\left(\tfrac{1}{3} - \mu^2\right)$$

$$= \frac{E}{r} + \frac{E\mathrm{a}^2}{r^3}\left(\epsilon - \tfrac{1}{2}am\right)\left(\tfrac{1}{3} - \mu^2\right),$$

E = mass of the Earth.

PROP. *To calculate the ellipticity of the surface in the two extreme cases of the mass being homogeneous, and of the density at the centre being infinitely greater than at any other point.*

537. This Proposition will verify the results of Arts. 525, 530.

Now by Art. 535, $\epsilon - \frac{1}{2}am = \dfrac{3}{5\mathrm{a}^2}\dfrac{\psi(\mathrm{a})}{\phi(\mathrm{a})}$.

I. When ρ is constant: $\phi(\mathrm{a}) = 3\int_0^{\mathrm{a}} \rho' a'^2 da' = D\mathrm{a}^3$: D the density at the surface;

$$\text{also } \psi(\mathrm{a}) = \int_0^{\mathrm{a}} \rho' \frac{d}{da'}(a'^5\epsilon')\, da' = D\mathrm{a}^5\,\epsilon;$$

$$\therefore\ \epsilon - \tfrac{1}{2}am = \tfrac{3}{5}\epsilon, \quad \therefore\ \epsilon = \tfrac{5}{4}am.$$

II. When the density at the centre is infinitely greater than that at any other point,

$$\phi(\mathrm{a}) = 3\int_0^{\mathrm{a}} \rho' a'^2 da' = \text{element at centre} = 3\rho' a'^2 da',$$

a' being indefinitely small.

Also $\psi(\mathrm{a}) = \rho' \dfrac{d}{da'}(a'^5\epsilon')\, da' = 5\rho' a'^4\epsilon' da' + \rho' a'^5 d\epsilon'$;

$$\therefore\ \epsilon - \tfrac{1}{2}\, am = \frac{a'^2}{\mathrm{a}^2}\left(\epsilon' + \tfrac{1}{5} a' \frac{d\epsilon'}{da'}\right) = 0;$$

$$\therefore\ \epsilon = \tfrac{1}{2}\, am.$$

In the case of the Earth observations shew that $am = \frac{1}{289}$: and therefore the ellipticity of the Earth in these two hypotheses is $\frac{1}{231}$ and $\frac{1}{578}$. The ellipticity of the Earth deduced from the measurements of degrees of the meridian is intermediate to these: and therefore we may conclude that the Earth is not homogeneous, and also that every particle of the Earth's mass attracts and not the central parts only (see Art. 260).

We shall presently seek for a law of density which is likely to be an approximation to the truth. Before this, however, we shall calculate the value of gravity and the length of a seconds pendulum at the equator, and the length of a degree of latitude, in order to compare the results of theory and observation.

PROP. *To calculate the value of gravity at any place on the Earth's surface, and the length of the seconds pendulum: and to shew that their changes as we pass from one place to another vary as the change in the square of the sine of the latitude.* Clairaut's Theorem.

538. Let g be the action of gravity at a place of which the latitude is l, or $\sin^{-1}\mu$.

By Art. 531 we have

$$\int (X dx + Y dy + Z dz) = V + \frac{4\pi}{9}\, am \frac{\phi(\mathrm{a})}{\mathrm{a}^3} r^2 + \frac{2\pi}{3}\, am \frac{\phi(\mathrm{a})}{\mathrm{a}^3} \left(\tfrac{1}{3} - \mu^2\right) r^2.$$

Now this expression is the sum of all the forces which act upon any particle multiplied by the elements of their respective directions. Hence the resolved part of the resultant attraction in the direction of r is obtained by differentiating this with respect to r and changing the sign; let the result be g: the resolved part perpendicular to this will be of the order α, let it be $\alpha g'$;

therefore gravity $= \sqrt{g^2 + \alpha^2 g'^2} = g$, neglecting α^2 ......

Now $g = -\dfrac{dV}{dr} - \dfrac{8\pi}{9} am \dfrac{\phi(\mathrm{a})}{\mathrm{a}^3} r - \dfrac{4\pi}{3} am \dfrac{\phi(\mathrm{a})}{\mathrm{a}^3} (\tfrac{1}{3} - \mu^2) r$

$$= \frac{4\pi}{r^2} \int_0^{\mathrm{a}} \rho' \left\{a'^2 + \frac{3a}{5r^3} \frac{d}{da'} (a'^5 Y_2')\right\} da'$$

$$- \frac{8\pi}{9} am \frac{\phi(\mathrm{a})}{\mathrm{a}^3} r - \frac{4\pi}{3} am \frac{\phi(\mathrm{a})}{\mathrm{a}^3} (\tfrac{1}{3} - \mu^2) r.$$

But $r = \mathrm{a}\left\{1 + \epsilon(\frac{1}{3} - \mu^2)\right\}$ at the surface. Also by Art. 538, we have

$$a \int_0^{\mathrm{a}} \rho' \frac{d}{da'} (a'^5 Y_2') \, da' = \psi(\mathrm{a}) (\tfrac{1}{3} - \mu^2) = \frac{5\mathrm{a}^2}{3} \phi(\mathrm{a}) (\epsilon - \tfrac{1}{2} am) (\tfrac{1}{3} - \mu^2).$$

$$\therefore g = \frac{4\pi}{3} \frac{\phi(\mathrm{a})}{\mathrm{a}^2} \left\{1 - \tfrac{2}{3} am + (\epsilon - \tfrac{5}{2} am)(\tfrac{1}{3} - \mu^2)\right\}$$

$$= \frac{4\pi}{3} \frac{\phi(\mathrm{a})}{\mathrm{a}^2} (1 + \tfrac{1}{3}\epsilon - \tfrac{3}{2} am) \left\{1 + (\tfrac{5}{2} am - \epsilon) \sin^2 l\right\}$$

$$= G \left\{1 + (\tfrac{5}{2} am - \epsilon) \sin^2 l\right\},$$

G being the equatorial gravity.

Hence, the ratio of the excess of polar over equatorial gravity to equatorial gravity (that is $\frac{5}{2} am - \epsilon$, by the formula just proved) *added to the ellipticity equals* $\frac{5}{2}$ × *the ratio of the centrifugal force at the equator to the equatorial gravity, whatever be the law of the density*. This is called Clairaut's Theorem, after its discoverer.

539. Let l and L be the lengths of the seconds pendulum at the places at which g and G are the values of gravity;

$$\therefore l = L \frac{g}{G} = L \left\{1 + (\tfrac{5}{2} am - \epsilon) \sin^2 l\right\}.$$

Prop. *To calculate the length of a degree of latitude at any place on the Earth's surface, in terms of the length of a degree at the equator.*

540. Let C and c be the lengths of a degree corresponding to the values of gravity G and g.

Now rad. of curv. $= \dfrac{\left\{r^2 + \dfrac{dr^2}{dl^2}\right\}^{\frac{3}{2}}}{r^2 + 2\dfrac{dr^2}{dl^2} - r\dfrac{d^2r}{dl^2}}$, $\sin l = \mu$, $r = \text{a}\,(1 + \alpha Y_2)$,

$$r = \text{a}\left\{1 + \epsilon\left(\tfrac{1}{3} - \mu^2\right)\right\} = \text{a}\left\{1 + \epsilon\left(\tfrac{1}{3} - \sin^2 l\right)\right\}.$$

$$\therefore \frac{dr}{dl} = -2\,\text{a}\sin l\cos l, \quad \frac{d^2r}{dl^2} = -2\text{a}\cos 2l.$$

$$\therefore \text{rad. of curv.} = r + \frac{d^2r}{dl^2}, \quad \text{neg}^s.\ \epsilon^2 \ldots\ldots,$$

$$= \text{a}\left\{1 + \epsilon\left(\tfrac{1}{3} - \sin^2 l - 2\cos 2l\right)\right\}$$

$$= \text{a}\left\{1 - \tfrac{5}{3}\epsilon + 3\epsilon\sin^2 l\right\}$$

$$= \text{a}\left(1 - \tfrac{5}{3}\epsilon\right)\left(1 + 3\epsilon\sin^2 l\right).$$

Now the length of the degree varies as the length of the radius of curvature;

$$\therefore c = C\,(1 + 3\epsilon\sin\ \ l).$$

These formulæ for the length of the seconds pendulum and the length of a degree of latitude enable us to test the truth of our calculations by comparing observed results with those given by these formulæ.

541. It will be observed, that throughout this calculation we have neglected α^2.... It is very desirable, then, to apply some independent test of the truth of our results. The two conclusions to which we are come are, that the figure of the Earth is an oblate spheroid, and that the law of variation in gravity in passing from one place to another on the surface of the Earth is represented by Clairaut's Theorem. Mr Airy has estimated the probable magnitude of the errors committed in these calculations, by considering the two following extreme laws of density: 1st, the case of homogeneity: 2nd, the case in which the matter at the centre only attracts; or, which amounts to the same, the density of the matter at the centre is infinitely greater than that of the rest of the mass. The deviations

from a spheroidal figure, in these cases, are found to be insensible. The same is found to be the case with Clairaut's Theorem. It follows, then, that the error in omitting small quantities in our calculations is inappreciable. (*Ency. Met. Figure of the Earth*, Arts. 69—72.) As far, then, as these two tests are concerned, in determining the law of density in the Earth's mass, we should be at a loss in attempting to decide what the actual law of nature is. But the magnitudes of the ellipticity being $\frac{1}{231}$ and $\frac{1}{577}$ in these extreme laws of density, we learn, that since the first is greater and the second less than that given by geodetic and other measures, we must suppose that the density increases towards the centre; but that it does not become infinitely great.

542. We are now able to calculate the principal moments of inertia of the Earth, which were required in Art. 470, in calculating the Precession of the Equinoxes and the Nutation of the Earth's Axis.

Prop. *To calculate the principal moments of inertia of the Earth, supposing it to consist of layers nearly spherical and of different densities.*

543. Let A, B, C be the principal moments of inertia about the axes of $x_, y_, z_,$ respectively: $x_, y_, z_,$ the co-ordinates to an element of the mass. Then the mass of this element $= -\rho' r^2 dr\, d\mu\, d\omega$ (Art. 183); also

$$y_,^2 + z_,^2 = r^2\{1 - (1-\mu^2)\cos^2\omega\} = r^2\{\tfrac{2}{3} + [\tfrac{1}{3} - (1-\mu^2)\cos^2\omega]\},$$

$$x_,^2 + z_,^2 = r^2\{\tfrac{2}{3} + [\tfrac{1}{3} - (1-\mu^2)\sin^2\omega]\},\quad x_,^2 + y_,^2 = r^2\{\tfrac{2}{3} + (\tfrac{1}{3} - \mu^2)\},$$

we have arranged these in this manner, because they are then each *of the form* $U_0 + U_2$. Hence

$$A = \int_{-1}^{1}\int_0^{2\pi}\int_0^{r}\rho' r^4\{\tfrac{2}{3} + \tfrac{1}{3} - (1-\mu^2)\cos^2\omega\}\, d\mu\, d\omega\, dr.$$

$$\text{But}\quad \int_0^{r}\rho' r^4\, dr = \tfrac{1}{5}\int_0^{a}\rho' \frac{d.r^5}{da'}\, da'$$

$$= \tfrac{1}{5}\int_0^{a}\rho' \frac{d}{da'}\{a'^5 + 5a'^5 a(Y_0' + Y_1' + Y_2' + \ldots)\}\, da'$$

$$= \tfrac{1}{5}\int_0^{a} \rho' \frac{d}{da'}\{a'^5 + 5a'^5\, a Y_2'\}\, da' \text{ by Art. 532,}$$

$$= \tfrac{1}{5}\int_0^{a} \rho' \frac{d}{da'}\{a'^5 + 5a'^4 \epsilon' (\tfrac{1}{3} - \mu^2)\}\, da' \text{ by Art. 533,}$$

$$= \tfrac{1}{5}\sigma(\mathrm{a}) + \psi(\mathrm{a})(\tfrac{1}{3} - \mu^2), \text{ where } \sigma(\mathrm{a}) = \int_0^{a} \rho' \frac{d.a'^5}{da'} da'.$$

$$\therefore A = \int_{-1}^{1}\int_0^{2\pi}\{\tfrac{2}{15}\sigma(\mathrm{a})$$

$$+ \psi(\mathrm{a})(\tfrac{1}{3} - \mu^2)[\tfrac{1}{3} - (1-\mu^2)\cos^2\omega]\}\, d\mu\, d\omega \text{ by Art. 176.}$$

$$= \tfrac{8}{15}\pi\sigma(\mathrm{a}) - \tfrac{8}{45}\pi\psi(\mathrm{a}).$$

In the same manner

$$B = \tfrac{8}{15}\pi\sigma(a) - \tfrac{8}{45}\pi\psi(\mathrm{a}).$$

Also we obtain $C = \int_{-1}^{1}\int_0^{2\pi}\int_0^{r}\rho' r'^4\{\tfrac{2}{3} + (\tfrac{1}{3} - \mu^2)\}$

$$= \tfrac{8}{15}\pi\sigma(\mathrm{a}) + \tfrac{16}{45}\pi\psi(\mathrm{a}).$$

Hence $\dfrac{C-A}{C} = \dfrac{\psi(\mathrm{a})}{\sigma(\mathrm{a})}$, neg$^{\text{g}}$. the square of $\psi(\mathrm{a})$.

544. We proceed to obtain an approximate law of the density of the strata, and then to prepare the formulæ deduced in the foregoing articles for numerical calculation.

PROP. *To obtain an approximate law of the density of the strata.*

545. By Art. 531, we have for calculating the pressure on the stratum of which the mean radius is a, neglecting α,

$$\int_{a}^{a} \frac{1}{\rho'}\frac{dp}{da'}da' = \frac{4\pi}{3}\frac{\phi(a)}{a} + 4\pi\int_{a}^{a}\rho' a' da'.$$

Laplace has integrated this equation upon the supposition that the change in pressure in descending through the strata varies as the change in the square of the density (*Mémoires de l'Institut*, Tom. III. p 496). This law of compression

differs from that of fluids, in which the change in pressure varies as the change in density. The law used by Laplace is *à priori* more probably true than the law of the compression of fluids, since tenacious and semi-fluid bodies must require a greater compressing force to produce a given compression and density under given circumstances than a fluid body does, in consequence of the greater cohesive force of the particles of the semi-fluid body. See also some remarks by Professor Challis on this subject in the *Phil. Mag.* Vol. XXXVIII. The approximate truth, however, of this law is shewn by the accuracy of the results to which it leads us. Putting, then, $dp = \frac{1}{2} k d.\rho'^2$, k being a constant, we have

$$\therefore \int_a^a \frac{1}{\rho'} \frac{dp}{da'} da' = k \int_a^a \frac{d\rho'}{da'} da' = k(\rho - D),$$

D being the density at the surface;

$$\therefore ka(\rho - D) = 4\pi \int_0^a \rho' a'^2 da' + 4\pi a \int_a^a \rho' a' da',$$

$$\text{because } \phi(a) = 3\int_0^a \rho' a'^2 da'.$$

Differentiate with respect to a, having regard to the note to Art. 534,

$$\therefore k\frac{d.\rho a}{da} - kD = 4\pi\rho a^2 + 4\pi \int_a^a \rho' a' da' - 4\pi\rho a^2 = 4\pi \int_a^a \rho' a' da';$$

$$\therefore \frac{d^2.\rho a}{da^2} = -q^2 \rho a, \text{ putting } \frac{4\pi}{k} = q^2;$$

$$\therefore \rho a = A \sin(qa + B), \quad \therefore \rho = \frac{A}{a} \sin(qa + B).$$

When $a = 0$, $\rho = \frac{A \sin B}{0}$; $\therefore B = 0$, otherwise ρ would be infinite at the centre, which cannot be;

$$\therefore \rho = \frac{A}{a} \sin qa,$$

A and q being unknown constants.

PROP. *To calculate the ellipticity of the strata on the approximate law of density deduced in the last Article.*

546. We must put $\rho = \dfrac{A}{a}\sin qa$ in the equation

$$\frac{d^2}{da^2}\{\phi(a)\varepsilon\} = \frac{6}{a^2}\phi(a)\varepsilon + 3a^2\varepsilon\frac{d\rho}{da} \text{ (see Art. 534.)}$$

Now $\phi(a) = 3\int_0^a \rho' a'^2 da' = 3A\int_0^a a' \sin qa' da$

$$= 3A\left\{-\frac{a}{q}\cos qa + \frac{1}{q^2}\sin qa\right\},$$

$$\text{also } \frac{d\rho}{da} = A\left\{\frac{q}{a}\cos qa - \frac{1}{a^2}\sin qa\right\} = -\frac{q^2}{3a^2}\phi(a),$$

and our equation becomes

$$\frac{d^2\{\phi(a)\varepsilon\}}{da^2} + q^2\phi(a)\varepsilon = \frac{6}{a^2}\phi(a)\varepsilon\text{*}.$$

The integral of this equation is

$$\phi(a)\varepsilon = C\left\{\left(1 - \frac{3}{q^2a^2}\right)\sin(qa + C') + \frac{3}{qa}\cos(qa + C')\right\}$$

C and C' being arbitrary constants. In our case $C' = 0$, otherwise the ellipticity at the centre would be infinite, as is easily shewn by expanding ε in powers of a.

Hence, if we substitute for $\phi(a)$, we have

$$\varepsilon = \frac{Cq^2}{3A}\,\frac{\left(1 - \dfrac{3}{q^2a^2}\right)\tan qa + \dfrac{3}{qa}}{\tan qa - qa};$$

$$\text{and } \therefore\ \frac{\varepsilon}{\epsilon} = \frac{\tan qa - qa}{\tan qa - qa}\,\frac{\left(1 - \dfrac{3}{q^2a^2}\right)\tan qa + \dfrac{3}{qa}}{\left(1 - \dfrac{3}{q^2a^2}\right)\tan qa + \dfrac{3}{qa}} \ \ldots\ (1).$$

* Mr O'Brien integrates this by putting

$$\phi(a)\varepsilon = \frac{1}{a^2}\int_0^a a'\int_0^a a'x'da'^2.$$

This gives the law of decrease of the ellipticity in the strata in passing from the surface to the centre.

To determine ε and ϵ we must combine with the above the following equation,

$$\epsilon - \frac{1}{2} am = \frac{3}{5a^2} \frac{\psi(a)}{\phi(a)} \text{ (see Art. 535).}$$

$$\text{Now } \psi(a) = \int_0^a \rho' \frac{d}{da'}(a'^5 \varepsilon') da' = A \int_0^a \frac{\sin qa'}{a'} \frac{d}{da'}(a'^5 \varepsilon') da'$$

$$= A \left\{ a^4 \epsilon \sin qa + \int_0^a a'^3 \varepsilon' (\sin qa' - qa' \cos qa') da' \right\}, \text{ by parts.}$$

The integral contained in this

$$= \epsilon \frac{\tan qa - qa}{\left(1 - \frac{3}{q^2 a^2}\right) \tan qa + \frac{3}{qa}} \int_0^a \left\{ \left(a'^3 - \frac{3a'}{q^2}\right) \sin qa' + \frac{3a'^2}{q} \cos qa' \right\} da'$$

$$= \epsilon \frac{\tan qa - qa}{\left(1 - \frac{3}{q^2 a^2}\right) \tan qa + \frac{3}{qa}} \frac{1}{q^4} \left\{ 3(2a^2 q^2 - 5) \sin qa - (q^3 a^3 - 15qa) \cos qa \right\}.$$

$$\text{Also } \phi(a) = \frac{3A}{q^3} \left\{ \sin qa - qa \cos qa \right\}.$$

Hence the equation $\epsilon - \frac{1}{2} am = \dfrac{3}{5a^2} \dfrac{\psi(a)}{\phi(a)}$ becomes

$$\frac{1}{2} am =$$

$$\epsilon \left\{ 1 - \frac{(q^4 a^4 - 3q^2 a^2) \tan^2 qa + 3q^3 a^3 \tan qa + (\tan qa - qa)\{(6q^2 a^2 - 15) \tan qa - (q^3 a^3 - 15qa)\}}{5q^2 a^2 (\tan qa - qa) \left\{ \left(1 - \frac{3}{q^2 a^2}\right) \tan qa + \frac{3}{qa} \right\}} \right\}$$

$$\frac{5am}{2} = \epsilon \frac{(2 - q^2 a^2) \tan^2 qa - qa \tan qa - q^2 a^2}{(\tan qa - qa) \left\{ \left(1 - \frac{3}{q^2 a^2}\right) \tan qa + \frac{3}{qa} \right\}}.$$

This equation determines ϵ: and consequently ε is also known by equation (1).

547. To facilitate the calculation of ϵ let $\dfrac{qa}{\tan qa} = 1 - x$;

$$\therefore \epsilon = \frac{5am}{2} \frac{1 - \dfrac{3x}{q^2a^2}}{3 - x - \dfrac{q^2a^2}{x}}.$$

Before we reduce this to numbers we shall calculate the principal moments of inertia deduced in Art. 543, in terms of the approximate law of density.

PROP. *To calculate* σ (a) *and* ψ (a) *and* $\dfrac{C-A}{C}$ *with the approximate law of density.*

548. Now $\sigma(a) = \int_0^a \rho' \dfrac{d}{da'}(a'^5)da' = 5A\int_0^a a'^3 \sin qa'da'$

$$= 5A\left\{-\frac{a^3}{q}\cos qa + \frac{3a^2}{q^2}\sin qa + \frac{6a}{q^3}\cos qa - \frac{6}{q^4}\sin qa\right\}$$

$$= \frac{5A}{q^4}\left\{(3q^2a^2 - 6)\sin qa - (q^3a^3 - 6qa)\cos qa\right\}.$$

Also $\psi(a) = \dfrac{5a^2}{3}\phi(a)(\epsilon - \frac{1}{2}am)$

$$= \frac{5Aa^3}{q^2}(\epsilon - \tfrac{1}{2}am)(\sin qa - qa\cos qa);$$

$$\therefore \frac{\psi(a)}{\sigma(a)}, \text{ or } \frac{C-A}{C}, = (\epsilon - \tfrac{1}{2}am)\frac{1 - \dfrac{qa}{\tan qa}}{3 - \dfrac{6}{q^2a^2} - \left(1 - \dfrac{6}{q^2a^2}\right)\dfrac{qa}{\tan qa}}$$

$$= \frac{(\epsilon - \frac{1}{2}am)x}{2 + \left(1 - \dfrac{6}{q^2a^2}\right)x}, \quad x = 1 - \frac{qa}{\tan qa}.$$

We now proceed to reduce these quantities to numbers.

PROP. *The law of density being represented by* $\rho = \dfrac{A}{a}\sin qa$; *it is required to find the value of* q *in the case of the Earth.*

549. The experiments of Cavendish and the observations of Maskelyne shew, that the mean density of the Earth is about five times and a half that of water. Now, since the superficial stratum of the Earth consists partly of the Ocean, we must not include this among the strata of the Earth when we speak of their law of variation, because the density of water is far less than that of rock or earth. We shall consequently suppose the superficial stratum to be of the density of granite or thereabouts; *i.e.* nearly in the ratio of 5 to 2 to that of water. We shall therefore suppose the mean density is to that of the superficial stratum as 11 to 5.

Let δ be the mean density: D the density of the outer stratum: then

$$\delta\int_0^{a} 4\pi a'^2 da' = \int_0^{a} 4\pi\rho' a'^2 da'$$

$$\frac{1}{3}a^3\delta = A\int_0^{a} a'\sin qa' \,.\, da' = A\left\{-\frac{a}{q}\cos qa + \frac{1}{q^2}\sin qa\right\};$$

$$\therefore\ \frac{\delta}{D} = \frac{3}{q^2a^2}\left(1 - \frac{qa}{\tan qa}\right) = \frac{11}{5} \text{ by hypothesis};$$

$$\therefore\ \tan qa = \frac{15qa}{15 - 11q^2a^2}.$$

A very small value of q will satisfy this equation; but a small value would give a very slow change of density in the strata of the Earth. After repeated trials we find that $qa = \frac{5}{6}\pi = 2.618$ nearly satisfies this equation: for then

$$\log_{10}\frac{15qa}{11q^2a^2 - 15} = \bar{1}.8130736 = \log_{10}\tan 33^0 2' = \log_{10}\tan\frac{\pi}{6} \text{ nearly};$$

$$\therefore\ qa = \frac{5}{6}\pi = 2.618 \text{ very nearly satisfies the equation.}$$

If we had taken $\delta = 2.4225D$, then $qa = 2.618$ would have satisfied the equation as far as four decimal places.

We shall take, then, $qa = \frac{5}{6}\pi = 2.618$.

The remarkable agreement found to subsist between the calculations made with this value of qa and by other means is the most satisfactory proof of its correctness.

COR. Density of stratum of which a is the mean radius equals $\frac{2D\text{a}}{a}\sin\left(\frac{a}{\text{a}}\frac{5\pi}{6}\right)$.

PROP. *To calculate numerically the ellipticity of the Earth with the value of* $q\text{a}$ *equal to* 2.618.

550. By Art. 547, $\epsilon = \frac{5am}{2}\dfrac{1-\dfrac{3x}{q^2\text{a}^2}}{3-x-\dfrac{q^2\text{a}^2}{x}}$,

$$\text{where } x = 1 - \frac{q\text{a}}{\tan q\text{a}};$$

hence $x = 5.5345$: also $am = \frac{1}{289}$ by observation;

$$\therefore \ \epsilon = .00325401 = \tfrac{1}{307.313},$$

this value of ϵ accords remarkably with the result of geodetic measures, which give $\epsilon = \frac{1}{306}$.

PROP. *To calculate numerically the value of* $\frac{C-A}{C}$ *and the Annual Precession of the Equinoxes with the approximate law of density.*

551. By Art. 548, $\dfrac{C-A}{C} = \dfrac{\psi(\text{a})}{\sigma(\text{a})} = \dfrac{(\epsilon - \frac{1}{2}am)x}{2+\left(1-\dfrac{6}{q^2\text{a}^2}\right)x}$

$$= .00313593.$$

Also the Annual Precession of the Equinoxes

$$= \frac{C-A}{C}\left(1+\frac{176.5906}{1+\nu}\right)4882''.05 \text{ (Art. 470.)}$$

$$= (.00313593)\,(3.3545)\,(4882''.05)\,;\ \nu = 74$$

$$= \log_{10}^{-1}\left\{\begin{matrix}\bar{3}.4963664\\ .5256278\\ 3.6886020\end{matrix}\right\}'' = \log^{-1}(1.7105962)'' = 51''.3566,$$

the observed precession is $50''.1$. The exceedingly remarkable agreement of the calculated values of the precession and ellipticity of the Earth with their observed values affords a convincing proof of the correctness of the principles involved in the calculation.

552. Before we quit this subject we shall give a few important Propositions which tend to throw additional light upon the determination of the figure of the Earth.

The permanent state of equilibrium of the heavenly bodies makes known to us some of the properties of their radii. If the planets did not revolve about one of their three principal axes, or very near to one of them, there would be produced in the position of the axes of rotation some variations which would become sensible, particularly in the Earth (see Art. 458). Now by the most accurate observations no such variations are perceived. Therefore we must infer, that a long period of time has elapsed since all parts of the heavenly bodies, and particularly the fluid particles on their sufaces, have been arranged in such a manner as to render their state of equilibrium permanent, consequently also their axes of rotation: for it is very natural to suppose, that after a great number of oscillations, the bodies must assume the forms corresponding to the state of equilibrium, on account of the resistances suffered by the particles of the fluid. We shall now examine into the conditions, arising from this supposition, in the expression of the radii of the heavenly bodies.

PROP. *To prove that in consequence of the permanence of the rotatory motion of the Earth* $Y_2 = K(\frac{1}{3} - \mu^2) + K'(1-\mu^2)\cos 2\omega$, K *and* K' *being constants which depend upon the internal structure of the Earth.*

553. Let the Earth be referred to its principal axes: and let r be the radius of any particle: θ the angle which r makes with the axis of $z_{,}$, ω the angle between the planes of $x_{,}z_{,}$ and $rz_{,}$: also let $\cos\theta = \mu$;

$$\therefore \; x_{,} = r\sin\theta\cos\omega = r\sqrt{1-\mu^2}\cos\omega,$$

$$y_{,} = r\sin\theta\sin\omega = r\sqrt{1-\mu^2}\sin\omega,$$

$$z_{,} = r\cos\theta \quad\quad = r\mu.$$

Let ρ be the density of the stratum of which the radius is r: then the element of the mass $(m) = \rho\delta r r\delta\theta r \sin\theta\delta\omega$ $= -\rho r^2 \delta r \delta\mu \delta\omega$, and by the properties of the principal axes, viz. $\Sigma . m z_i y_i = 0$, $\Sigma . m x_i z_i = 0$, $\Sigma . m x_i y_i = 0$, we have

$$\left.\begin{array}{l} \int_0^r \int_{-1}^1 \int_0^{2\pi} \rho r^4 \mu \sqrt{1-\mu^2} \sin\omega \, dr \, d\mu \, d\omega = 0 \\ \int_0^r \int_{-1}^1 \int_0^{2\pi} \rho r^4 \mu \sqrt{1-\mu^2} \cos\omega \, dr \, d\mu \, d\omega = 0 \\ \int_0^r \int_{-1}^1 \int_0^{2\pi} \rho r^4 (1-\mu^2) \cos\omega \sin\omega \, dr \, d\mu \, d\omega = 0 \end{array}\right\} \ldots (1),$$

Now $r = a \left\{1 + \alpha Y_1 + \alpha Y_2 + \ldots + \alpha Y_i + \ldots\right\}$

$$\text{and } \therefore \int_0^r \rho r^4 dr = \int_0^a \rho r^4 \frac{dr}{da} da = \frac{1}{5}\int_0^a \rho \frac{d . r^5}{da} da,$$

$$= \int_0^a \rho \frac{d}{da} \left\{\tfrac{1}{5} a^5 + \alpha a^5 (Y_1 + Y_2 + \ldots + Y_i + \ldots)\right\} da$$

$$= U_0 + \alpha U_1 + \alpha U_2 + \ldots \alpha U_i + \ldots \text{ suppose.}$$

Then by substituting this series in equations (1), bearing in mind the property of Laplace's Coefficients proved in Art. 176, we have

$$\int_{-1}^1 \int_0^{2\pi} U_2 \mu \sqrt{1-\mu^2} \sin\omega \, d\mu \, d\omega = 0,$$

$$\int_{-1}^1 \int_0^{2\pi} U_2 \mu \sqrt{1-\mu^2} \cos\omega \, d\mu \, d\omega = 0,$$

$$\text{and } \int_{-1}^1 \int_0^{2\pi} U_2 (1-\mu^2) \sin\omega \cos\omega \, d\mu \, d\omega = 0.$$

Now since U_2 is a function of μ, $\sqrt{1-\mu^2}\cos\omega$, and $\sqrt{1-\mu^2}\sin\omega$ of the second order, and satifies the Equation of Laplace's Coefficients, it is of the form $H(\frac{1}{3} - \mu^2)$ $+ H'\mu\sqrt{1-\mu^2}\sin\omega + H''\mu\sqrt{1-\mu^2}\cos\omega + H'''(1-\mu^2)\sin 2\omega$ $+ H^{iv}(1-\mu^2)\cos 2\omega$. Then by putting this for U_2 in the last three equations we find

$$H' = 0, \quad H'' = 0, \quad H''' = 0,$$

$$\therefore U_2 = H(\tfrac{1}{3} - \mu^2) + H^{iv}(1-\mu^2)\cos 2\omega.$$

But U_2 was written for $\int_0^a \rho \frac{d}{da}(a^5 Y_2)\,da$, and by Art. 533 this

$$= \tfrac{5}{3}\mathrm{a}^2\phi(\mathrm{a})\left\{\mathrm{Y}_2 - \tfrac{1}{2}m\left(\tfrac{1}{3} - \mu^2\right)\right\},$$

$$\therefore\ \mathrm{Y}_2 = K\left(\tfrac{1}{3} - \mu^2\right) + K'(1 - \mu^2)\cos 2\omega,$$

K and K' being constants which depend upon the internal structure of the Earth.

Thus far, then, the condition that the Earth rotates about a principal axis, determines the form of the function Y_2 in the radius of the surface, $\mathrm{r} = \mathrm{a}(1 + \alpha\mathrm{Y}_1 + \alpha\mathrm{Y}_2 + \ldots + \alpha\mathrm{Y}_i + \ldots)$; shewing that three of the terms in the most general form of Y_2 vanish.

554. Laplace has obtained some remarkable formulæ for the value of gravity, the length of a seconds pendulum, and the length of a degree of latitude, which are independent of any supposition of the internal structure of the Earth except that it consists of nearly spherical strata. They are deduced as in Arts. 538, 539, 540; but $\mathrm{a}\{1 + \alpha\mathrm{Y}_1 + \alpha\mathrm{Y}_2 + \alpha\mathrm{Y}_3 + \ldots\ldots\}$ is put for r instead of $\mathrm{a}\{1 + \alpha\mathrm{Y}_2\}$. The results are

$$g = G\left\{1 + \alpha\mathrm{Y}_2 + 2\alpha\mathrm{Y}_3 + \ldots\ldots + (i-1)\alpha\mathrm{Y}_i + \ldots\ldots\right\}$$
$$- \tfrac{10}{3}\pi\alpha m\frac{\phi(\mathrm{a})}{\mathrm{a}^2}\left(\tfrac{1}{3} - \mu^2\right),$$

$$l = L\left\{1 + \alpha\mathrm{Y}_2 + 2\alpha\mathrm{Y}_3 + \ldots\ldots + (i-1)\alpha\mathrm{Y}_i + \ldots\ldots\right.$$
$$\left. - \tfrac{1}{2}\alpha m\left(\tfrac{1}{3} - \mu^2\right)\right\},$$

$$c = C\left\{1 - 5\alpha Y_2 - 11\alpha\mathrm{Y}_3 \ldots\ldots - (i^2 + i - 1)\alpha\mathrm{Y}_i - \ldots\right.$$
$$+ \alpha\mu\left(\frac{d\mathrm{Y}_2}{d\mu} + \frac{d\mathrm{Y}_3}{d\mu} + \ldots\ldots + \frac{d\mathrm{Y}_i}{d\mu} + \ldots\right)$$
$$\left. - \frac{\alpha}{1-\mu^2}\left(\frac{d^2\mathrm{Y}_2}{d\omega^2} + \frac{d^2\mathrm{Y}_3}{d\omega^2} + \ldots\ldots + \frac{d^2\mathrm{Y}_i}{d\omega^2} + \ldots\right)\right\}.$$

If we compare the expressions for the radius of the Earth with that for the length of the pendulum and the length of a degree of the meridian we shall perceive, that the term $\alpha\mathrm{Y}_i$ in the expression for the radius is multiplied by $i - 1$

in the length of the pendulum, and by $i^2 + i - 1$ in the degree of the meridian. It follows, then, that however small $i - 1$ may be this term will be more sensible in the lengths of the pendulum than in the horizontal parallax of the Moon, which is proportional to the radius of the Earth: and it will be still more sensible in the measures of the degrees than in the lengths of the pendulum.

These three expressions are very important, inasmuch as they are independent of the internal structure of the Earth; that is, they are independent of the figure and density of the strata, since the functions $Y_1 Y_2 \ldots$ all refer to the surface. It follows, then, that if we can determine the functions $Y_1 Y_2 \ldots$ by the measures of degrees and parallaxes, we shall obtain directly the length of the pendulum. We may by this means ascertain whether the law of universal gravitation agrees with the figure of the Earth, and with the observed variations of gravity at its surface. These remarkable relations connecting the degrees of the meridian and the length of the pendulum also serve to verify any hypothesis, assumed to represent the measures of the degrees of the meridian. Laplace (*Méc. Céles.* Liv. III. §. 33.) makes an application of these formulæ to an hypothesis of Bouguer with respect to the lengths of degrees: and his calculation shews that it must be rejected. The hypothesis is that the variation of a degree of the meridian is proportional to the fourth power of the sine of the latitude.

555. Laplace shews, that the greatest minimum probable error in calculating the length of a degree from observations made at seven places is 97.2 toises, the mean length of the degrees being about 51307.4 toises (Liv. III. §. 41). The ratio the error bears to this = 0.00189.

It is also shewn (Liv. III. §. 42.) that the greatest probable error in the calculation of the length of a seconds pendulum is 0.00018, the mean of the lengths of the pendulum at the fifteen places at which the observations were made being 0.99922: the ratio the error bears to this = 0.00018. Now this is more than ten times less than the error of the measures of the degrees, and remarkably confirms the theory in Art. 554; *viz.* that the terms of the expression of the terrestrial radius, which cause the Earth to vary from an elliptical figure, are

much less sensible in the lengths of the pendulum than in the lengths of the degrees of the meridian. Later observations, as Mr Bowditch observes in his Commentary on this part of the *Mécanique Céleste*, do not confirm this result. The discrepancies among the observations of the length of the pendulum are greater than in those of the best observations of the measured arcs of the meridian. Various causes have been assigned for these differences in the observations of the pendulum: as the local attractions of neighbouring bodies; the peculiar action of the substance of the stratum of the Earth over which the pendulum is placed; and finally magnetic action. With respect to the first Bouguer has found by observation that the attraction of the mountain Chimboraço produced a deviation of $7''.5$ in the plumb line: and Dr Maskelyne observed the attraction of the mountain Schehallien to $5''.8$.

The results at which we have arrived respecting the ellipticity of the Earth are confirmed by the comparison of the calculated and observed effects which the spheroidal form of the Earth has upon the motion of the Moon. We proceed to shew this.

PROP. *To find the effect of the ellipticity of the Earth upon the motion of the Moon in latitude: and thence to deduce the numerical value of the ellipticity.*

556. Let λ be the latitude of the Moon's centre; i the inclination of her orbit to the ecliptic; n her mean motion; ϵ' the epoch; Ω the longitude of her ascending node: then

$$\tan\lambda = \tan i \sin(nt + \epsilon' - \Omega)$$

$$\text{or } \lambda = i \sin(nt + \epsilon' - \Omega) \text{ since } \lambda \text{ and } i \text{ are small.}$$

This formula enables us to find the change in λ in terms of the changes in the elements of the Moon's orbit; these we shall now calculate from the principles of the Planetary Theory.

Since the coefficient i is very small, we may neglect the variations of n and ϵ' in the formula for λ; but not the variation of Ω, because the expression for this has i in its denominator, as we shall see.

By Art. 366 we have the equations

$$\frac{di_{,}}{dt} = \frac{na}{\mu i}\frac{dR}{d\Omega} \text{ and } \frac{d\Omega_{,}}{dt} = -\frac{na}{\mu i}\frac{dR}{di}$$

the subscript accents being used when the quantities under which they are placed are considered variable.

The function R must now be found.

Let V be the sum of the quotients formed by dividing each particle of the Earth's mass by its distance from the centre of the Moon: then $-\frac{dV}{dx}$ is the attraction of the Earth on the Moon parallel to the axis of x (see Art. 167). Also, supposing the Moon's mass condensed into its centre, the attraction of the Moon on the Earth parallel to x equals $-\frac{M}{E}\frac{dV}{dx}$, M and E being the masses of the Moon and Earth. The equations of motion to the Moon's centre, referred to the Earth's centre, are

$$\frac{d^2x}{dt^2} = \frac{E+M}{E}\frac{dV}{dx}, \quad \frac{d^2y}{dt^2} = \frac{E+M}{E}\frac{dV}{dy}, \quad \frac{d^2z}{dt^2} = \frac{E+M}{E}\frac{dV}{dz}.$$

By comparing these with the equations of Art. 355, in the Planetary Theory,

we have $\mu = E + M$, and $R = \frac{E+M}{r} - \frac{E+M}{E}V$.

and by the value of V in Art. 536, Cor., we have

$$R = -\frac{(E+M)\,a^2}{r^3}\left(\epsilon - \tfrac{1}{2}am\right)\left(\tfrac{1}{3} - \mu^2\right).$$

We must now find μ. Let θ be the longitude of the Moon and I the obliquity of the ecliptic.

$\therefore\ \mu = \cos$ (Moon's North Polar Distance)

$= \sin I \cos\lambda \sin\theta + \cos I \sin\lambda.$

But $\theta = nt + \epsilon' + 2e\sin(nt + \epsilon' - \varpi)$;

$$\therefore\ \mu = \sin I \sin(nt + \epsilon') + i \cos I \sin(nt + \epsilon' - \Omega)$$
$$+ 2e \sin I \cos(nt + \epsilon') \sin(nt + \epsilon' - \varpi), \text{neg}^s.\ e^2, i^2, \ldots$$
$$= \sin I \sin(nt + \epsilon') + i \cos I \sin(nt + \epsilon' - \Omega)$$
$$- e \sin I \sin \varpi + e \sin I \sin(2nt + 2\epsilon' - \varpi).$$

Substituting this in R, and preserving only the terms which are periodical and also independent of $nt + \epsilon'$, since these last go through their changes so rapidly as to neutralize their effects very quickly, we have

$$R = \frac{(E + M)\,\text{a}^2}{2r^3}(\epsilon - \tfrac{1}{2}am)\, i \sin 2I \cos \Omega$$

$$= (E + M)\, Ai \cos \Omega \text{ suppose.}$$

$$\therefore\ \frac{di_{\prime}}{dt} = -naA \sin \Omega,\quad \frac{d\Omega_{\prime}}{dt} = -\frac{na}{i} A \cos \Omega.$$

Since the node on the whole progresses pretty steadily (see Art. 338), we may put $\Omega = -ht$ on the second side of these equations: h = the mean motion of the node:

$$\therefore\ i_{\prime} = i - \frac{na}{h} A \cos ht,\ \Omega_{\prime} = \Omega - \frac{na}{hi} A \sin ht:$$

and the change in λ, arising from these changes in i and Ω,

$$= \delta i \sin(nt + \epsilon' - \Omega) - i \cos(nt + \epsilon' - \Omega)\,\delta\Omega$$

$$= -\frac{na}{h} A \sin(nt + \epsilon'),\ \text{putting } \Omega = -ht,$$

$$= -\frac{n\text{a}^2}{2ha^2}(\epsilon - \tfrac{1}{2}am) \sin 2I \sin(nt + \epsilon'),\ \text{putting } r = a.$$

Burg makes this term $-8''$ sin $(nt + \epsilon')$ by observation. Hence we have the following equation for calculating the ellipticity:

$$\epsilon - \tfrac{1}{2}am = \frac{2ha^2\pi}{n\text{a}^2 \sin 2I}\,\frac{8''}{180^0} = \frac{ha^2\pi}{40500\, n\text{a}^2 \sin 2I}.$$

$$\text{Now } I = 23^0\,28',\ \frac{h}{n} = 0.0040217,\ \frac{a}{\text{a}} = 60.197.$$

$$\therefore\ \epsilon - \tfrac{1}{2}am = \log^{-1}\left\{\begin{array}{l}\bar{3}.6044097 - 4.6074550 \\ 3.5591496 - \bar{1}.8636557 \\ .4971509\end{array}\right\}$$

$$= \log^{-1}\{1.6607102 - 4.4711107\}$$

$$= \log^{-1}(\bar{3}.1895995) = .0015474;$$

$$\text{and } \tfrac{1}{2}am = \frac{1}{578} = .0017476;$$

$$\therefore\ \epsilon = .0032950 = \frac{1}{303} \text{ nearly.}$$

This coincides remarkably with the values already obtained for ϵ.

CHAPTER III.

FORM OF EQUILIBRIUM OF THE OCEAN UNDER THE MOON'S ATTRACTION, AND THE FORM OF THE ATMOSPHERE.

THE following Proposition we shall find of use when we come to the Chapter on the Tides.

PROP. *Supposing that the Earth is a sphere surrounded by a sea of small depth: required to determine whether the form of the sea attracted by the Moon will be spheroidal, the Earth and Moon being both supposed held at rest.*

557. If the spheroid be the form of equilibrium it must evidently be prolate, the axis of the spheroid passing through the Moon. Let c' be the distance between the Earth and Moon, a the mean radius of the Earth, a' the radius of the solid nucleus of the Earth, r and r' the distances of any particle (xyz) from the Earth's centre and the Moon, E and M the masses of the Earth and Moon, ρ and ρ' the mean density of the Earth and the density of the sea, ϵ the ellipticity of the spheroidal figure of the ocean caused by the Moon.

Then the excess of attraction of the nucleus above what it would be supposing it of the density of the sea, gives the following resolved forces on a particle (xyz) parallel to the axes

$$-\frac{4\pi(\rho-\rho')a'^3}{3}\frac{x}{r^3}, \quad -\frac{4\pi(\rho-\rho')a'^3}{3}\frac{y}{r^3}, \quad -\frac{4\pi(\rho-\rho')a'^3}{3}\frac{z}{r^3}$$

The attraction of a whole fluid spheroid gives

$$-\tfrac{4}{3}\pi\rho'(1+\tfrac{2}{5}\epsilon)\,x, \quad -\tfrac{4}{3}\pi\rho'(1+\tfrac{2}{5}\epsilon)\,y, \quad -\tfrac{4}{3}\pi\rho'(1-\tfrac{4}{5}\epsilon)\,z.$$

The difference of the attraction of the Moon on the particle and centre of the Earth gives

$$-\frac{Mx}{r'^3},\quad -\frac{My}{r'^3},\quad -\frac{M(z-c')}{r'^3}-\frac{M}{c'^2}.$$

Adding the respective forces together and substituting in the equation

$$Xdx + Ydy + Zdz = 0 \text{ (Art. 521)},$$

we have

$$\frac{4\pi(\rho-\rho')a'^3}{3}\,\frac{xdx+ydy+zdz}{r^3}$$

$$+\tfrac{4}{3}\pi\rho'(1+\tfrac{2}{5}\epsilon)(xdx+ydy)+\tfrac{4}{3}\pi\rho'(1-\tfrac{4}{5}\epsilon)zdz$$

$$+\frac{M}{r'^3}\{xdx+ydy+(z-c')dz\}+\frac{Mdz}{c'^2}=0.$$

But $r^2 = x^2+y^2+z^2$ and $r'^2 = x^2+y^2+(z-c')^2$.

Hence substituting and integrating,

$$-\frac{4\pi(\rho-\rho')a'^3}{3}\frac{1}{r}+\frac{2\pi\rho'}{3}\left(1+\frac{2\epsilon}{5}\right)(x^2+y^2)$$

$$+\frac{2\pi\rho'}{3}\left(1-\frac{4\epsilon}{5}\right)z^2-\frac{M}{r'}+\frac{Mz}{c'^2}=\text{constant}.$$

Now if b be the semi-axis minor of the spheroid, then $b^3(1+\epsilon)=a^3$, and $b=a(1-\frac{1}{3}\epsilon)$, and the equation to the spheroid is

$$x^2+y^2+(1-2\epsilon)z^2=b^2=a^2(1-\tfrac{2}{3}\epsilon);$$

$$\therefore\ \frac{1}{r}=(x^2+y^2+z^2)^{-\frac{1}{2}}=\frac{1}{a}\left(1-\frac{2\epsilon}{3}+2\epsilon\frac{z^2}{a^2}\right)^{-\frac{1}{2}}=a\left(1+\frac{\epsilon}{3}-\frac{\epsilon z^2}{a^2}\right)$$

$$\frac{1}{r'}=(x^2+y^2+z^2-2zc'+c'^2)^{-\frac{1}{2}}$$

$$=\frac{1}{c'}\left(1-\frac{2z}{c'}+\frac{x^2+y^2+z^2}{c'^2}\right)^{-\frac{1}{2}}=\frac{1}{c'}\left(1+\frac{z}{c'}-\frac{x^2+y^2-2z^2}{2c'^2}\right).$$

Substituting these in the above equation to the surface of the ocean, we have

$$(x^2+y^2)\left\{\frac{2\pi\rho'}{3}\left(1+\frac{2\epsilon}{5}\right)+\frac{M}{2c'^3}\right\}$$

$$+z^2\left\{\frac{2\pi\rho'}{3}\left(1-\frac{4\epsilon}{5}\right)+\frac{4\pi(\rho-\rho')a'^3\epsilon}{3a^3}-\frac{M}{c'^3}\right\}=\text{constant}.$$

Then in order that this may coincide with the equation $x^2+y^2+(1-2\epsilon)z^2=\text{constant}$, we must have

$$(1-2\epsilon)\left\{\frac{2\pi\rho'}{3}\left(1+\frac{2\epsilon}{5}\right)+\frac{M}{2c'^3}\right\}$$

$$=\frac{2\pi\rho'}{3}\left(1-\frac{4\epsilon}{5}\right)+\frac{4\pi(\rho-\rho')a'^3\epsilon}{3a^3}-\frac{M}{c'^3};$$

$$\therefore\ \epsilon=\frac{\dfrac{3M}{2c'^3}}{\dfrac{8\pi\rho'}{15}+\dfrac{4\pi(\rho-\rho')a'^3}{3a^3}+\dfrac{M}{c'^3}}$$

$$=\frac{\dfrac{3M}{2c'^3}}{\dfrac{8\pi\rho'}{15}+\dfrac{4\pi(\rho-\rho')a'^3}{3a^3}},\ \text{(neglecting } \epsilon^2 \text{ as before).}$$

Now E = mass of the Earth

$$=\tfrac{4}{3}\pi\rho a'^3+\tfrac{4}{3}\pi\rho'(a^3-a'^3)=\tfrac{4}{3}\pi(\rho-\rho')a'^3+\tfrac{4}{3}\pi\rho' a^3;$$

$$\therefore\ \frac{8\pi\rho'}{15}+\frac{4\pi(\rho-\rho')a'^3}{3a^3}=\frac{2E}{5a^3}+\frac{4\pi(\rho-\rho')a'^3}{5a^3}$$

$$=\frac{2E}{5a^3}\left\{1+\frac{2\pi(\rho-\rho')a'^3}{E}\right\};$$

$$\therefore\ \epsilon=\frac{\dfrac{15}{4}\dfrac{M}{E}\left(\dfrac{a}{c'}\right)^3}{1+\dfrac{2\pi a'^3}{E}(\rho-\rho')},$$

$$\frac{a}{c'} = \text{the parallax of the Moon} = \frac{1}{60} \text{ nearly,}$$

$$\frac{M}{E} = \frac{1}{74} \text{ nearly,} \quad E = \frac{4\pi}{3}\rho a^3 \text{ nearly;}$$

$$\therefore \frac{2\pi a'^3}{E}(\rho - \rho') = \frac{3}{2}\left(1 - \frac{\rho'}{\rho}\right) \text{ nearly, since } a = a' \text{ nearly,}$$

$$\rho = 5\rho' \text{ nearly.}$$

These values shew that ϵ will be very small: and that the spheroidal figure may be taken as the form of equilibrium. We must remark that the spheroidal figure of the nucleus has been neglected in this calculation; also the centrifugal force of the particles arising not only from the rotation of the Earth about the centre of gravity of the Earth and Moon; but also that arising from the rotation about its own axis, which is far more important.

558. It is deserving of observation, however, that the greater the mean depth of the ocean is (i. e. the less a' is) the greater is ϵ. This shews that the *depth* of the sea affects the Tides. Also ϵ is greater the greater the density of the sea is in comparison with the mean density of the Earth.

If $\rho' = \rho$ the value of ϵ is more than double what it is when $\rho' = \frac{1}{5}\rho$ (the value in nature). If ρ' be greater than ρ, but a little less than $\frac{4}{3}\rho$ we find $\epsilon = 1$, when the above numerical values are substituted. In this case our solution is not even an approximation: but it shews that if the density of the sea were somewhat greater than the mean density of the Earth the figure of the Ocean would be very prolate and consequently the Tides much greater than they are in nature.

Prop. *To determine the form of equilibrium of the atmosphere of the heavenly bodies.*

559. Let θ be the co-latitude of any particle of the atmosphere, r its distance from the centre of gravity of the spheroid. Then if w be the angular velocity the centrifugal force of the particle is $w^2 r \sin\theta$: and therefore, supposing that

the attraction of the mass of the air upon itself is neglected in comparison with the attraction of the Earth's mass,

$$\frac{dp}{\rho} = -\frac{E}{r^2}\,dr + w^2(x\,dx + y\,dy),$$

if we neglect the ellipticity of the spheroid;

$$\therefore \text{ const.} = \frac{E}{r} + \frac{w^2}{2}r^2\sin^2\theta,\ E = \text{mass of the Earth.}$$

Let R, R' be the polar and equatorial radii of the atmosphere: hence

$$\text{const.} = \frac{E}{R}, \quad \text{const.} = \frac{E}{R'} + \frac{w^2}{2}R'^2;$$

$$\therefore \frac{E}{R} = \frac{E}{r} + \tfrac{1}{2}w^2r^2\sin^2\theta;$$

$$\text{and } \frac{w^2}{2E}R'^3 = \frac{R'-R}{R}.$$

At the equator the centrifugal force equals gravity: therefore $w^2R' = \dfrac{E}{R'^2}$, or $w^2R'^3 = E$, $\therefore R' = \tfrac{3}{2}R$.

It appears also that the equatorial is the greatest radius of the atmosphere: for by differentiating the equation between r and θ

$$\frac{dr}{d\theta} = \frac{w^2r^4\sin\theta\cos\theta}{E - w^2r^3\sin^2\theta};$$

but the centrifugal force resolved in the direction of the radius is $w^2r\sin^2\theta$, and this must not be greater than $\dfrac{E}{r^2}$, and therefore $\dfrac{dr}{d\theta}$ is always positive, or r increases from the pole to the equator.

560. There is but one form of equilibrium. For the equation to the surface of the atmosphere may be written thus:

$$r^3 - \frac{2E}{Rw^2 \sin^2\theta} r + \frac{2E}{w^2 \sin^2\theta} = 0,$$

and, since the last term is positive, one value of r must be negative, and therefore does not serve our purpose: let r_1 and r_2 be the other roots, then since the second term does not appear the third root is $-(r_1 + r_2)$. Now r_1 and r_2 cannot both be less than $\sqrt[3]{\frac{E}{w^2 \sin^2\theta}}$, for then the third root would (disregarding its sign) be less than $2\sqrt[3]{\frac{E}{w^2 \sin^2\theta}}$ and their product would be less than $\frac{2E}{w^2 \sin^2\theta}$: but this is the product of the roots. Hence only one positive value of r is less than $\sqrt[3]{\frac{E}{w^2 \sin^2\theta}}$, or there is only one form of equilibrium.

Laplace draws the following conclusions with respect to the Zodiacal Light from these results (Liv. III. §. 47). The Sun's atmosphere can extend no farther than to the orbit of a planet, of which the periodic revolution is performed in the same time as the Sun's rotatory motion about its axis; or in twenty-five days and a half. Therefore it does not extend so far as the orbits of Mercury and Venus: and we know that the Zodiacal Light extends much beyond them. The ratio of the polar to the equatorial diameter of the solar atmosphere cannot be less than $\frac{2}{3}$: and the Zodiacal Light appears under the form of a very flat lens, the apex of which is in the plane of the solar equator. Therefore the fluid which reflects to us the Zodiacal Light is not the atmosphere of the Sun: and since it surrounds the Sun, it must revolve about it according to the same laws as the planets: perhaps this is the reason why its resistance to their motions is so insensible.

HYDRODYNAMICS.

CHAPTER I.

EQUATIONS OF MOTION.

561. THE equations of the equilibrium of fluids, which we have found in Art. 521, are deduced from the characteristic property of fluids, both incompressible and elastic, *viz.* the equable transmission in every direction of pressures applied at their surface. This property arises from the fact, that the molecules of the fluid when compressed or dilated rapidly assume the same relative positions that they previously had. The time that the particles occupy in passing into this state has no influence on the laws of equilibrium, since these are observed only after the fluid has attained its state of rest. But this time, small as it may be, must influence the laws of motion of fluids, so that the principle of the equality of pressure in every direction is true in Hydrostatics, but is not always applicable in Hydrodynamics; this is Poisson's view of the subject, *Traité de Mécanique*.

Laplace remarked an analogous difference in the state of rest and motion of fluids relative to Mariotte's law. This law, which teaches that the density of an elastic fluid varies as the pressure, requires that the temperature of the fluid should become the same after the change in volume that it was before. It is ascertained that heat is given out or absorbed when a volume of air is suddenly compressed or dilated; and in this way the elasticity of the air is greatly modified by the nature of the motion. This circumstance introduces into the equa-

tions of motion terms, which cannot be deduced from the equations of equilibrium. In the present work we shall suppose, as is ordinarily made the supposition, that the equality of pressure holds equally in the state of rest and motion of fluids; when we adopt this hypothesis the equations of equilibrium conduct immediately to those of the motion of fluids.

PROP. *To determine the equations of motion of a mass of fluid, the molecules of which are acted on by given forces.*

562. Let xyz be the co-ordinates to any molecule at the time t, and X, Y, Z the sums of the resolved parts of the accelerating forces which act upon this molecule parallel to the axes of co-ordinates respectively. Now in accordance with the Principle enunciated in Art. 224, the acceleration of the molecules would cease if that at the point (xyz) were acted on by the accelerating forces

$$X - \frac{d^2x}{dt^2}, \quad Y - \frac{d^2y}{dt^2}, \quad Z - \frac{d^2z}{dt^2},$$

and all the other molecules acted on simultaneously by similar forces.

Hence if p be the pressure at the point (xyz) at the time t referred to a unit of surface and ρ be the density, we have by the equation of equilibrium of a fluid mass

$$\frac{\delta p}{\rho} = \left(X - \frac{d^2x}{dt^2}\right)\delta x + \left(Y - \frac{d^2y}{dt^2}\right)\delta y + \left(Z - \frac{d^2z}{dt^2}\right)\delta z,$$

where the differentials δx, δy, δz do not refer to the motion, but are arbitrary; and may therefore be taken equal to the differentials of the spaces described by the molecule parallel to the axes. Hence

$$\frac{1}{\rho}\frac{dp}{dx} = X - \frac{d^2x}{dt^2}, \quad \frac{1}{\rho}\frac{dp}{dy} = Y - \frac{d^2y}{dt^2}, \quad \frac{1}{\rho}\frac{dp}{dz} = Z - \frac{d^2z}{dt^2}.$$

Now let u, v, w be the velocities of the molecule (xyz) parallel to x, y, z respectively at the time t. Then each of these will be a function of the time and the position of the molecule;

$$\therefore \frac{d^2x}{dt^2} = \frac{du}{dt} = \left(\frac{du}{dt}\right) + \frac{du}{dx}u + \frac{du}{dy}v + \frac{du}{dz}w,$$

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \left(\frac{dv}{dt}\right) + \frac{dv}{dx}u + \frac{dv}{dy}v + \frac{dv}{dz}w,$$

$$\frac{d^2z}{dt^2} = \frac{dw}{dt} = \left(\frac{dw}{dt}\right) + \frac{dw}{dx}u + \frac{dw}{dy}v + \frac{dw}{dz}w.$$

Hence the three equations involving p become

$$\frac{1}{\rho}\frac{dp}{dx} = X - \left(\frac{du}{dt}\right) - \frac{du}{dx}u - \frac{du}{dy}v - \frac{du}{dz}w,$$

$$\frac{1}{\rho}\frac{dp}{dy} = Y - \left(\frac{dv}{dt}\right) - \frac{dv}{dx}u - \frac{dv}{dy}v - \frac{dv}{dz}w,$$

$$\frac{1}{\rho}\frac{dp}{dz} = Z - \left(\frac{dw}{dt}\right) - \frac{dw}{dx}u - \frac{dw}{dy}v - \frac{dw}{dz}w.$$

These are three equations connecting the five unknown quantites u, v, w, p, ρ which we wish to determine in terms of x, y, z, and t.

563. Two more equations will be furnished by the following consideration. Suppose we consider the motion of the molecules, which at the instant t form an indefinitely small parallelopiped with its sides parallel to the co-ordinate planes. The various molecules will change their situation, and we can determine the volume of the figure which they form after a short time: but since the number of molecules remains the same the volume which these molecules occupy when multiplied by the density of the fluid must remain the same during the motion.

Let δx, δy, δz be the sides of the parallelopiped at the time t: m the molecule nearest the origin; n the molecule immediately over this. Then at time $t + \delta t$ the co-ordinates of m are

$$x + u\delta t, \quad y + v\delta t, \quad z + w\delta t,$$

and the co-ordinates of n change from x, y, $z + \delta z$ to

$$x + u'\delta t, \quad y + v'\delta t, \quad z + \delta z + w'\delta t,$$

u', v', w' being the values of u, v, w at n;

$$\therefore\ u' = u + \frac{du}{dz}\delta z, \quad v' = v + \frac{dv}{dz}\delta z, \quad w' = w + \frac{dw}{dz}\delta z.$$

Hence the co-ordinates of n at the time $t + \delta t$ are

$$x + u\delta t + \frac{du}{dz}\delta z\delta t, \quad y + v\delta t + \frac{dv}{dz}\delta z\delta t, \text{ and}$$

$$z + \delta z + w\delta t + \frac{dw}{dz}\delta z\delta t;$$

and the distance between m and n at that time is

$$\sqrt{\frac{du^2}{dz^2}\delta z^2\delta t^2 + \frac{dv^2}{dz^2}\delta z^2\delta t^2 + \left(\delta z + \frac{dw}{dz}\delta z\delta t\right)^2} = \delta z + \frac{dw}{dz}\delta z\delta t,$$

neglecting small quantities of the third order.

Now let us consider the two angular points of the parallelopiped which lie in the same diagonal plane with m and n: we shall call them m' and n': we shall obtain their distance at the time $t + \delta t$ from that of m and n by putting $x + \delta x$, $y + \delta y$ for x and y: then the distance between m' and n' is

$$\delta z + \left\{\frac{dw}{dz} + \frac{d^2w}{dz\,dx}\delta x + \frac{d^2w}{dz\,dy}\delta y\right\}\delta z\delta t = \delta z + \frac{dw}{dz}\delta z\delta t.$$

Hence the distance between m' and n' is the same as that between m and n, when we neglect small quantities of the third order. In the same manner it may be shewn, that all the edges of the parallelopiped which are parallel at time t are equal to each other at time $t + \delta t$ and therefore still form a parallelopiped, the sides being

$$\delta z + \frac{dw}{dz}\delta z\delta t, \quad \delta y + \frac{dv}{dy}\delta y\delta t, \quad \delta x + \frac{du}{dx}\delta x\delta t.$$

Also the density at the time $t + \delta t$ is

$$\rho + \left\{\frac{d\rho}{dt} + \frac{d\rho}{dx}u + \frac{d\rho}{dy}v + \frac{d\rho}{dz}w\right\}\delta t,$$

and therefore the mass of the parallelopiped, which at time t is $\rho\delta x\delta y\delta z$, becomes at time $t+\delta t$,

$$(\rho + \frac{d\rho}{dt}\delta t + \frac{d\rho}{dx}u\delta t + \frac{d\rho}{dy}v\delta t + \frac{d\rho}{dz}w\delta t)$$

$$\times (1 + \frac{du}{dx}\delta t)(1 + \frac{dv}{dy}\delta t)(1 + \frac{du}{dz}\delta t)\delta x\delta y\delta z.$$

Equating these expressions, dividing by $\delta x\delta y\delta z\delta t$, and then taking the limit, we have

$$0 = \rho\left\{\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right\} + \frac{d\rho}{dt} + \frac{d\rho}{dx}u + \frac{d\rho}{dy}v + \frac{d\rho}{dz}w.$$

This equation is called the *Equation of the continuity of the fluid;* since it expresses analytically the relation between the velocity of the molecules and the density of the fluid, which are necessarily dependent on each other.

564. If the fluid be incompressible then the variation of ρ equals zero, and the above equation gives two

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0,$$

$$\text{and } \frac{d\rho}{dt} + \frac{d\rho}{dx}u + \frac{d\rho}{dy}v + \frac{d\rho}{dz}w = 0;$$

these complete the five equations for computing u, v, w, p, ρ in terms of x, y, z, t when the fluid is incompressible.

When the fluid is homogeneous and incompressible then ρ is constant throughout the fluid and given in value: and therefore the last equation becomes identical.

565. If the fluid be compressible the fourth and fifth equations are

$$0 = \rho\left\{\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right\} + \frac{d\rho}{dt} + \frac{d\rho}{dx}u + \frac{d\rho}{dy}v + \frac{d\rho}{dz}w;$$

$$\text{and } 0 = F(\rho, p),$$

the function F depending upon the nature of the fluid.

The equations admit of great simplification in the case of an incompressible homogeneous fluid mass, when $u\,dx + v\,dy + w\,dz$ is a perfect differential.

PROP. *To find the pressure at any point of a homogeneous and incompressible fluid mass in motion.*

566. Assuming $u\,dx + v\,dy + w\,dz = d\phi$ a perfect differential, we have

$$u = \frac{d\phi}{dx}, \quad v = \frac{d\phi}{dy}, \quad w = \frac{d\phi}{dz}.$$

Hence the equation of the continuity of the fluid becomes

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

By the integration of this equation ϕ is to be found. We shall proceed to eliminate u, v, w from the equations involving p.

$$\frac{du}{dt}dx + \frac{dv}{dt}dy + \frac{dw}{dt}dz = \frac{d^2\phi}{dt\,dx}dx + \frac{d^2\phi}{dt\,dy}dy + \frac{d^2\phi}{dt\,dz}dz$$

$$= \frac{d\frac{d\phi}{dt}}{dx}dx + \frac{d\frac{d\phi}{dt}}{dy}dy + \frac{d\frac{d\phi}{dt}}{dz}dz = d\frac{d\phi}{dt},$$

$$\frac{du}{dx}dx + \frac{dv}{dx}dy + \frac{dw}{dx}dz = d\frac{d\phi}{dx} \text{ in the same way}$$

$$\frac{du}{dy}dx + \frac{dv}{dy}dy + \frac{dw}{dy}dz = d\frac{d\phi}{dy}$$

$$\frac{du}{dz}dx + \frac{dv}{dz}dy + \frac{dw}{dz}dz = d\frac{d\phi}{dz}.$$

Now let the three equations in p be multiplied respectively by dx, dy, dz and added together;

$$\therefore \frac{dp}{\rho} = X\,dx + Y\,dy + Z\,dz$$

$$- d\frac{d\phi}{dt} - \frac{d\phi}{dx} d\frac{d\phi}{dx} - \frac{d\phi}{dy} d\frac{d\phi}{dy} - \frac{d\phi}{dz} d\frac{d\phi}{dz}$$

$$= Xdx + Ydy + Zdz - d\frac{d\phi}{dt} - \tfrac{1}{2} d\left\{\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right\}.$$

567. Cor. When the excursions of the molecules are small we may neglect the squares of the velocities and the equation becomes

$$\frac{dp}{\rho} = Xdx + Ydy + Zdz - d\frac{d\phi}{dt}.$$

Prop. *To prove that if* $udx + vdy + wdz$ *be a perfect differential at any instant it is so during the whole time of the motion.*

568. For at the time $t + \delta t$ the value of $udx + vdy + wdz$ becomes

$$udx + vdy + wdz + \left\{\frac{du}{dt} dx + \frac{dv}{dt} dy + \frac{dw}{dt} dz\right\} \delta t$$

$$= d\phi + d\frac{d\phi}{dt} \delta t = d\phi + (Xdx + Ydy + Zdz)\, \delta t - \frac{dp}{\rho} \delta t$$

$$- \frac{\delta t}{2} d\left\{\frac{d\phi^2}{dx^2} + \frac{d\phi^2}{dy^2} + \frac{d\phi^2}{dz^2}\right\};$$

wherefore, if $udx + vdy + wdz$ be a perfect differential $d\phi$ at the time t, it will be so also at the time $t + \delta t$, and will consequently be so throughout the motion.

Prop. *To determine equations for calculating the motion of an elastic fluid, the excursions of the molecules being supposed small, no extraneous forces acting.*

569. We shall suppose $udx + vdy + wdz = d\phi$, then the equation of continuity is

$$\rho\left\{\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right\} + \frac{d\rho}{dt} = 0,$$

neglecting small quantities of the second order;

$$\therefore \frac{d \log_e \rho}{dt} + \frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

We shall suppose that $p = a^2\rho$, which is the law of nature if the motion be so slow as not to absorb or develop heat. Also since the excursions are small we have, as in Art. 567,

$$\frac{dp}{\rho} = -d\frac{d\phi}{dt};$$

$$\therefore a^2 \frac{d \log_e \rho}{dt} = -\frac{d^2\phi}{dt^2};$$

$$\therefore \frac{d^2\phi}{dt^2} = a^2 \left\{\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2}\right\}.$$

From this equation ϕ must be determined and then p and ρ calculated by the equations given above.

570. For the Theories of Sound and Light which depend upon the principles in this Chapter we must refer the reader to two Articles on those subjects in the *Encyclopædia Metropolitana*, by Sir John W. F. Herschel, and to Mr Airy's *Tracts*. We shall proceed with applications of the principles more peculiarly adapted to the nature of the present work, viz. to the Tides and the Stability of the Ocean.

CHAPTER II.

TIDES AND THE STABILITY OF THE OCEAN.

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571. THERE remains yet another phenomenon, which is evidently connected with the mechanism of the Solar System, the Tides of our Ocean. In the calmest weather the vast body of waters that wash our coasts advances on our shores, inundating all the flat sands, rising to a considerable height, and then as gradually retiring to their former level; and all this without any visible cause to impel the waters to our shores or again to draw them off. Twice every day is this repeated. In many places this motion of the waters is tremendous, the sea advancing, even in the calmest weather, with a high surge, rolling along the flats with resistless violence, and rising to the height of many fathoms.

572. In searching for the cause of this remarkable phenomenon philosophers readily conceived, that, since the Sun and Moon each cross the meridian twice in twenty-four hours, these bodies may by their attraction influence the waters of the ocean. Accordingly various theories have been adopted for the calculation of the tides upon this hypothesis of lunar and solar attraction, of which the most noted have been those of D. Bernoulli and Laplace. If the hypothesis of universal gravitation be adopted, there can be but one correct theory based upon it for calculating the oscillations of the Ocean; but in consequence of the difficulties of the analysis, which have hitherto been insurmountable, other hypotheses must be resorted to in addition to that of gravitation in order to obtain an approximate solution of the problem. The irregularity of the depth of the Ocean, the manner in which it is spread over the earth, the position and declivity

of the shores, and their connections with the adjoining coasts, the currents, and the resistances which the waters suffer, cannot possibly be subjected to an accurate calculation, though these causes modify the oscillations of the great fluid mass. All we can do is to analyze the general phenomena, which must result from the attractions of the Sun and Moon, and to deduce from the observations such data as are indispensable for completing in each port the theory of the ebb and flow of the tides. These data are the arbitrary quantities, depending on the extent of the surface of the sea, its depth, and the local circumstances of the port. In the absence of these data we must resort to the best expedients that can be found.

Bernoulli, in his theory, assumed that the attraction of the Moon causes the Ocean to assume at every instant the form it would have if the Earth and Moon were stationary. It is found, by calculating the tides upon this hypothesis, supposing the pole of the prolate spheroid (which is the form of equilibrium nearly, see Art. 557.) to lag behind the Moon, that this hypothesis gives results, which accord very well with observations in some of the more ordinary phenomena of the tides. This theory is termed by Mr Whewell the *Equilibrium Theory*. Laplace, however, has taken a different course. He calculates the attractive forces of the Sun and Moon upon the Ocean, and finds them to contain constant terms and periodical terms. He states, that in consequence of the resistance and friction of the waters they would soon have assumed a form of equilibrium under the forces which are represented by the constant terms: and then, assuming this as a general dynamical principle, that the state of a system of bodies, in which the primitive conditions of the motion have disappeared by the resistances it suffers, is periodical when the forces themselves are periodical, he obtains an expression for the height of the tide the same as that obtained from the Equilibrium Theory of Bernoulli. But there are so many assumptions in this, that we may, as far as we know *à priori*, as readily adopt the equilibrium theory as Laplace's: we must test their accuracy by comparing their results with observations. With this laborious task many

calculators and observers are at this time employed under the superintendence of Mr Whewell and Mr Lubbock: and we must look to the general empiric laws to be deduced from the enormous mass of observations, which is in the progress of accumulation to guide us in adopting such proper hypotheses as shall bring the subject under the dominion of analysis without materially vitiating the rigour of the approximation. A most interesting Essay *towards a first approximation to a map of cotidal lines* by Mr Whewell will be found in the *Philosophical Transactions* for 1833. In the present state of our knowledge of the tides we are constrained to confess that the laws we possess are only empiric. All we shall attempt in this work will be to obtain the formulæ for the calculation of the tides upon the Equilibrium Theory of Bernoulli.

PROP. *To calculate the height of the tide at any place at a given time upon Bernoulli's hypothesis.*

573. Let c, $c(1+\epsilon)$ be the semi-axes of the prolate spheroid into which the Moon attracts the Ocean: a the radius of the sphere the volume of which equals that of the Earth: h the elevation of the pole of the spheroid above the mean level of the sea: r the distance of any point of the surface from the centre of the ocean, and ϕ the angle r makes with the axis of the spheroid: n the angular velocity of rotation of the Earth about its axis: ω, $\omega_{\prime}$ the right ascensions of the point on the surface and of the Moon: θ, $\theta_{\prime}$ their north polar distances. Hence we have

$$c^3(1+\epsilon) = a^3, \quad h = c(1+\epsilon) - a;$$

$$\therefore c = a\left(1 - \tfrac{1}{3}\epsilon\right), \quad h = \tfrac{2}{3}a\epsilon; \quad \therefore a\epsilon = \tfrac{3}{2}h, \quad c = a - \tfrac{1}{2}h.$$

$$\text{Also } r = c(1 + \epsilon\cos^2\phi)$$

$$= a - \tfrac{1}{2}h + \tfrac{3}{2}h\cos^2\phi;$$

therefore height of tide at the place of which θ and ω are the co-ordinates at the previous noon

$$= r - a = \tfrac{1}{2}h(3\cos^2\phi - 1).$$

Let λ = the difference between the right ascensions of the Moon and the pole of the tidal spheroid (see Art. 572), then $nt + \omega - \omega_{,} - \lambda$ is the hour angle of the pole, t being reckoned from noon;

$$\therefore \cos\phi = \cos\theta\cos\theta_{,} + \sin\theta\sin\theta_{,}\cos(nt + \omega - \omega_{,} - \lambda);$$

therefore height of tide =

$$\tfrac{3}{2}h\left\{\left[\cos\theta\cos\theta_{,} + \sin\theta\sin\theta_{,}\cos(nt + \omega - \omega_{,} - \lambda)\right]^2 - \tfrac{1}{3}\right\}.$$

Then considering $\theta_{,}$, the north polar distance, nearly 90°, we shall neglect $\cos\theta'$. Also we shall put

$$1 + \cos 2(nt + \omega - \omega_{,} - \lambda) \text{ for } 2\cos^2(nt + \omega - \omega_{,} - \lambda);$$

therefore height of tide =

$$\tfrac{3}{4}h\left\{\sin^2\theta\sin^2\theta_{,} - \tfrac{2}{3} + \sin^2\theta\sin^2\theta_{,}\cos 2(nt + \omega - \omega_{,} - \lambda)\right\},$$

and since during a day $\theta_{,}$ remains nearly constant, we have

$$\text{change in height} = \tfrac{3}{4}h\sin^2\theta\sin^2\theta_{,}\cos 2(nt + \omega - \omega_{,} - \lambda).$$

Supposing accented letters to apply to the Sun in the same way that the unaccented letters apply to the Moon, we have the whole variation in the height of the tide arising from the combined action of these two luminaries =

$$\tfrac{3}{4}h\sin^2\theta\sin^2\theta_{,}\cos 2(nt + \omega - \omega_{,} - \lambda) + \tfrac{3}{4}h'\sin^2\theta\sin^2\theta_{,}'\cos 2(nt + \omega - \omega_{,}' - \lambda').$$

Prop. *To find the time of High Tide at a given place.*

574. The height of the tide at a given place (by the last Article)

$$= \tfrac{3}{4}\sin^2\theta\left\{h\sin^2\theta_{,}\cos 2(nt + \omega - \omega_{,} - \lambda) + h'\sin^2\theta_{,}'\cos 2(nt + \omega - \omega_{,}' - \lambda')\right\},$$

and when the tide is full the differential coefficient of this vanishes. We shall suppose the angular velocities of the Sun and Moon to be the same, or $\dfrac{d(\omega_{,} + \lambda)}{dt} = \dfrac{d(\omega_{,}' + \lambda')}{dt}$;

$$\therefore\ 0 = h\sin^2\theta_{,}\sin 2(nt+\omega-\omega_{,}-\lambda)$$
$$+ h'\sin^2\theta_{,}'\sin 2(nt+\omega-\omega_{,}'-\lambda'),$$

or $$0 = h\sin^2\theta_{,}\sin 2(nt+\omega-\omega_{,}-\lambda)$$
$$+ h'\sin^2\theta_{,}'\sin 2(nt+\omega-\omega_{,}-\lambda+\omega_{,}-\omega_{,}'+\lambda-\lambda');$$

$$\therefore\ \tan 2(nt+\omega-\omega_{,}-\lambda) = \frac{h'\sin^2\theta_{,}'\sin 2(\omega_{,}'-\omega_{,}+\lambda'-\lambda)}{h\sin^2\theta_{,}+h'\sin^2\theta_{,}'\cos 2(\omega_{,}'-\omega_{,}+\lambda'-\lambda)}.$$

It will be seen by referring to Art. 295, that the force of the Sun on the Ocean varies as $\frac{S}{r'^3}$ and that of the Moon as $\frac{M}{r^3}$:

$$\text{hence } \frac{h'}{h} = \frac{S}{M}\frac{r^3}{r'^3} = \frac{S}{M}\frac{\Pi'^3}{\Pi^3},$$

Π, Π' being the parallax of the Moon and Sun;

$$\therefore \tan 2(nt+\omega-\omega_{,}-\lambda) = \frac{\frac{S}{M}\left(\frac{\Pi'}{\Pi}\right)^3\sin^2\theta_{,}'\sin 2(\omega_{,}'-\omega_{,}+\lambda'-\lambda)}{\sin^2\theta_{,}+\frac{S}{M}\left(\frac{\Pi'}{\Pi}\right)^3\sin^2\theta_{,}'\cos 2(\omega_{,}'-\omega_{,}+\lambda'-\lambda)}.$$

This expression shews, that the time of the Moon's meridian passage precedes the High Tide by an interval, which is not the same for all ages of the Moon.

The mean of all these intervals is λ: and $nt+\omega-\omega_{,}-\lambda$ is the excess of any interval above the mean: and $\lambda'-\lambda$ is the time of the Moon's meridian passage when λ (the mean) is the interval of time between that event and high tide. We may remark, that λ will be subject to much smaller variations than λ', because the action of the Moon on the Ocean is much greater than that of the Sun.

The value of the interval, at any port, when the Moon is full or new is called the *Establishment of the Port*.

Let $\theta_{,} = \theta_{,}' = 90^0$ or the Sun and Moon be supposed in the equator. Then the above formula leads to the following table as the result of calculation compared with good observations of

the time of high water made at London Docks. See *Companion to the British Almanack*, 1830.

Time of Moon's Meridian Passage $\varpi_i - \varpi_i'$.	Time by which the Moon's Meridian Passage precedes the time of High Water. Observed.		Calculated.	
h.	h.	m.	h.	m.
0	2	2	2	0
1	1	47	1	47
2	1	32	1	32
3	1	18	1	17
4	1	5	1	4
5	0	55	0	55
6	0	52	0	54
7	1	43	1	6
8	1	32	1	32
9	1	59	1	58
10	2	9	2	10
11	2	10	2	9

The mean of the observed results gives $\lambda = 1$ hour, 32 minutes; and this (by the Table) corresponds with 2 hours for the time of the Moon's passage: and $\therefore \lambda' - \lambda = 2$ hours. Hence the greatest tide is $\frac{1}{12}$th of a month, or nearly $2\frac{1}{2}$ days, after new and full Moon in the Port of London.

PROP. *To calculate the tide at a Port at which the tidal wave arrives by two distinct routs.*

575. We shall consider the action of the Moon only. Let $n - \dfrac{d\varpi_i}{dt} - \dfrac{d\lambda}{dt} = m$: T the time of transmission up the

first channel, then at the time t the tide at the port produced by the tidal wave up the first channel

$$= \frac{DM}{r^3} \cos 2\,(nt + \omega - \omega_{,} - \lambda - mT)$$

D depending upon the height of this wave when it was at the mouth of the channel. T' the time the tidal wave takes to move to the second mouth and up the second channel: then the tide at the time t at the port arising from this second wave

$$= \frac{EM}{r^3} \cos 2\,(nt + \omega - \omega_{,} - \lambda - mT')$$

E depending upon the height of the tidal wave when it reached the mouth of the second channel.

Hence the height at the port

$$= \frac{M}{r^3}\{D\cos 2(nt+\omega-\omega_{,}-\lambda-mT)+E\cos 2(nt+\omega-\omega_{,}-\lambda-mT')\}$$

$$= \frac{MF}{r^3} \cos 2\,(nt + \omega - \omega_{,} - \lambda - G),$$

$$\text{where } F^2 = D^2 + E^2 + 2DE \cos 2m\,(T' - T),$$

$$\text{and } \sin 2G = \frac{D}{F} \sin 2mT + \frac{E}{F} \sin 2mT'.$$

Hence F and G depend upon m and therefore on the rapidity of the Moon.

If $F = 0$, that is, if $T = T'$ and $D = -E$, or if it be high water at one mouth when it is low water at the other, and if the tides require the same time to reach the port after the great tidal wave has reached the first mouth, then there will be a complete interference at the port, or no tide at all, if we consider the height of the two poles of the spheroid above the mean level of the Ocean to be the same. Now this is not strictly the case, the height of the pole furthest from the Moon is less (as might be shewn by a nearer approximation in Art. 557) than the other. Hence there will only be one ebb and one flow in twenty-four hours, and that very small. This

singular fact has been observed at Batsham, a port of the kingdom of Tonquin, $20^0\ 50'$ north latitude. The two waves seem to come by two channels which run, one from the China seas between the continent and the island Luconia, the other from the Indian sea between the continent and the island of Borneo. (Principia, Tom. III. Prop. 24).

Prop. *To determine equations for calculating the motion of the surface of an incompressible fluid mass surrounding a body nearly spherical, the body having a uniform rotatory motion about a fixed axis, and the fluid being supposed to be deranged but very little from the state of equilibrium by very small forces.*

576. We shall refer the fluid mass to polar co-ordinates. Let the axis of z be the axis of rotation, n the angular velocity: $r_{\prime}$ the distance of any molecule at the commencement of t from the centre of gravity of the body which the fluid covers; this centre of gravity we shall suppose at rest (see Art. 430); $\theta_{\prime}$ the angle between $r_{\prime}$ and z, $\omega_{\prime}$ the angle between the planes rz and xz at the commencement of t. Suppose that at the end of the time t, $r_{\prime}$, $\theta_{\prime}$, $\omega_{\prime}$ are become $r_{\prime} + ar$, $\theta_{\prime} + a\theta$, $nt + \omega_{\prime} + a\omega$, a being a very small fraction of which the square and higher powers may be neglected;

$$\therefore\ x = (r_{\prime} + ar) \sin (\theta_{\prime} + a\theta) \cos (nt + \omega_{\prime} + a\omega),$$

$$y = (r_{\prime} + ar) \sin (\theta_{\prime} + a\theta) \sin (nt + \omega_{\prime} + a\omega),$$

$$z = (r_{\prime} + ar) \cos (\theta_{\prime} + a\theta.$$

We shall substitute the expressions in the equation of Art. 562, neglecting a^2......

$$\frac{dx}{dt} = a\left\{\sin\theta_{\prime} \cos (nt + \omega_{\prime}) \frac{dr}{dt} + r_{\prime} \cos\theta_{\prime} \cos (nt + \omega_{\prime}) \frac{d\theta}{dt}\right.$$

$$\left. - r_{\prime} \sin\theta_{\prime} \sin (nt + \omega_{\prime}) \frac{d\omega}{dt}\right\} - nr_{\prime} \sin\theta_{\prime} \sin (nt + \omega_{\prime}),$$

$$\frac{d^2x}{dt^2} = a\left\{\sin\theta_{\prime} \cos (nt + \omega_{\prime}) \frac{d^2r}{dt^2} + r_{\prime} \cos\theta_{\prime} \cos (nt + \omega_{\prime}) \frac{d^2\theta}{dt^2}\right.$$

$$- r_{,} \sin\theta_{,} \sin(nt + \omega_{,}) \frac{d^2\omega}{dt^2} - n \sin\theta_{,} \sin(nt + \omega_{,}) \frac{dr}{dt}$$

$$\left. - nr_{,} \cos\theta_{,} \sin(nt + \omega_{,}) \frac{d\theta}{dt} - nr_{,} \sin\theta_{,} \cos(nt + \omega_{,}) \frac{d\omega}{dt} \right\}$$

$$- n^2 r_{,} \sin\theta_{,} \cos(nt + \omega_{,}).$$

From this we can obtain $\dfrac{d^2y}{dt^2}$ by putting $\omega_{,} + nt + a\omega - \frac{1}{2}\pi$ for $\omega_{,} + nt + a\omega$, and $\dfrac{d^2z}{dt^2}$ by putting $\omega_{,} + nt + a\omega = 0$ and $\theta_{,} + a\theta + \frac{1}{2}\pi$ for $\theta_{,} + a\theta$;

$$\therefore \frac{d^2y}{dt^2} = a \left\{ \sin\theta_{,} \sin(nt + \omega_{,}) \frac{d^2r}{dt^2} + r_{,} \cos\theta_{,} \sin(nt + \omega_{,}) \frac{d^2\theta}{dt^2} \right.$$

$$+ r_{,} \sin\theta_{,} \cos(nt + \omega_{,}) \frac{d^2\omega}{dt^2} + n \sin\theta_{,} \cos(nt + \omega_{,}) \frac{dr}{dt}$$

$$\left. + nr_{,} \cos\theta_{,} \cos(nt + \omega_{,}) \frac{d\theta}{dt} - nr_{,} \sin\theta_{,} \sin(nt + \omega_{,}) \frac{d\omega}{dt} \right\}$$

$$- n^2 r_{,} \sin\theta_{,} \sin(nt + \omega_{,}),$$

$$\frac{d^2z}{dt^2} = a \left\{ \cos\theta_{,} \frac{d^2r}{dt^2} - r_{,} \sin\theta_{,} \frac{d^2\theta}{dt^2} - nr_{,} \cos\theta_{,} \frac{d\omega}{dt} \right\} - n^2 r_{,} \cos\theta_{,}.$$

Hence, making the substitutions and putting $X\delta x + Y\delta y + Z\delta z = \delta V$ for the attractions, we have

$$\delta V - \frac{\delta p}{\rho} = \frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z$$

$$= ar_{,}^2 \delta\theta_{,} \left\{ \frac{d^2\theta}{dt^2} - 2n \sin\theta_{,} \cos\theta_{,} \frac{d\omega}{dt} \right\}$$

$$+ ar_{,}^2 \delta\omega_{,} \left\{ \sin^2\theta_{,} \frac{d^2\omega}{dt^2} + 2n \sin\theta_{,} \cos\theta_{,} \frac{d\theta}{dt} + \frac{2n}{r_{,}} \sin^2\theta_{,} \frac{dr}{dt} \right\}$$

$$+ a\delta r_{,} \left\{ \frac{d^2r}{dt^2} - 2nr_{,} \sin^2\theta_{,} \frac{d\omega}{dt} \right\} - \frac{n^2}{2} \delta \left\{ (r_{,} + ar) \sin(\theta_{,} + a\theta) \right\}^2.$$

At the external surface of the fluid we have $\delta p = 0$: moreover, in the state of equilibrium, we have

$$0 = \tfrac{1}{2}n^2\delta.\{(r_{,} + ar)\sin(\theta_{,} + a\theta)\}^2 + (\delta V),$$

since r, θ, ω are constant when there is equilibrium: (δV) is the value of δV corresponding to this state, it is therefore the force of gravity multiplied by the element of its direction. Let g be the force of gravity, ay the small elevation of a particle of the fluid above the surface of equilibrium, which we shall consider as the true level of the sea. The variation (δV) will increase by this elevation in the state of motion, by the quantity $-ag\delta y$, since gravity acts nearly in the direction of y towards the origin of that line: then

$$\delta V = (\delta V) - ag\delta y + a\delta V',$$

where $a\delta V'$ is a variation depending upon the new forces which in the state of motion act upon the particle.

Likewise $\frac{1}{2}n^2\delta.\{(r_{,} + ar)\sin(\theta_{,} + a\theta)\}^2$ will be increased by the quantity $an^2\delta y r_{,}\sin^2\theta_{,}$ by means of the elevation of the particle above the level of the sea: this quantity may, however, be neglected in comparison with $-ag\delta y$, because the ratio of centrifugal force to gravity at the equator, which equals $\frac{n^2 r_{,}}{g}$, is a very small fraction, being nearly $\frac{1}{289}$. Lastly, the radius $r_{,}$ is very nearly constant at the surface of the sea, since it differs but slightly from a spherical surface; we may therefore neglect $\delta r_{,}$. Then the equation becomes, at the *surface of the sea*,

$$r_{,}^2\delta\theta_{,}\left\{\frac{d^2\theta}{dt^2} - 2n\sin\theta_{,}\cos\theta_{,}\frac{d\omega}{dt}\right\}$$

$$+ r_{,}^2\delta\omega_{,}\left\{\sin^2\theta_{,}\frac{d^2\omega}{dt^2} + 2n\sin\theta_{,}\cos\theta_{,}\frac{d\theta}{dt}\right\}$$

$$= -g\delta y + \delta V',$$

the variations δy and $\delta V'$ being in reference to $\delta\theta_{,}$ and $\delta\omega_{,}$.

577. We shall now consider the equation relative to the continuity of the fluid.

Volume of the element at the commencement of t,

$$= \delta r_{,} r_{,} \delta\theta_{,} r_{,} \sin\theta_{,} \delta\omega_{,}, \text{ or } r_{,}^{2} \sin\theta_{,} \delta r_{,} \delta\theta_{,} \delta\omega_{,}.$$

And the volume of the same element after a time t,

$$= (r_{,} + ar)^{2} \sin(\theta_{,} + a\theta)\, \delta(r_{,} + ar)\, \delta(\theta_{,} + a\theta)\, \delta(\omega_{,} + a\omega + nt).$$

But since the density of the sea is supposed to remain the same, these volumes must be the same,

$$\therefore (r_{,} + ar)^{2} \sin(\theta_{,} + a\theta)\, \delta(r_{,} + ar)\, \delta(\theta_{,} + a\theta)\, \delta(\omega_{,} + a\omega + nt)$$
$$= r_{,}^{2} \sin\theta_{,} \delta r_{,} \delta\theta_{,} \delta\omega_{,}.$$

We have then, by equating, expanding, and neglecting the squares and higher powers of a,

$$0 = r_{,}^{2} \sin\theta_{,} \left\{\frac{dr}{dr_{,}} + \frac{d\theta}{d\theta_{,}} + \frac{d\omega}{d\omega_{,}}\right\} + r_{,}^{2}\theta\cos\theta_{,} + 2r_{,} r \sin\theta_{,},$$

$$\text{or } 0 = \frac{d \,.\, r_{,}^{2} r}{dr_{,}} + r_{,}^{2} \left\{\frac{\theta\cos\theta_{,}}{\sin\theta_{,}} + \frac{d\omega}{d\omega_{,}} + \frac{d\theta}{d\theta_{,}}\right\}.$$

Let γ be the mean depth of the sea corresponding to $\theta_{,}$ and $\omega_{,}$. Now since the oscillations are small we may assume that all the particles which are on any one radius, will remain on the same radius when $\theta_{,}$, $\omega_{,}$ change to $\theta_{,} + a\theta$ and $\omega_{,} + a\omega + nt$: *i.e.* the relative change of situation of the particles will be chiefly in the direction of the radius vector.

Hence the integral of the above equation is

$$0 = r_{,}^{2} r - (r_{,}^{2} r) + r_{,}^{2} \gamma \left\{\frac{\theta\cos\theta_{,}}{\sin\theta_{,}} + \frac{d\omega}{d\omega_{,}} + \frac{d\theta}{d\theta_{,}}\right\},$$

where $(r_{,}^{2} r)$ is the value of $r_{,}^{2} r$ at the bottom of the sea: and equals $r_{,}^{2}(r)$ nearly; since the change in the radius of the Earth between the bottom and surface of the sea is so small. The mean depth even of the Pacific Ocean is only about $\frac{1}{1000}$th of the radius of the Earth. Wherefore,

$$0 = r - (r) + \gamma \left\{\frac{\theta\cos\theta_{,}}{\sin\theta_{,}} + \frac{d\omega}{d\omega_{,}} + \frac{d\theta}{d\theta_{,}}\right\}.$$

Now the depth of the sea corresponding to $\theta_{,} + a\theta$ and $nt + \theta_{,} + a\omega = \gamma + a\{r - (r)\}$.

$$\text{Also the depth} = \gamma + \frac{d\gamma}{d\theta_{,}} a\theta + \frac{d\gamma}{d\omega_{,}} a\omega + ay,$$

where ay is the elevation of the particle above the mean surface.

$$\therefore\ r - (r) = \frac{d\gamma}{d\theta_{,}}\theta + \frac{d\gamma}{d\omega_{,}}\omega + y;$$

$$\therefore y = -\frac{d.\gamma\theta}{d\theta_{,}} - \frac{d.\gamma\omega}{d\omega_{,}} - \frac{\gamma\theta\cos\theta_{,}}{\sin\theta_{,}} = -\frac{d.\gamma\theta\sin\theta_{,}}{d\theta_{,}}\frac{1}{\sin\theta_{,}} - \frac{d.\gamma\omega}{d\omega_{,}}.$$

Let $\cos\theta_{,} = \mu_{,}$;

$$\therefore\ y = \frac{d.\gamma\theta\sqrt{1-\mu^2_{,}}}{d\mu_{,}} - \frac{d.\gamma\omega}{d\omega_{,}}.$$

By means of this and equation of last Article, we have to determine the oscillations of the Ocean.

Prop. *The depth of the sea being supposed uniform, the Earth to have no rotatory motion, and its figure to be a sphere; required to prove the stability of the Ocean.*

578. By the last Article we have, since γ is constant,

$$y = \gamma\frac{d.\theta\sqrt{1-\mu_{,}^2}}{d\mu_{,}} - \gamma\frac{d\omega}{d\omega_{,}};$$

$$\therefore\ \frac{d^2y}{dt^2} = \gamma\frac{d\left\{\dfrac{d^2\theta}{dt^2}\sqrt{1-\mu_{,}^2}\right\}}{d\mu_{,}} - \gamma\frac{d\dfrac{d^2\omega}{dt^2}}{d\omega_{,}}.$$

But if we put $\cos\theta_{,} = \mu_{,}$ in the equation of Art. 576, and observe that $\delta y = \dfrac{dy}{d\theta_{,}}\delta\theta_{,} + \dfrac{dy}{d\omega_{,}}\delta\omega_{,}$ and $\delta V' = \dfrac{dV'}{d\theta_{,}}\delta\theta_{,} + \dfrac{dV'}{d\omega_{,}}\delta\omega_{,}$, and equate the coefficients of $\delta\theta_{,}$ and also of $\delta\omega_{,}$ in that equation, we obtain (putting $n = 0$,)

$$r_{,}^2\frac{d^2\theta}{dt^2} = -g\frac{dy}{d\theta_{,}} + \frac{dV'}{d\theta_{,}} = g\frac{dy}{d\mu_{,}}\sqrt{1-\mu^2_{,}} - \frac{dV'}{d\mu_{,}}\sqrt{1-\mu_{,}^2};$$

$$r_{,}^2(1-\mu_{,}^2)\frac{d^2\omega}{dt^2} = -g\frac{dy}{d\omega_{,}} + \frac{dV'}{d\omega_{,}}.$$

Let these be substituted in the value of $\dfrac{d^2 y}{dt^2}$ then

$$\frac{d^2 y}{dt^2} = \frac{g\gamma}{r_{\prime}^2}\frac{d}{d\mu_{\prime}}\left\{(1-\mu_{\prime}^2)\frac{dy}{d\mu_{\prime}}\right\} - \frac{\gamma}{r_{\prime}^2}\frac{d}{d\mu_{\prime}}\left\{(1-\mu_{\prime}^2)\frac{dV'}{d\mu_{\prime}}\right\}$$

$$+ \frac{g\gamma}{r_{\prime}^2}\frac{\dfrac{d^2 y}{d\omega_{\prime}^2}}{1-\mu_{\prime}^2} - \frac{\gamma}{r_{\prime}^2}\frac{\dfrac{d^2 V'}{d\omega_{\prime}^2}}{1-\mu_{\prime}^2} \;\ldots\ldots\ldots\ldots (1).$$

We have already shewn (see Art. 177) that y can be expanded in a series of Laplace's Coefficients of the form

$$a\left\{Y_0 + Y_1 + Y_2 + \ldots\ldots + Y_i + \ldots\ldots\right\}.$$

The part of aV' relative to the nearly spherical stratum of fluid of which the general radius is $r_{\prime}$ or $(a + ay)$, is by Art. 182,

$$4\pi\rho a a\left\{Y_0 + \tfrac{1}{3} Y_1 + \tfrac{1}{5} Y_2 + \ldots\ldots + \frac{1}{2i+1} Y_i + \ldots\ldots\right\}.$$

Likewise the part of aV' which depends upon the action of the Sun and Moon can be expanded in a series of Laplace's Coefficients,

$$aU_0 + aU_1 + \ldots\ldots + aU_i + \ldots\ldots$$

This being premised, let us substitute the values of $r_{\prime}$, y, V' in the equation (1): then by Arts. 170, 181, Cor. we have

$$\frac{d^2 Y_i}{dt^2} + \frac{i(i+1)}{2i+1}\frac{g\gamma}{a^2}\left\{(2i+1) - \frac{4\pi\rho a}{g}\right\}Y_i = \frac{\gamma}{a^2} i(i+1) U_i.$$

Let ρ' be the mean density of the Earth, then

$$g = \frac{4\pi\rho' a}{3}, \quad \therefore \frac{4\pi a}{g} = \frac{3}{\rho'},$$

then putting $\lambda_i^2 = \dfrac{i(i+1)}{2i+1}\dfrac{g\gamma}{a^2}\left\{2i + 1 - \dfrac{3\rho}{\rho'}\right\}$;

the above equation becomes

$$\frac{d^2Y_i}{dt^2} + \lambda_i^2 Y_i = \frac{\gamma}{a^3} i(i+1) U_i,$$

and the integral of this is

$$Y_i = M_i \sin \lambda_i t + N_i \cos \lambda_i t$$

$$+ \frac{i(i+1)}{\lambda_i} \frac{\gamma}{a^3} \sin \lambda_i t \int U_i \cos \lambda_i t\, dt - \frac{i(i+1)}{\lambda_i} \frac{\gamma}{a^3} \cos \lambda_i t \int U_i \sin \lambda_i t\, dt.$$

M_i and N_i being two of Laplace's Coefficients.

When $i = 0$, the differential equation in Y_i becomes

$$\frac{d^2Y_0}{dt^2} = 0, \quad \therefore Y_0 = M\ t + N_0;$$

$$\therefore \frac{y}{a} = Y_0 + Y_1 + Y_2 + \ldots$$

$$\begin{aligned} &= M_0 t + N_0 + M_1 \sin \lambda_1 t + N_1 \cos \lambda_1 t \\ &\qquad + M_2 \sin \lambda_2 t + N_2 \cos \lambda_2 t \\ &\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\qquad + M_i \sin \lambda_i t + N_i \cos \lambda_i t \\ &\qquad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

If the quantity M_0 be finite, then y will increase without any limit, and the oscillations of the sea will not be stable. But it is easy to shew that M_0 and N_0 vanish, because the mass of the fluid is constant.

This condition gives $\int_{-1}^{1} \int_0^{2\pi} y\, d\mu\, d\omega = 0$;

$$\therefore \int_{-1}^{1} \int_0^{2\pi} \{Y_0 + Y_1 + Y_2 + \ldots\ldots\}\, d\mu\, d\omega = 0,$$

but by Art. 176, $\int_{-1}^{1} \int_0^{2\pi} Y_i\, d\mu\, d\omega = 0$ when i is > 0;

$$\therefore \text{also} \int_{-1}^{1} \int_0^{2\pi} Y_0\, d\mu\, d\omega = 0, \quad \text{or } 4\pi Y_0 = 0,$$

$$\therefore Y_0 = 0, \quad \therefore M_0 t + N_0 = 0.$$

The stability of the sea depends then on the signs of

$$\lambda_1^2,\ \lambda_2^2, \ldots\ldots \lambda_i^2 \ldots\ldots$$

for if one of these be negative the value of y will have an exponential term in its expansion, and all the terms will not be periodic functions of t. The condition that this should not happen is, that $2i + 1 - \frac{3\rho}{\rho'}$ shall not be negative for any positive integral value of i,

$$\therefore\ \frac{\rho}{\rho'} \text{ must not be greater than any value of } \frac{2i+1}{3},$$

$$\therefore\ \frac{\rho}{\rho'} \text{ must not be greater than } 1,$$

or the density of the sea must not be greater than the mean density of the Earth: otherwise the equilibrium of the waters of the ocean would not be stable.

579. Laplace extends this investigation to the case, where the rotatory motion of the Earth is taken into consideration, and the depth of the sea is not uniform: and arrives at the same result as before (Liv. IV. §. 3, also §. 13). He likewise shews in §. 14, of the second Chapter that the converse Proposition is true in many cases.

The part of the oscillations, which depends on the primitive state of the sea must have quickly disappeared by the resistances of different kinds, which the waters of the Ocean suffer in their motions: so that if it were not for the action of the Sun and Moon the sea would long since have subsided into a permanent state of equilibrium. It seems pretty evident, that the same would be the case if the ellipticity of the Earth and the rotatory motion were taken into account, the only permanent effect of the rotatory motion being to modify the action of the Sun and Moon, and so to alter the *period* of the oscillations, but not their nature. For these reasons we may consider the solution in Art. 578, as applicable to the case of nature.

Now the experiments made by Maskelyne on the attraction of the mountain Schehallien in Scotland and by Cavendish

on the attraction of leaden balls shew, that the mean density of the Earth is somewhere about five times that of the sea. We are hereby assured, then, that, provided no geological convulsion change the form of the Earth's surface, no inundating catastrophe can overwhelm us, so long as matter obeys the laws, which at present regulate its motion: a striking illustration of the words "Hitherto shalt thou come, but no further." Job xxxviii. 11.

CHAPTER III.

THE MOTION OF BODIES IN A RESISTING MEDIUM.

580. It is found by experiment, that when bodies move in a fluid, whether incompressible or aeriform, they meet with a resistance which tends continually to diminish their velocity. In consequence of the great difficulty of making accurate experiments on the resistance of media, and also because of the extreme complication of the analysis, which prohibits our making any extensive use of the facts which are brought to light, the laws of the resistance of fluids have not yet been very satisfactorily ascertained.

The general approximate law seems to be that the resistance on a plane surface, moving with its plane at right angles to the line of motion, is proportional to the extent of surface, the density of the resisting medium, and the square of the velocity taken conjointly. Some recent experiments upon the motion of boats on canals seem to indicate, that beyond a certain degree of velocity this law is not even an approximation to the truth, but the simple velocity better suits the experiments than the square of the volocity. See Mr Russell's experiments recorded in the Reports of the British Association for the Advancement of Science.

In a work of the nature of the present we should not have thought of entering upon this subject were it not intimately concerned with celestial phenomena. It has been computed that Encke's comet has since its appearance in 1786 been moving round the Sun with an increasing mean motion. Encke attributes this to the resistance of a medium pervading space. We shall therefore proceed to calculate the effect, that such a medium must have upon the motion of the planets, and then explain the process of calculating the perturbation produced by this cause in the motion of comets.

PROP. *Supposing that a resisting medium pervades the planetary spaces, and that its resistance varies as the square of the velocity, required to find equations for determining the effect on the motion of the planets.*

581. Let x and y be the co-ordinates to a planet and r its distance measured from the Sun: S the mass of the Sun and $V\dfrac{ds^2}{dt^2}$ the resistance of the medium. The equations of motion are

$$\frac{d^2x}{dt^2} = -\frac{Sx}{r^3} - V\frac{ds^2}{dt^2}\frac{dx}{ds}, \quad \frac{d^2y}{dt^2} = -\frac{Sy}{r^3} - V\frac{ds^2}{dt^2}\frac{dy}{ds},$$

multiply by $2\dfrac{dx}{dt}$ and $2\dfrac{dy}{dt}$, add, and integrate,

$$\therefore \frac{ds^2}{dt^2} = C + \frac{2S}{r} - 2\int V\frac{ds^3}{dt^3}dt.$$

But in the instantaneous ellipse (Arts. 350, 352), the first differential coefficients are the same as in the actual orbit, because the contact of this ellipse with the orbit is of the first order; hence

$$\frac{2S}{r} - \frac{S}{a} = \frac{ds^2}{dt^2} \text{ in ellipse } (2a = \text{axis-major})$$

$$= C + \frac{2S}{r} - 2\int V\frac{ds^3}{dt^3}dt,$$

$$\therefore\ -\frac{1}{a} = \frac{C}{S} - \frac{2}{S}\int V\frac{ds^3}{dt^3}dt;$$

$$\therefore\ \frac{da}{dt} = -\frac{2a^2V}{S}\frac{ds^3}{dt^3} = -2a^2VS^{\frac{1}{2}}\left\{\frac{2}{r} - \frac{1}{a}\right\}^{\frac{3}{2}}.$$

Again, $$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = -V\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)\frac{ds}{dt},$$

but in the ellipse $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = \sqrt{Sa(1-e^2)}$;

$$\therefore \frac{1}{\sqrt{a(1-e^2)}} \frac{d\sqrt{a(1-e^2)}}{dt} = -V\frac{ds}{dt},$$

$$\therefore \frac{d.a(1-e^2)}{dt} = -2a(1-e^2)V\frac{ds}{dt},$$

performing the differentiation, and substituting,

$$\frac{de}{dt} = 2V\frac{1-e^2}{e}\left(1-\frac{a}{r}\right)\frac{ds}{dt} = 2V\frac{1-e^2}{e}\left(1-\frac{a}{r}\right)S^{\frac{1}{2}}\left(\frac{2}{r}-\frac{1}{a}\right)^{\frac{1}{2}}.$$

Also, since $\dfrac{a(1-e^2)}{r} = 1 + e\cos(\theta - \varpi)$,

$$\frac{d\varpi}{dt} = \frac{1-e^2}{re\sin(\theta-\varpi)}\frac{da}{dt} - \left\{\frac{2a}{r\sin(\theta-\varpi)} + \frac{1}{e}\cot(\theta-\varpi)\right\}\frac{de}{dt}$$

$$= -\frac{2V}{e}\sin(\theta-\varpi)\frac{ds}{dt}, \text{ after all reductions.}$$

Prop. *Supposing the orbit nearly circular, and V very small and constant, required to find the effect of the medium on the elements of the orbit.*

582. Since the orbit is nearly circular we shall neglect $e^2, e^3 \ldots$: hence $r = a\{1 - e\cos(\theta-\varpi)\}$;

$$\therefore \frac{dt}{d\theta} = \frac{r^2}{h} \text{ (Art. 242)} = \frac{a^{\frac{3}{2}}}{\sqrt{S}}\{1 - 2e\cos(\theta-\varpi)\};$$

$$\therefore nt = \theta - \varpi - 2e\sin(\theta-\varpi), \quad \sqrt{\frac{S}{a^3}} = n,$$

t being reckoned from the perihelion passage;

$$\therefore \theta - \varpi = nt + 2e\sin nt.$$

$$\text{Hence } \frac{da}{dt} = -2V\sqrt{Sa}\{1 + 3e\cos(\theta-\varpi)\}$$

$$= -2Vna^2\{1 + 3e\cos nt\};$$

$$\therefore a = \text{const.} - 2Va^2\{nt + 3e\sin nt\}.$$

Hence the effect of the medium is to diminish the mean distance by $4\pi Va^2$ during each successive revolution, and therefore to increase the mean motion $\left(n, \text{ which} = \sqrt{\dfrac{S}{a^3}}\right)$ by $6\pi Vna$.

Also $\dfrac{de}{dt} = \dfrac{2Vna}{e}\{1-e^2-1-e\cos(\theta-\varpi)\}\{1+e\cos(\theta-\varpi)\}$

$$= -Vna\{2\cos(\theta-\varpi)+e[3+\cos 2(\theta-\varpi)]\}$$

$$= -Vna\{2\cos nt + e(1+3\cos 2nt)\};$$

$$\therefore\ e = \text{const.} - Va\{2\sin nt + e(nt + \tfrac{3}{2}\sin 2nt)\}.$$

Hence the eccentricity is diminished during each revolution by $2\pi Vae$.

Lastly, $\dfrac{d\varpi}{dt} = -\dfrac{2Vna}{e}\sin(\theta-\varpi)\left\{\dfrac{1+e^2+2e\cos(\theta-\varpi)}{1-e^2}\right\}^{\frac{1}{2}}$

$$= -nA_1\sin nt - nA_2\sin 2nt + \ldots$$

A_1, A_2 ... being functions of V, a, and e;

$$\therefore\ \varpi = \text{const.} + A_1\cos nt + \tfrac{1}{2}A_2\cos 2nt + \ldots$$

hence during a revolution ϖ is unaffected, or the axis-major remains stationary.

583. The variations of the elements of the planetary orbits calculated on the principles of the Planetary Theory give the position of the planets very accurately: hence the variations arising from a resisting medium, if one exist, are insensible in the motion of the planets: hence V is extremely small. But V varies directly as the density of the medium and inversely as the mass of the body acted on, consequently although the medium produces no effect on the motion of the planets, yet its influence on the comets may be sensible, since their mass is extremely small, Art. 478.

Cor. If we suppose V to be a function of r, and neglect the square of the eccentricity, the above formulæ will lead to the equation

$$e = q\sqrt{\frac{a}{\phi(a)}},\quad qa \text{ const., and } V = \phi(r):$$

also ϖ is unaffected.

584. In order to apply the variations in Art. 581, to find the change in a comet's orbit, we must use the method

of quadratures; that is, we must calculate the differential coefficients for values of the elements at certain instants not far apart, and then multiply these results by the respective intervals of time between these instants and add them all together: see Mr. Airy's Translation of *Encke's Dissertation* on his comet in the *Astronomische Nachrichten*, Nos. 210, 211.

The result deduced in Art. 582, with respect to the position of the perihelion is true for all orbits, as there shewn.

CONCLUSION.

SUMMARY OF ARGUMENTS IN FAVOUR OF UNIVERSAL GRAVITATION.

585. The perpetual recurrence of similar phenomena under similar circumstances suggests the idea, that the material world is regulated in its movements by laws, that the changes and vicissitudes we witness, whether in the heavens or in the planet we inhabit, are not the results of mere chance and caprice, but spring from the secret influence and operation of certain principles and properties with which matter is endowed. The ordinary observer cannot fail to trace in many instances the connexion between one set of phenomena and another; as for example, the relation between the length of the day, and the interval of time, which elapses between the times of high tide; the connexion which this bears with the position of the Sun and Moon in the heavens: then again the analogy, which exists between the notion, that the Sun and Moon have an affinity for the waters of the Ocean, and the fact that terrestrial bodies are drawn towards the Earth when left to themselves. In this way our conception of order and regularity in the changes of the natural world is strengthened, and by carrying on our researches we begin to discover that many effects, which seemed to be independent of each other, and linked by no natural connexion, are collateral results of one and the same principle.

Our object in the present Chapter is to gather the arguments which convince us of the truth of the Theory of Universal Gravitation, arranging them in order and presenting them under one view. By the labours of philosophers, extended over a long period of time, the celestial phenomena

have been traced to the action of a few simple laws. These laws we have pointed out in the course of this work.

586. All our knowledge of external objects is the result of experience; by experience we accumulate facts; and by the comparison and classification of facts we are led through a process of induction to the discovery of the general laws, from the operation of which these facts spring merely as limited and individual results.

Experience teaches us that bodies, when left to themselves and when unresisted by external objects, fall downwards. This constant tendency downwards, in preference to any other direction, suggests the first idea of an affinity, which one portion of matter has for other portions. The greatness of the size of the Earth, when compared with that of any body upon which we can perform experiments, sufficiently accounts for the fact, that bodies do not appear to influence each other. Experiments shew, however, that when two bodies are placed at rest and near each other on the surface of a fluid (under which circumstances the least possible resistance is offered to their motion) they will begin to move and finally come in contact. Also Cavendish's experiments with leaden balls prove the same.

Let us examine the consequences to which this conception of an attractive property of matter leads us.

587. The examination of numerous experiments led us to conceive that the following laws regulate the motion of bodies; That a body in motion will continue in motion and move uniformly in a straight line when not acted on by external forces; and That when a force acts upon a body in motion the change in motion is the same as if the force acted on the body originally at rest (Arts. 191, 204).

Now the bodies of the Solar System do not move in straight lines. The laws just enunciated shew, then, that these bodies are acted on by external forces; and, since we have seen that a principle of attraction does prevail in matter on the Earth's surface, we are fully justified in adopting as an hypothesis, to stand or fall by the comparison of calculated results with the observed phenomena, that this principle of attraction prevails throughout the Universe. We shall now enquire into the nature of this force of attraction.

588. It is by a combination of the first and second laws of motion, that we calculate, by the use of mathematical symbols, the relations connecting the path described by a body moving in space, and the forces which act upon the body. In order, then, to discover the nature of the forces we must examine the nature of the orbit described by the body. Kepler has furnished us with the necessary facts respecting the configuration of the Solar System; he deduced them from observations made upon the motion of the planets. These, as we shall shew, incontrovertibly strengthen the testimony in favour of the universal gravitation of matter, and moreover point out the law of variation of the attraction. The following are the laws which Kepler discovered: they are very nearly verified in the Solar System.

Each planet describes an ellipse about the Sun, the Sun being in one of the foci.

The areas described by the radius-vector of each planet about the Sun vary as the times of describing them.

The squares of the periodic times of the planets about the Sun bear to each other the same relation as the cubes of their major axes.

These laws have been shewn to hold for the satellites revolving about their primaries, if small inequalities be neglected. The same gravitating principle, therefore, that retains (as we shall see) the planets in their orbits about the Sun, binds the satellites to their primaries.

589. The Sun's magnitude is very enormous in comparison with that of the other heavenly bodies; also the mutual distances of the heavenly bodies are never very small in comparison with their distance from the Sun: this is shewn by astronomical observations which are independent of all theory. For this reason we may neglect, at least for a first approximation, the mutual action of the heavenly bodies in comparison with the action of the Sun upon them. Also the diameters of the Sun and planets bear a very small ratio to the distances of the planets from the Sun: therefore we shall not be very far from the truth if we consider the Sun and planets as intense particles, condensed into their centres. If we adopt this supposition, Kepler's second law proves, that the forces acting on

the planets pass through the Sun's centre (Art. 257): and consequently confirms the notion of a principle of attraction, and shews, that the attraction of the planets, as we have supposed, is far more feeble than the attraction of the Sun. Kepler's first law proves, that the attraction between the Sun and planets varies in intensity inversely as the square of the distance of the bodies. The third law shews, that, not only must the force on each planet pass through the Sun's centre, and the law of attraction be the same, but the intensity of the attractive force on each planet must be the same at the same distance for every planet.

590. Thus far then the laws, which Kepler discovered to prevail in the Solar System, give great weight to the evidence in favour of the universality of the principle of attraction; and moreover they point out the law of variation of the attraction when the distance varies, and shew that it is the inverse square of the distance. But we have hitherto considered the heavenly bodies to be merely intense particles, whereas the diameter of the Earth is nearly 8000 miles, and that of the Sun between 800,000 and 900,000 miles. And, moreover, we are convinced, that it is not the centres alone of the heavenly bodies that attract, since Dr Maskelyne ascertained by observations on the stars, made near the mountain Schehallien, in Scotland, that the direction of the plumb line was affected by the attraction of the mountain, and consequently the Earth's attraction is not directed towards its centre. The same is likewise proved by measuring degrees of latitude near the pole and equator. It becomes necessary, then, to enquire more minutely into the legitimacy of the hypothesis we have adopted in Art. 589.

591. The force of the objection, that all the particles of a body attract and not the centres only, will be considerably weakened by referring to Art. 165, in which we proved that a mass of matter, composed of particles attracting according to a certain law, will have nearly the same attraction for a distant body, as if we considered the particles to be condensed into their common centre of gravity. But upon a further examination we find, that the objection is, as far as the accuracy of our results in this stage of the question is concerned, quite

removed. For in the Article following that last cited we shew, that the law of attraction of the constituent particles and the resultant law of attraction are always the same. But the resultant law is that of the inverse square of the distance, as we have proved by Kepler's laws: this, then, is the law of attraction of the constituent particles. If we turn now to Art. 153, we shall find that the inverse square of the distance is one of those three laws of attraction, which give accurately the same resultant for a spherical shell as if we conceive it condensed into its centre.

When we bear in mind, then, that astronomical observations have proved that the figures of the Sun and planets and their satellites do not differ much from spheres, the objection which arises from effecting the calculations of the motion of the heavenly bodies on the hypothesis of their being intense particles, is entirely removed.

592. We come, then, to this conclusion, that, if we neglect the minute errors, which accurate observations on the heavenly bodies detected in Kepler's laws, if we neglect the errors arising from the deviation of the figures of these bodies from spheres, and the probable variation of density in their interior, of which we have made no account, then the only simple hypothesis which will account for the phenomena is, that all particles of the universe attract each other with a force which varies inversely as the square of the distance and directly as the mass of the attracting particle.

It remains to be seen whether this law, simple as it is, will upon effecting the calculations give correct numerical results: for the test of a theory consists in a comparison of the exact results to which it leads with the observed phenomena; and not solely in its power of explaining the nature of the phenomena.

593. Now the calculations of the position of the planets made upon this hypothesis of their gravitating towards the Sun, with a force directly as their mass and inversely as the square of their distance from the Sun, are found to agree very well with the observed positions, if the calculations extend over only a few years. After the lapse however of a considerable interval of time, as a century, minute errors are

detected in the calculations, and they are then found not to agree exactly with the observed positions of the planets. But this disagreement is in fact precisely what we should have anticipated; since, if the principle of gravitation be universal, the planets would attract each other and consequently disturb the elliptic motion and the equable description of areas; likewise the deviation of the figures of the planets from perfect sphericity and their heterogeneous structure give rise to additional errors. An idea of the extreme smallness of the perturbations may be learned from the fact, that if we trace on paper an ellipse ten feet in diameter to represent the orbit in which the Earth is moving at any instant about the Sun, and if we trace by its side the path actually described in its revolution round the Sun, the difference between the original ellipse and the curve actually described is so excessively minute, that the nicest examination with microscopes, continued along the outlines of the two curves, would hardly detect any perceptible interval between them: Herschel's *Astronomy*.

594. Our next enquiry should therefore be, whether the magnitudes of the minute deviations from elliptic motion accord with the calculations, effected on the hypothesis of the universality of the gravitation of matter. And here we enter upon an investigation, so complicated, and depending upon such a variety of disturbing causes all in simultaneous operation, that it is desirable to seek for a compendious method of treating the subject.

The course we shall pursue is exactly the reverse of that we have hitherto followed: for we shall now assume the truth of the Law of Universal Gravitation, and calculate by means of mathematical reasoning the phenomena, which would result from the operation of this Law in combination with the Laws of Motion. In adopting this method we entirely disentangle ourselves from the multitude of difficulties, which were surrounding us; we have not now to consider the influence which this or that observed fact may have upon our calculations; we no longer have to modify our original notions upon each discovery: we commence entirely anew, and, assuming the Laws of Motion and of Universal Gravitation, we investigate by

means of the rigorous and infallible engine of mathematical calculation, the phenomena, which would naturally arise from the action of these laws. The complete accordance which is found to exist, in all the instances submitted to this test, between the calculated and the observed phenomena of the Solar System, is the surest proof of the truth of the assumed laws.

We were unable to proceed in this way from the beginning, since mathematical reasoning is incapable of application, unless we have laws to reason upon: and in consequence of the number and complexity of the observed facts, it was impossible, *à priori*, even to conjecture, that the law of attraction was that of the inverse square of the distance. But having been led by a process of induction step by step to this great principle, we descend through a process of deduction to examine its consequences, and so, in the end, incontrovertibly to establish its truth. So strong is the evidence in its favour, and so firm the basis on which it rests, that in many cases we attribute slight discrepancies to errors in the observations rather than in the assumed law.

595. Notwithstanding what we have said of the advantage of considering the subject in this point of view, it must not be denied, that great difficulties beset our path in effecting the mathematical calculation. Were it not indeed for the peculiar configuration of our system, the enormous preponderance of the mass of the Sun over that of the planets, the grouping of the heavenly bodies into clusters consisting of a primary and its satellites, the mass of the central body greatly exceeding that of its attendants, were it not for these peculiar arrangements, the calculation would baffle the powers of analysis.

It must not be imagined, however, that these difficulties are such, as to vitiate the results to which they lead us: they add to the labour of the calculation, but do not subtract from their certainty; since we may by successive approximation arrive within an appreciable distance of the exact results.

596. The motion of a body moving freely in space consists of two parts, one a motion of translation from one

situation to another, and the other a motion of rotation: and these may be considered independently of each other; for the motion of the centre of gravity would be precisely the same, if the whole mass were concentrated in that point, and all the forces which act upon the body were transferred to the same point: and the motion of rotation would be the same, if the centre of gravity were held at rest. These principles we have proved in Arts. 429, 430, 501, 502: and they enable us to divide the investigation of the motion of the planets and other heavenly bodies into two branches, which we shall consider separately.

597. The forces which act upon a heavenly body are the attractions of the Sun, planets, satellites, and comets upon every particle of its mass, and the resistance of the medium in which the body moves.

But in consequence of the enormous magnitude of the Sun in comparison of that of the other bodies, and in consequence also of the tenuity of the medium, which pervades the planetary spaces, the great preponderating force is the attraction of the Sun: and since the figure of the Sun differs but little from a sphere, and since the same is likewise true of the other bodies of the solar system, it follows, that the resultant of all the forces which act upon the various particles of the body, the motion of which we are considering, when transferred to the centre of gravity of the body will differ very slightly from a force varying as the sum of the masses of the Sun and body divided by the square of the distance of their centres: the small additional forces are called *disturbing forces*.

598. Now if the disturbing forces did not exist, the centres of the bodies of the Solar System would each move in a conic section: and all the orbits would have one common focus in the Sun's centre: (Art. 252.) But, in consequence of the disturbing forces, slight deviations from these paths will arise, which it will be necessary to calculate.

Lagrange has put the subject in a most lucid point of view. Let us imagine the disturbing forces to cease acting, the centre of the body would ever after move in a conic section, the magnitude and position of which would depend

upon the state of things at the instant the disturbing forces ceased to act.

Now this curve and the path actually described have a common tangent, and the velocity at the instant under consideration is the same as that in the conic section calculated according to the principles of elliptic motion. For these reasons we may suppose, that for an instant the body is moving in the conic section, the elements of its position and magnitude depending upon the time. And in short the entire motion of the centre of the body may be represented by supposing it to move in a conic section, of which the elements are subject to continual variation. The paths of all the heavenly bodies (with the exception perhaps of some of the comets) are nearly elliptical, and the ellipses in which they are moving at any instant are called the *instantaneous* ellipses.

We thus reduce the calculation of the motion of the heavenly bodies to that of the elements of the instantaneous ellipse, in which the body is moving at the instant we wish to calculate the position of the body. Nothing remains to be done after this, but to substitute the values of these elements in the expressions of the radius vector, latitude, and longitude. See Dynamics, Chap. vi.

599. When a number of small disturbing forces act upon a body and alter its motion, the aggregate effect is very nearly equal to the algebraical sum of the separate effects, which they would produce if they acted independently of each other. For the real effect differs from this sum merely by the effect which the disturbing forces produce in modifying each others action, and must consequently be of the second order of magnitude, and therefore so trifling as to be inappreciable except in extreme cases. (Art. 288.) This principle we find of vast importance, since it greatly facilitates the calculations.

600. In order to determine the elliptic elements of a heavenly body corresponding to any instant, we investigate their value in terms of a small arbitrary disturbing force expressed in general symbols, as will be seen by turning to Chapter vi. of Dynamics.

The various disturbing forces are then calculated and substituted singly in the formulæ, which give the variations

of the elements: and the variations being added together with their correct signs the whole variation is known: the value of each element being then determined we are prepared to apply the formulæ of Arts. 278, 280, already mentioned.

Having thus explained the analytical machinery, so to speak, which we use to calculate the motions of the heavenly bodies, we must shew how the results of the calculation bear comparison with the observed phenomena.

601. The planets are subject to perturbations of two kinds; both depending upon their reciprocal action; a full explanation of these will be found in Art. 377. They are termed *periodic variations* and *secular variations*. Some of these remarkably prove the correctness of the principles which have guided the calculations; such as the great inequality of Jupiter and Saturn, which at one time was the great stumbling block in the way of receiving Newton's theory of gravitation; but which Laplace so entirely removed by shewing, that the defect lay in the analysis and not in the principle of gravitation: again, the inequality of the Earth and Venus, discovered by Mr Airy, as clearly demonstrates the truth of the law of gravitation, especially when we consider the extreme minuteness of this error and the complex character of the analysis. (Art. 376.)

There is a remarkable perturbation in the motion of the Moon arising from the secular inequality of the eccentricity of the Earth's orbit: it is called the Secular Acceleration of the Moon's mean motion. It had been observed by Halley, upon comparing together the records of the most ancient lunar eclipses of Chaldean Astronomers with those of modern times, that the periodic time of the Moon about the Earth is now sensibly shorter than it was at that distant epoch. This result was confirmed by a further comparison of both sets of observations with those of the Arabian Astronomers of the eighth and ninth centuries; and it was proved, that the mean motion is increasing by about 11″ per century, a quantity small in itself, but becoming of importance by the accumulation of ages. This had long been a stumbling block to mathematicians, and so difficult did it appear to render an exact account of it, that the theory of gravity was declared to be

inadequate satisfactorily to remove the difficulty by explaining the cause of the phenomenon. It was in this dilemma, that the penetrating sagacity of Laplace was once more called into action to rescue Physical Astronomy from its reproach. If the solar ellipse were invariable the alternate dilatation and contraction of the lunar orbit would, in the course of a great many revolutions of the Sun, at length bring about an exact compensation in the distance and periodic time of the Moon.

But the solar ellipse is not invariable, its eccentricity has been decreasing since the earliest ages, and will continue to do so till the orbit becomes a circle, after which epoch the orbit will again dilate and increase in eccentricity. It was from this variation of the eccentricity of the solar orbit, that Laplace shewed, that the variation in the Moon's mean motion arose. This phenomenon is a very striking instance of the propagation of a periodic inequality from one part of a system to another. The masses of the planets are too small and their distances from the Earth are too great for their difference of action on the Earth and Moon ever to become sensible. Yet their effect on the Earth's orbit is propagated (as we see) through the Sun to the Moon's orbit; and, what is very remarkable, the transmitted effect thus indirectly produced on the angle described by the Moon round the Earth is more sensible to observation, than that directly produced by them on the angle described by the Earth round the Sun.

602. But without adding more suffice it to say, that the calculations of the inequalities of the planets and of Jupiter's satellites is arrived at such a degree of precision, as to agree exactly with the observations, omitting only unavoidable instrumental errors.

603. Again in the Lunar Theory, we have many proofs of the truth of Newton's law (Arts. 337—346). The Variation, the Evection, and Annual Equation, the motion of the perigee and node, all agree in their numerical results with observation: numerous other inequalities, a few of which are mentioned in Arts. 342—346, give additional evidence in support of our theory.

604. Likewise the calculation of the motion of comets very remarkably agrees with observation. The time of the

re-appearance of Halley's Comet in the year 1835, after an absence of 76 years, was predicted correctly within nine days of its actual appearance. A most astonishing fact, when we consider that the light of the comet is diminishing, and that consequently we could not expect to have the *time* of re-appearance very accurately calculated.

605. In short, when we consider the simplicity of the law to which we are led, the variety and different characters of the tests we use, the labyrinth of calculations through which we have to wind our way, and the exact character of the results upon which any reliance can be placed, we are irresistibly constrained to admit, that no theory has ever been based upon a firmer foundation than that of Universal Gravitation. In many instances, if the law departed in the slightest degree from that of the inverse square, inequalities, which are now calculated numerically in the theory and agree with observation, would not give results near the truth: this is the case with the motion of the Moon's perigee.

606. But we have hitherto gathered our evidence solely from the motion of the heavenly bodies considered as intense particles: when, however, we descend deeper into the consequences of the law of gravity, and enquire into the minute errors caused by the attraction of the various particles, which form the masses of the heavenly bodies, we obtain an accession of sound arguments in favour of Universal Gravitation.

607. By measuring degrees of latitude as near as possible to the pole and equator it is found, that they increase in length as we pass from the equator to the pole; this shews that the vertical lines (or normal lines) to the Earth's surface are less and less inclined to each other as we proceed towards the pole from the equator. We learn from this that the form of the Earth is not spherical, but flattened at the poles: and when the calculations and observations are combined, the geodetic measures shew, that the Earth is very nearly spheroidal, having an ellipticity $\frac{1}{306}$. This is a result not of theory, but observation and trigonometrical calculations.

608. With this fact to guide us we entered upon an enquiry whether this form were given to the Earth in a fluid or semi-fluid state, since we easily see *à priori*, that the rota-

tory motion would, in that case, cause the parts near the equator to bulge. We proved in Art. 525, that this would not give the proper numerical value of the ellipticity if we suppose, that the mass of the Earth were homogeneous, an hypothesis in itself highly improbable, since the pressure of the upper strata must produce a condensation of the lower: also it is contrary to the results of Maskelyne's and Cavendish's results respecting the mean density of the Earth. We here have, then, a negative argument in favour of gravitation. We therefore proceeded (Art. 531) to investigate the figure of the Earth upon the hypothesis, that it consists of strata differing but little from spherical shells (an hypothesis extremely probable, since the ellipticity of the surface is $\frac{1}{306}$ by observations), and increasing in density towards the centre according to an unknown law. The result we arrived at was, that the form of all the shells is spheroidal, decreasing in ellipticity towards the centre (Art. 533). An equation was obtained for calculating the ellipticity of the surface, when the law of density was discovered: this law we obtained upon hydrostatic principles in terms of arbitrary constants (Art. 545); and having determined the values of one of these constants by means of the facts given us by Maskelyne and Cavendish respecting the mean density of the Earth, we reduced to numbers the formula for the ellipticity, and obtained a result according most remarkably with that given by geodetic measurements.

609. As a further test we reduced to numbers the formula for the Precession of the Equinoxes, which had been previously calculated in Arts. 463—470, and obtained a result according with remarkable exactness with the observed precession (Art. 551).

610. We have calculated the error caused in the latitude of the Moon by the bulging portion of the Earth at the equator, and obtained results very closely agreeing with those we have given above (see Art. 556).

611. Also pendulum experiments give, by the formula of Art 539, an ellipticity according very well with the other values.

612. It will be readily granted, then, that we have an abundance of evidence (and more might be given) to justify

the conviction, that not only the heavenly bodies attract each other with forces varying as the attracting mass and inversely as the square of the distance, but that the individual particles of which they are composed attract according to the same law.

613. The theory of the Tides is at present in so imperfect a state, that we must not look for evidence in that quarter. Nevertheless some of the observations collected by Mr Whewell and Mr Lubbock seem to indicate, that the force of attraction of the Moon in raising the waters varies inversely as the cube of the distance, the theoretical law according to the theory of gravitation (Art. 574).

614. We may well draw this epitome to a conclusion in the words of Sir John Herschel. "There is one feature in Physical Astronomy which renders it remarkable among the Sciences, and has been the chief, if not the only, source of the perfection it has attained. It is this, that the fundamental law embracing all the minutiæ of the phenomena so far as we yet know them, presents itself at once, on the consideration of broad features and general facts, deduced by observations of even a rude and imperfect kind, in such a form as to require no modification, extension, or addition when applied to minute detail. In other sciences, when an induction of a moderate extent has led us to the knowledge of a law which we conceive to be general, the further progress of our enquiries frequently obliges us either to limit its extent or modify its expression. ...In Physical Astronomy, however,...our first conclusion is our last. The law, on which all its phenomena depend, flows naturally and easily from the simplest among them, as presented by the rudest observation; and, in point of fact, such has really been the order of investigation in this science. The rude supposition of the uniform revolution of the Moon in a circle about the Earth as a centre led Newton at once to the true law of gravity, as extending from the Earth to its companion. The uniform circular motions of the planets about the Sun, in times following the progression assigned by observation in Kepler's rule, confirmed the law, and extended its influence to the boundaries of our system. Every thing more refined than this, the elliptic motions of the planets and satellites, their mutual perturbations, the slow changes

of their orbits and motion denominated secular variations, the deviation of their figures from the spherical form, the oscillatory motions of their axes, which produce nutation and the precession of the equinoxes, the theory of the tides both of the ocean and the atmosphere, have all in succession been so many trials for life and death in which this law has been, as it were, pitted against nature; trials, of which the event no human foresight could predict, and where it was impossible even to conjecture what modifications it might be found to need." *Enc. Met. Physical Astronomy*, p. 647.

APPENDIX.

615. In this Appendix we have brought together a number of Problems of a miscellaneous character, and some of them of considerable difficulty. The object aimed at in giving this collection is twofold: first, to practise the skill of the student in the application of the Principles developed in this Work; but, secondly and chiefly, to bring before him some more problems of considerable interest, and which have engaged the attention of Philosophers, without adding much to the bulk of the Volume; this we could do by breaking up long investigations into a series of shorter ones, and giving copious hints to help to their solution.

Prob. 1. Three bodies, attracting each other according to the law of gravitation, are projected in the same plane, from the points of an equilateral triangle, in directions making the same angles with the straight lines drawn from the centres of the bodies to their common centre of gravity, and with velocities proportional to the lengths of those lines. Required to find the motion.

The equations of motion cannot be integrated: but the problem may be solved in the following manner. Shew that at the commencement of the motion the resultant forces acting on the bodies all pass through the common centre of gravity; that these forces each vary inversely as the square of the distance from the centre of gravity; but vary among themselves directly as the distance. Then by composition of motion, by a simple application of the Second Law of Motion, shew that the distances of the bodies from each other will at each successive instant, and therefore always, form an equilateral triangle. The forces will therefore always vary inversely as the square of the distance: and the motion may easily be found. This is true for any law of force.

Prob. 2. Three bodies situated in a straight line, and attracting each other according to the law of gravity, are projected in directions parallel to each other, with velocities proportional to their respective distances from their common centre of gravity. Find their relative distances at first, that they may continue to lie in a straight line during their motion.

Let x be the ratio of the distances of the middle body from the two extreme ones: the equation for finding x is of the fifth degree.

Prob. 3. Apply the above to the case of the Sun, Earth, and Moon; the Earth being the middle body: and shew, that if the distance of the Moon from the Earth had been equal to a quantity, which differs very slightly from $\frac{1}{100}$th part of the distance of the Sun, we should have had full Moon every night. What disadvantages would have attended such an arrangement?

The equation of last Prob. gives $x^3 = (E + M) \div 3S$, nearly.

616. It is supposed, that Light is the effect of the vibrations of an elastic medium pervading space. The comparison of the results of Art. 582 with observation shews, that if this be a true theory the tenuity of the medium is such as not to produce any change in the motion of the planets, perceptible at least within the range of years through which we have good recorded observations. The only Theory of Light besides that mentioned above is, that it is the effect of the impulse of luminous particles emanating with a constant velocity from the Sun in all directions. Laplace has calculated the effect of such a medium upon the motion of the Earth and Moon. The results are embodied in the solutions of the following Problems.

617. The resistance which the moving body meets with may thus be found. Suppose for an instant, that a velocity equal and opposite to that of the body is applied to the body itself, and also to the particles of the medium; the relative action of the medium and the body on each other will not be altered. The body will then be at rest, and the resultant

impulse of the medium will be inclined to its former direction; but we may resolve it in the directions of the radius vector of the body and of the body's motion: these two forces will be proportional to the velocity of light and the velocity of the body respectively. The effect of the first is slightly to diminish the gravitation of the body towards the Sun; this we shall therefore neglect.

PROB. 4. To find the effect of the impulse of light, on the theory of emanations, on the mean motion of the Earth.

This Prob. may be solved in a manner similar to the more general one which follows. The result is, secular inequality of mean motion of the Earth $= 3Hnt \div 2a^2$.

PROB. 5. To find the effect on the mean motion of the Moon about the Earth.

Let xyz be co-ordinates to the Earth from the Sun; $x'y'z'$ co-ordinates to the Moon from the Earth; r and f the distances of the Earth and Moon from the Sun; H and H' the quantities which measure the resistances the Earth and Moon meet with for a unit of velocity and a unit of distance from the Sun; S, E, M the masses of the Sun, Earth, and Moon. Then the disturbing force of the Moon's motion relative to the Earth, parallel to x, is

$$\frac{H'}{f^2}\,\frac{d(x+x')}{dt}-\frac{H}{r^2}\,\frac{dx}{dt},$$

and those parallel to y and z are similar. Introduce these into the equations of Art. 355; and we have by Arts. 360, 366,

$$\frac{d(R)}{dt} \text{ or, } \frac{n'a'^2}{3}\,\frac{dn'}{dt}=\frac{H'}{f^2}\,\frac{ds'^2}{dt^2}+\left(\frac{H'}{f^2}-\frac{H}{r^2}\right)\frac{dx\,dx'+dy\,dy'+dz\,dz'}{dt^2},$$

neglecting e^2 ... and periodic terms, we have secular inequality in the mean motion = the variable term in

$$\int n'dt \div t = 3H'(n'-n)\,t \div 2a^2.$$

PROB. 6. To compare the above secular inequalities.

In calculating $H' \div H$, we may suppose the actions of the Sun's light on the Earth and Moon directly proportional to

the area of a great circle of the body acted on, and inversely proportional to the mass. The result is, that the ratio $= 63.169$.

Prob. 7. To find the effect on these secular inequalities arising from the diminution of the mass of the Sun, in consequence of the continual emanation of matter, which takes place according to the hypothesis.

We shall neglect the eccentricity of the orbit. The principle of areas holds, and therefore a^2n is constant. Let α be the fraction of mass that the Sun loses in a unit of time: a_0, n_0 the values of a and n when $t = 0$. Then neglecting $\alpha^2 \ldots$, $a = a_0(1 + \alpha t)$; $n = n_0(1 - \alpha t)^2$; and the inequality in the mean motion $= -n_0 \alpha t$. We may compare this with the inequality arising from the impulse, in Probs. 4, 5. Let i be the ratio of velocity of light to velocity of Earth; ρ the density of the medium at the distance of the Earth from the Sun; Π the parallax of the Sun. Then it is not difficult to shew that $H = \pi i \rho a^5 \Pi^2 n \div E$, by calculating the momentum of the mass of light which is absorbed by the Earth in a unit of time, and equating it to the expression for the force, mentioned in Art. 617, as slightly diminishing gravity, viz.

$$(H \div a^2) \times \text{vel. of light}.$$

Hence, after some simple calculations, we obtain

$$\frac{\text{seq. ineq. of Earth's mean mot. from dim. of Sun's mass}}{\text{sec. ineq. of Earth's mean mot. from impulse of light}}$$

$$= -\frac{8E}{3\Pi^2 S} = -46970,$$

$$\frac{\text{sec ineq. of Earth's mean mot. from dim. of Sun's mass}}{\text{sec. ineq. of Moon's mean mot. from impulse of light}} = -74.35.$$

By observation we are sure, that the Earth's secular inequality is not $6''$ (see *Mec. Cel.* X. 7. 22.): it follows that the impulse of the Sun's light on the Moon cannot produce a secular inequality of $\frac{1''}{12}$. Also it easily follows, from the simple expression for the Earth's secular equation, and the fact that observation shews that this quantity cannot have amounted to

$0''.36$ in 2000 years, that during that period the Sun's mass has not varied a 2,000,000[th] part.

PROB. 8. Supposing that the attraction of the Sun and planets is caused by the motion of a fluid from the planetary spaces towards the Sun, required to shew that its velocity of transmission must be at least 7,000,000 times as great as that of light.

This is easily done by supposing the Moon's secular equation (viz. $10''.181621$ in 100 years,) to arise *wholly* from this cause: then the formula of Prob. 5 will lead to the result.

618. The Moon in revolving about the Earth keeps very nearly the same face towards us: this proves, that the mean motion about the Earth exactly equals the motion of revolution of the Moon about her axis; and that her axis is nearly perpendicular to the plane of the ecliptic. This has led astronomers to suppose, that the Moon is not an exact figure of revolution, but that it is prolate towards the Earth: in the following calculations we assume, that the principal axis x_1 always nearly points to the Earth.

From observations made on the motion of the spots on the Moon's disk, Dominic Cassini made the remarkable discovery, that the descending node of the lunar equator always coincides with the ascending node of the lunar orbit. He also found that the lunar equator is inclined about $2^\circ\, 30'$ to the plane of the ecliptic. Tobias Mayer confirmed the coincidence of the nodes by many observations; but made the above angle equal $1^\circ\, 29'$; and this has been confirmed by Bouvard and Nicollet, who make it $1^\circ\, 28'\, 45''$.

619. In the following Problems $x_1 y_1 z_1$ are co-ordinates to the Earth from the Moon, referred to the Moon's principal axes: r the distance of the Earth from the Moon: $\theta =$ inclination of the lunar equator to the ecliptic: $i =$ inclination of the orbit of the Earth and Moon to the ecliptic: we shall neglect the squares and higher powers of θ and i because they are both very small: ϕ the angle between the axis of x_1 in the Moon and the descending node of the lunar equator: l the latitude of the Earth measured from the ascending node of the

Earth and Moon's orbit. The rest of the notation is as in Arts. 446, &c.

PROB. 9. To transform the equations of rotation to calculate the libration of the Moon in latitude and longitude.

We shall easily obtain $x_1 = r\cos(\phi - l)$, $y_1 = -r\sin(\phi - l)$, $z = r(i-\theta)\sin l$; let $C - A = f^2 C$, $C - B = g^2 C$: then Art. 459 gives

$$\frac{d\omega_1}{dt} + g^2\omega_2\omega_3 = 0, \quad \frac{d\omega_2}{dt} - f^2\omega_1\omega_3 = \frac{3E}{r^3} f^2 (i+\theta)\sin l,$$

$$\frac{d\omega_3}{dt} + (f^2 - g^2)\,\omega_1\omega_2 = -\frac{3E}{2r^3}(f^2 - g^2)\sin 2(\phi - l); \quad \frac{E}{r^3} = m^2;$$

in which we neglect the *squares* of $C - A$ and $C - B$.

PROB. 10. To find the libration in longitude.

Let u = the difference of longitude of the Earth and the axis of x_1 in the Moon; this is always very small, and is called the libration in longitude. Then by Art. 447, the last equation of last Prob. may be reduced to

$$\frac{d^2u}{dt^2} + 3m^2(f^2 - g^2)\,u = 3m^2(f^2 - g^2)\,\Sigma\,.\,H\sin(ht + h'):$$

where $\Sigma\,.\,H\sin(ht + h')$ is the sum of the periodical terms in the longitude of the Earth as seen from the Moon: hence

$$u = K\sin\left\{mt\sqrt{3(f^2 - g^2)} + k\right\} + \Sigma\,.\,L\sin(ht + h');$$

where $L = \{3m^2(f^2 - g^2)\,H\} \div \{3m^2(f^2 - g^2) - h^2\}$; and K, k depend upon the circumstances at the epoch from which t is measured: K must be zero or extremely small, because no inequality corresponding with the first term in u has ever been detected by observation.

PROB. 11. To obtain a value of $f^2 - g^2$ from observations made on the Moon's libration in longitude.

The greatest value of H corresponds to the equation to the centre: in that case $H = 6^\circ\,18'\,2''$ and $h^2 = m^2\,(0.98317)$; and therefore $f^2 - g^2 = 0''.32772\,L \div (L - 6^\circ\,18'\,2'')$.

The term in u, which is capable of rising into most importance in consequence of its introducing a small divisor in L, is that which depends on the annual equation. In this case $H = 11' 9''$ and $h = m (0.0748)$;

$$\text{and } f^2 - g^2 = 0''.001865\, L \div (L - 11' 9'').$$

Now by a discussion of 174 observations of the libration of the Moon in longitude MM. Bouvard and Nicollet found the value of L depending on the annual equation $= -4' 45''$: this gives $f^2 - g^2 = 0.0005567$: if this be substituted for $f^2 - g^2$ in the expression given by the equation to the centre, $L = -39''$. It follows, that if the inequality arising from the annual equation be so difficult to detect by observation, that arising from the equation to the centre (which is less than a seventh part of the former) will never be observed. Even the inequality $4' 45''$ when seen from the Earth on the Moon's surface amounts only to $1''.3$, and may therefore very easily elude detection.

PROB. 12. To find the mean inclination of the Moon's equator to the ecliptic.

Put $\theta \cos \phi = s$ and $\theta \sin \phi = s'$ in the first two equations of Prob. 9: then by Art. 447 (since $\omega_3 = m$ in small terms),

$$\omega_1 = \frac{ds}{dt} + ms' \text{ and } \omega_2 = -\frac{ds'}{dt} + ms.$$

substitute these in the first two equations of Prob. 9, observing that $\theta \sin l = s \sin (l - \phi) + s' \cos (l - \phi)$:

$$\therefore\ \frac{d^2 s}{dt^2} + m(1 - g^2)\frac{ds'}{dt} + m^2 g^2 s = 0,$$

$$\frac{d^2 s'}{dt^2} - m(1 - f^2)\frac{ds}{dt} + 4m^2 f^2 s'$$

$$= -3m^2 f^2 i \sin l = -3m^2 f^2 i \sin (ht + h').$$

Let $h = (1 + a) m$; am is the mean motion of regression of the node, a is very small. Put

$$s = M \cos (kt + p) + F \cos (ht + h');$$

$$\text{and } s' = N \sin (kt + q) + G \sin (ht + h').$$

By substitution we have $k = 0$, $M = 0$; $k^2 = m^2 (1 + 3f^2)$, $M = (m \div k) N$; $G = 3f^2 i \div (2a - 3f^2)$: and $G = F$ in

small terms. Putting k, $-k$ for these last values of k, and $M'N'$, $M''N''$ for the corresponding values of M and N (their connection being determined by the above conditions), we have

$$\theta \cos \phi = s = M' \cos (kt+p) + M'' \cos (kt - p') + F \cos (ht + h'),$$
$$\theta \sin \phi = s' = N + N' \sin (kt+p) - M'' \sin (kt-p') + F \sin (ht+h').$$

Now it is known by observation, that the ascending node of the Moon's orbit always coincides with the descending node of the Moon's equator: hence $\phi = l = ht + h'$ for all values of t, in the small quantities s and s'. Therefore the above equations can be satisfied only by $M' = 0$, $M'' = 0$, $N = 0$, $N' = 0$, $N'' = 0$, $\theta = F$. Now $\theta = 1^\circ 28' 45''$, $i = 5^\circ 8' 49''$, $a = 0.004022$:

$$\text{hence } \; 3f^2 = \frac{2a\theta}{\theta + i} = 0.0017956.$$

Mayer in 1749 by numerous observations found $\theta = 1^\circ 29'$. MM. Bouvard and Nicollet have more recently made it

$$= 1^\circ 28' 45''.$$

Prob. 13. To calculate f^2 and g^2 on the supposition of the form of the Moon being one of fluid equilibrium.

Let c be the distance of the Earth from the Moon: r_1 its distance from any particle (xyz) of the Moon's mass. We shall have to add to X, Y, Z in Art. 531 the terms

$$\frac{E(c-x)}{r_{\prime}^3} - \frac{E}{c^2}, \quad -\frac{Ey}{r_{\prime}^3}, \quad -\frac{Ez}{r_{\prime}^3};$$

$$\therefore \text{ to } V \text{ the terms } \frac{E}{r_{\prime}} - \frac{Ex}{c^2}, \; = \frac{E}{c} - \frac{3Er^2}{c^3}\left\{\tfrac{1}{3} - (1-\mu^2)\cos^2\omega\right\}:$$

we reject the further powers of c; because, since mean motion of Earth about the Moon = motion of revolution of the Moon, we have, by Art. 531, $E \div c^3 = \frac{4}{3}\pi a m \phi(\mathrm{a}) \div \mathrm{a}^3$, which is of the order of a. Hence the second side of the equation in Art. 533 for calculating Y_2 becomes

$$\frac{m}{6}\frac{a^2\phi(\mathrm{a})}{\mathrm{a}^3}\left\{\tfrac{8}{3}(\tfrac{1}{3} - \mu^2) + \tfrac{4}{3}(1-\mu^2)\cos 2\omega\right\}.$$

By Art. 553 $\mathrm{Y}_2 = \mathrm{K}(\frac{1}{3} - \mu^2) + \mathrm{H}(1-\mu^2)\cos 2\omega$; by substituting this in the equation in Y_2 at the surface it is easy to

see, that Y_2' must be of the same form as Y_2 with regard to μ and ω. Let $Y_2' = K'(\frac{1}{3} - \mu^2) + H'(1 - \mu^2)\cos 2\omega$. Now we should introduce but a very small error by supposing that the strata, which in fluid equilibrium are very nearly spherical, are similar to the superficial stratum: à fortiori the error will be insensible if we only suppose that $\mathrm{H} \div \mathrm{K} = H' \div K' = \beta$. Substitute these in the equation, equate coefficients of $\frac{1}{3} - \mu^2$ and $(1 - \mu^2)\cos 2\omega$, and take their ratio, and we have $\beta = \frac{3}{4}$: hence $\mathrm{Y}_2 = \mathrm{K}\left\{\frac{1}{3} - \mu^2 + \frac{3}{4}(1 - \mu^2)\cos 2\omega\right\}$: and the equation in Y_2 becomes at the surface,

$$\int_0^{a} \rho' \frac{d}{da'}(a'^5 Y_2')\,da' = \tfrac{5}{3}a^2\phi(a)(\mathrm{K} - \tfrac{3}{4}m)\left\{\tfrac{1}{3} - \mu^2 + \tfrac{3}{4}(1 - \mu^2)\cos 2\omega\right\}:$$

from this we can easily obtain (since $am = Ea^3 \div Mc^3$)

$$f^2 = \frac{C - A}{C} = \frac{8a^2\phi(a)}{3\sigma(a)}\left\{a\mathrm{K} - \frac{5Ea^3}{4Mc^3}\right\}:$$

$$g^2 = \frac{C - B}{C} = \frac{2a^2\phi(a)}{3\sigma(a)}\left\{a\mathrm{K} - \frac{5Ea^3}{4Mc^3}\right\}.$$

It is evident from these, that f^2 and g^2 are greatest if we suppose the Moon's mass to be homogeneous: (when we suppose it heterogeneous the density is assumed to increase with the depth). In the case of homogeneity the above equation leads to $f^2 = 5Ea^3 \div Mc^3$, $g^2 = 5Ea^3 \div 4Mc^3$: now, $E \div M = 74$; $a \div c = \sin$ (angular radius of the Moon) $= \sin 15' \, 43''.5 = 0.0045742$:

then $f^2 = 0.00003542050$, and $g^2 = 0.00000882756$;

these are very much smaller than those given by observation (Probs. 11, 12; $f^2 = 0.0005985$ and $g^2 = 0.0000418$), hence the Moon is not homogeneous, and à fortiori has not the form of equilibrium even if we suppose it to be heterogeneous.

620. The planet Saturn has two rings surrounding it: each ring is a solid body, whose centre of figure coincides nearly with the centre of Saturn: but the centre of gravity of the ring does not so nearly coincide with the centre of Saturn. This centre of gravity revolves about the planet in the same time as the ring; and therefore the ring revolves about its centre of gravity in the same time as about Saturn. The

action of the Sun and the Satellites upon these rings must produce in their planes a motion of precession similar to that of the equator of the Earth. This action being different for the two rings, we should expect to find their motions different, and the rings not generally in the same plane. But ever since they were first observed they have remained nearly in the planes of Saturn's equator. We proceed to shew the physical cause of this.

We shall make the calculations for one ring, neglecting the mutual action of the rings in comparison of the action of Saturn. We shall suppose the ring to be a circle (radius $= a$) of variable density: in that case $C = A + B$. Let $\theta =$ inclination of plane of ring to Saturn's equator; this has always hitherto been small, we shall therefore neglect its square.

PROB. 14. To shew that the rings of Saturn remain nearly in the plane of Saturn's equator, in consequence of the attraction of the protuberant parts of the mass of that planet.

We have, by Art. 447 and the transformation used in PROB. 12,

$$\frac{d^2s}{dt^2} + m^2 s = \frac{L_{,}}{A}; \quad \frac{d^2s'}{dt^2} + m^2 s' = -\frac{M_{,}}{B} \ldots\ldots (1);$$

also $\phi - \psi = mt +$ very small terms, since $\omega_3 = m$.

We must now calculate $L_{,}$ and $M_{,}$: we shall neglect all periodical terms in them except those which rise into importance by integration; that is, those in which the coefficient of t in the argument $= m$ nearly. The forces which act on the ring are the attractions of Saturn and the Sun. We shall begin with the first.

Let the centre of gravity of the ring be the origin of co-ordinates, and the principal axes the axes of co-ordinates: XYZ the co-ordinates to the centre of Saturn: $X'Y'Z'$ the co-ordinates to the centre of the circumference of the ring: $x'y'z'$ co-ordinates to particle m of the ring: r the distance of m from the centre of Saturn: S' the mass and a the equatorial radius of Saturn: the angles ϕ, ψ as in Art. 447. We shall neglect the squares and products of XYZ $X'Y'Z'\theta$ and a.

Now $a^2 = (x' - X')^2 + (y' - Y')^2 + (z' - Z')^2$,

$$r^2 = (x' - X)^2 + (y' - Y)^2 + (z' - Z)^2;$$

$$\therefore\ r = a + \frac{x'(X' - X) + y'(Y' - Y) + z'(Z' - Z)}{a}.$$

Also by Art. 536, Cor. $V = \frac{S'}{r} + \frac{S'a^2}{a^3}(\epsilon - \frac{1}{2}am)(\frac{1}{3} - \mu^2)$:

$\mu = \cos(\angle$ between r and axis of Saturn)

$$= \frac{(x' - X)\sin\theta\sin\phi + (y' - Y)\sin\theta\cos\phi + (z' - Z)\cos\theta}{r}.$$

From these we obtain

$$L_{,} = \Sigma . m\left\{y'\frac{dV}{dz'} - z'\frac{dV}{dy'}\right\} = -\frac{2Sa^2A}{a^5}(\epsilon - \tfrac{1}{2}am)\,\theta\cos\phi,$$

$$M_{,} = \frac{2Sa^2B}{a^5}(\epsilon - \tfrac{1}{2}am)\,\theta\sin\phi, \text{ and } N_{,} = 0.$$

We shall now calculate $L_{,}M_{,}N_{,}$ for the attraction of the Sun: in this the final equations in Art. 459 will help us. Let $x_{,}y_{,}z_{,}r_{,}$ be co-ordinates to the Sun: c the mean value of $r_{,}$; let v be the longitude of the Sun measured from the descending node of Saturn's equator: θ' the inclination of Saturn's equator to the ecliptic: the other angles as before: then

$$x_{,} = r_{,}\{\cos v\cos(\phi - \psi) + \sin v\sin(\phi - \psi)\cos\theta' \\ + \theta\sin\phi\sin v\sin\theta'\},$$

$$y_{,} = r_{,}\{-\cos v\sin(\phi - \psi) + \sin v\cos(\phi - \psi)\cos\theta' \\ + \theta\cos\phi\sin v\sin\theta'\},$$

$$z_{,} = r_{,}\{\sin v\sin\theta' - \theta\cos v\sin\psi - \theta\sin v\cos\psi\cos\theta'\}.$$

In calculating $L_{,}M_{,}$ we shall neglect all periodical terms except those which depend on $\phi - \psi$, *i. e.* on mt + const.; these are retained because the integration introduces a small divisor. We therefore have

$$L_{,} = \frac{3SA}{2c^3}\{-s(\cos^2\theta' - \tfrac{1}{2}\sin^2\theta') + \cos\theta'\sin\theta'\cos(\phi - \psi)\},$$

$$M_{,} = \frac{3SB}{2c^3}\{s'(\cos^2\theta' - \tfrac{1}{2}\sin^2\theta') - \cos\theta'\sin\theta'\sin(\phi - \psi)\}.$$

Hence the equations (1) become

$$\frac{d^2 s}{dt^2} + p^2 s = \frac{3S}{2c^3} \cos\theta' \sin\theta' \cos(\phi - \psi),$$

$$\frac{d^2 s'}{dt^2} + p^2 s' = \frac{3S}{2c^3} \cos\theta' \sin\theta' \sin(\phi - \psi),$$

where $p^2 = m^2 + \frac{2S'a^2}{a^5}(\epsilon - \frac{1}{2}am) + \frac{3S}{2c^3}(\cos^2\theta' - \frac{1}{2}\sin^2\theta')$;

$$\therefore \quad s = M\cos(pt + N) + \frac{3S\cos\theta'\sin\theta'}{2c^3(p^2 - m^2)}\cos(\phi - \psi),$$

$$s' = M'\cos(pt + N') + \frac{3S\cos\theta'\sin\theta'}{2c^3(p^2 - m^2)}\sin(\phi - \psi).$$

In order that θ may always remain small, M and M' must be small: and also the coefficient of the last terms must be small. Now this coefficient equals

$$\frac{\cos\theta'\sin\theta'}{\cos^2\theta' - \frac{1}{2}\sin^2\theta' + \frac{4S'a^2c^3}{3Sa^5}(\epsilon - \frac{1}{2}am)}.$$

To reduce this to numbers, let b be the distance of the outer satellite from the centre of Saturn: T, T' the times of a sidereal revolution of the Sun and this satellite: then

$$(S' \div S)(c^3 \div b^3) = T^2 \div T'^2: \; T = 10759.08 \text{ days},$$

$$T' = 79.3296 \text{ days}, \; b \div a = 59.154, \; a \div a = 2, \; \theta' = 30^\circ:$$

and therefore

$$\text{coefficient} = \frac{0''.005632}{\epsilon - \frac{1}{2}am + 0.00000000394}.$$

If $\epsilon - \frac{1}{2}am = 0$, this coefficient is not small, but very large. If however $\epsilon - \frac{1}{2}am$ have a sensible value such as 0.0602 (the value Laplace uses in *Mec. Cel.* Liv. VIII. §. 36.), the coefficient becomes exceedingly small.

Thus the action of Saturn, arising from the oblateness of its form, constantly retains the rings, so as to keep them nearly in the plane of its equator. Laplace came to this result before it was known by observation that Saturn had

any rotatory motion: he therefore inferred that Saturn revolved about its axis from the fact, that the rings always lie nearly in the plane of the equator.

621. The following Problems are formed from the general investigation of M. Mossotti *On the Forces which regulate the internal constitution of Bodies*, translated in Vol. I. of Taylor's *Scientific Memoirs*. They are here framed so that they may be solved by analysis more simple than that used by the Author. These problems contain the results at which M. Mossotti has arrived; for his general formulæ, in the paper above referred to, are in the end applied only to particular cases.

622. A given number of molecules repelling each other are placed in a boundless ether, the particles of which also repel each other; but the mutual action of the molecules and ether is attractive. The molecules are supposed to be spherical, very small, each in itself homogeneous, though they differ from each other in density. The law of elasticity of the ether is supposed to be such, that the change in the fluid-pressure varies as the change in the square of the density. We shall call the molecules M, M_1, M_2, ... Let $q =$ density of ether at the point (xyz); p the pressure; then $dp = kqdq$, k being a constant: $V =$ the sum of the quotients of the particles of ether divided by their distances from (xyz) taken throughout infinite space: $G, G_1, G_2, \ldots$ similar expressions for the actions of the molecules on (xyz); $r, r_1, r_2, \ldots$ the distances of (xyz) from the centres of the molecules. For convenience we suppose the integral V is taken throughout space, as though the ether pervaded the spaces occupied by the molecules: this will introduce an error; but the error may be counteracted by supposing that the mass of the molecules is altered in proportion to the mass of ether which they actually displace. Suppose $D, D_1, D_2, \ldots$ are the mean densities of the ether thus displaced; and $m, m_1, m_2, \ldots$ the densities, and $v, v_1, v_2, \ldots$ the volumes of the molecules.

Prob. 15. To find the law of density of the ether when there is only *one* molecule.

By the equation of fluid pressure (Art. 520.)

$$kq = \text{const.} - V + G.$$

We cannot find q from this, because V cannot be integrated without knowing q. But by differentiation we have (Art. 168, note,)

$$\frac{d^2q}{dx^2} + \frac{d^2q}{dy^2} + \frac{d^2q}{dz^2} = \left(-\frac{4\pi}{k}q =\right) - a^2q \ldots\ldots(1).$$

The nature of the case shews, that q is a function of r only: transform the equation and integrate, observing that $q = 0$ when $r = \infty$, and

$$q = \frac{Ae^{-ar}}{r}, \quad \therefore\ V = \frac{4\pi A}{a^2}\,\frac{1 - e^{-ar}}{r}, \text{ and } G = \frac{v\,(D + m)}{r}.$$

Substitute these values in the first equation, and we have

$$q = \frac{a^2 v\,(D + m)}{4\pi}\,\frac{e^{-ar}}{r}, \quad V = \frac{v\,(D + m)\,(1 - e^{-ar})}{r}.$$

Prob. 16. Suppose there are only *two* molecules.

By proceeding as before, we arrive at equation (1): q is a function only of r and r_1; transform equation in q, and it becomes

$$\frac{1}{r}\,\frac{d^2.qr}{dr^2} + \frac{1}{r_1}\,\frac{d^2.qr_1}{dr_1^{\,2}} = -a^2q:$$

the complete integral of this, under the circumstances of the problem, is

$$q = \frac{A}{r}\,e^{-ar} + \frac{A_1}{r_1}\,e^{-ar_1} = Q + Q_1, \text{ suppose;}$$

$$\therefore\ V = \Sigma\,.\,q\,\{\text{vol. of element of ether} \div \text{dist. from } (xyz)\}$$
$$= \Sigma\,.\,Q\,(\text{vol. of el.} \div \text{dist.}) + \Sigma\,.\,Q_1\,(\text{vol. of el.} \div \text{dist.}).$$

Q is a function of r only, and Q_1 of r_1 only; hence by choosing the spherical co-ordinates differently for the different terms, we have

$$V = \frac{4\pi A}{a^2}\,\frac{1 - e^{-ar}}{r} + \frac{4\pi A_1}{a^2}\,\frac{1 - e^{-ar_1}}{r_1}, \text{ and}$$

$$q = \frac{a^2}{4\pi}\left\{v\,(D + m)\,\frac{e^{-ar}}{r} + v_1\,(D_1 + m_1)\,\frac{e^{-ar_1}}{r_1}\right\}.$$

PROB. 17. In the case of Prob. 16, to find the attraction of the ether on the centre of the molecule M.

We have explained (Art. 621.) how compensation may be made for our supposing the ether to pervade the molecules. But since the first term of $q = \infty$ at the centre of M, we shall not obtain a correct result by differentiating V with respect to r (Art. 167.), and putting $r = 0$. This difficulty, however, may be overcome by entirely omitting the first term of V, because that part of the attraction of the ether upon the centre of M, which depends upon this term, is the same in all directions. It may then be easily proved, that the attraction of the ether on M

$$= v_1 (D_1 + m_1) \left\{ \frac{1}{R^2} - \frac{(1 + aR)}{R^2} \varepsilon^{-aR} \right\}.$$

R = the distance of the centres of the molecules.

PROB. 18. In the case of two molecules only, find the moving force with which they act on each other: and shew, that if a be a very large constant, the action of the molecules is repulsive from their point of contact to a certain small distance of separation, which is the distance of equilibrium; that at a greater distance their action is always attractive; that this attraction increases as they separate still more, though through a very small space; and that after this, as they separate more and more the force follows the Law of Universal Gravitation.

The molecules are so small, that the resultant of the fluid-pressure upon them may be neglected. The above points all flow with ease from the expression for the attraction, which is as follows: the moving force estimated positive from M towards M_1

$$= vv_1 \{(D + m)(D_1 + m_1) - mm_1\} \frac{1}{R^2}$$

$$- vv_1 (D + m)(D_1 + m_1) \frac{(1 + aR) e^{-aR}}{R^2}.$$

PROB. 19. When there are any number of molecules, to find q, V, and the attraction of the ether on M.

These values are

$$q = \frac{a^2}{4\pi} \Sigma . v (D + m) \frac{e^{-ar}}{r}, \quad V = \Sigma . v (D + m) \frac{1 - e^{-ar}}{r},$$

and the attraction of the ether on M

$$= \Sigma . \left\{ \frac{v (D + m)}{R^2} \left(1 - \frac{1 + aR}{e^{aR}}\right) \right\} - \frac{v (D + m)}{R^2} \left(1 - \frac{1 + aR}{e^{aR}}\right),$$

Σ being extended, in each case, throughout the whole system of molecules.

Prob. 20. Four homogeneous and equal spherical molecules, placed at the points of a regular tetrahedron, are in equilibrium under their mutual action, and that of the ether. Find the conditions of equilibrium, and prove that the greater the density of the ether, the greater will be the mutual distance of the molecules.

N.B. This illustrates the expansion and contraction of bodies by change of temperature.

Take the molecule, the equilibrium of which is to be considered, as the origin, and the axis of z perpendicular to the plane passing through the other three: the forces parallel to x and y vanish of themselves, and the equation of equilibrium parallel to z is

$$- v^2 (D + m)^2 \frac{1 + aR}{R^2} e^{-aR} + \{v^2 (D + m)^2 - v^2 m^2\} \frac{1}{R^2} = 0,$$

from which the fact above stated is evidently true.

623. The subject of Terrestrial Magnetism has of late excited much interest, and observations are now being made on a new plan in many parts of the world to help philosophers to a complete theory of the law of distribution of magnetic power through the Earth. The celebrated Professor Gauss has published a theory based on general calculations, from which we have framed the following problems. See the *Scientific Memoirs*, Vol. II. The foundation of these researches is the assumption, that the terrestrial magnetic force is the collective action of all the magnetized particles of the Earth's mass. Magnetization is represented to be a separation of the magnetic fluids. Admitting this representation, the mode of action of

the fluids (repulsion of similar and attraction of dissimilar particles inversely as the square of the distance,) belongs to the number of established physical truths.

By the magnetic force which is produced in any point of space by the action of the magnetic fluid elsewhere, we always mean to speak of the moving force, which is there exercised on a unit of the positive magnetic fluid; therefore in this sense the supposed magnetic fluid m concentrated in a point exercises at the distance ρ the magnetic force $m \div \rho^2$, of either repulsion or attraction in the direction ρ, according as μ is positive or negative.

Prob. 21. To find expressions for the magnetic force exercised in each point of space by the Earth.

Conceive the whole volume of the Earth, as far as it contains free magnetism (that is to say, separated magnetic fluids), to be divided into infinitely small elements; suppose dm is the quantity of free magnetism in any of the elements: ρ the distance of this element from the point (xyz) in space: $V =$ the sum of the quotients of dm and ρ with the sign changed through the whole Earth. Then the attraction or repulsion in any direction (Art. 167.) depends upon the calculation of V. If X, Y, Z be the forces, at right angles to each other, in the meridian, perpendicular to the meridian, and vertically downwards, and if u and λ be the north polar distance and latitude of the point in space, the action at which is under consideration, and r its distance from the centre of the Earth, then

$$X = -\frac{1}{r}\frac{dV}{du}, \quad Y = -\frac{1}{r \sin u}\frac{dV}{d\lambda}, \quad Z = -\frac{dV}{dr}:$$

and if V be expanded in a series of Laplace's Coefficients, as

$$V = \frac{R^2 P_0}{r} + \frac{R^3 P_1}{r^2} + \dots + \frac{R^{i+2} P_i}{r^{i+1}} + \dots$$

($R =$ mean rad. of the Earth:) from which X, Y, Z may easily be expressed in series.

Prob. 22. To shew, that if the figure of the Earth be considered a sphere, the law of distribution of the magnetic

force all over the surface of the Earth may be deduced from any one of the following sets of observations; *first*, a knowledge of the value of V at all points of the surface; *secondly*, a knowledge of the value of X alone for the whole surface; *thirdly*, a knowledge of Y for the whole surface, together with a knowledge of X merely for any continuous line running from pole to pole of the Earth's axis of revolution; *fourthly*, a knowledge of Z all over the surface. Any one of these will lead to a complete theory.

At the surface $r = R$, and therefore

$$V = R\left\{P_0 + P_1 + \ldots. + P_i + \ldots.\right\},$$

$$X = -\left\{\frac{dP_0}{du} + \frac{dP_1}{du} + \ldots. + \frac{dP_i}{du} + \ldots.\right\},$$

$$Y = -\frac{1}{\sin u}\left\{\frac{dP_0}{d\lambda} + \frac{dP_1}{d\lambda} + \ldots. + \frac{dP_i}{d\lambda} + \ldots.\right\},$$

$$Z = P_0 + 2P_1 + \ldots. + (i+1)P_i + \ldots.$$

Hence, by the property of Laplace's Coefficients, proved in Arts. 181, 182, we see, *first*, that if we know V, we know P_0, P_1, ... and therefore X, Y, Z. *Secondly*,

$$\int_0^u X\,du = F - P_0 - P_1 - \ldots. - P_i - \ldots.$$

where F is independent of u, and would in the general case be a function of λ: but since F is independent of u, it must be the same for every point of a given meridian, but all meridians have a common point in the two poles, therefore F is an absolute constant, and a knowledge of X leads to a knowledge of $\int_0^u X\,du$, and therefore to P_0, P_1 .... P_i .... *Thirdly*, we have

$$\int_0^\lambda \sin u\, Y d\lambda = G - P_0 - P_1 - \ldots. - P_i - \ldots.$$

where G is independent of λ, but may be a function of u. Hence

$$G = \frac{V}{R} + \int_0^\lambda Y \sin u\, d\lambda,$$

$$\frac{dG}{du} = -X + \int_0^\lambda \left\{\frac{dY}{du}\sin u + Y\cos u\right\} d\lambda.$$

Now G does not involve λ, and therefore if the values of the second side were calculated merely for any *particular line* joining the poles of the Earth's axis, the result will come out the same as if we calculated it for the whole surface. Hence if (in combination with our knowing Y all over the surface) we know X for any single line on the surface from pole to pole, we have G, and therefore P_0, P_1 .... P_i .... *Fourthly*, it is clear that a knowledge of Z all over the surface leads to the knowledge of P_0, P_1 .... P_2 ....

624. For a developement of this Theory, and its remarkable success when applied numerically to the case of the Earth, we must refer to the Original Memoir, or to the translated version in the *Scientific Memoirs*.

THE END.

ERRATA ET CORRIGENDA.

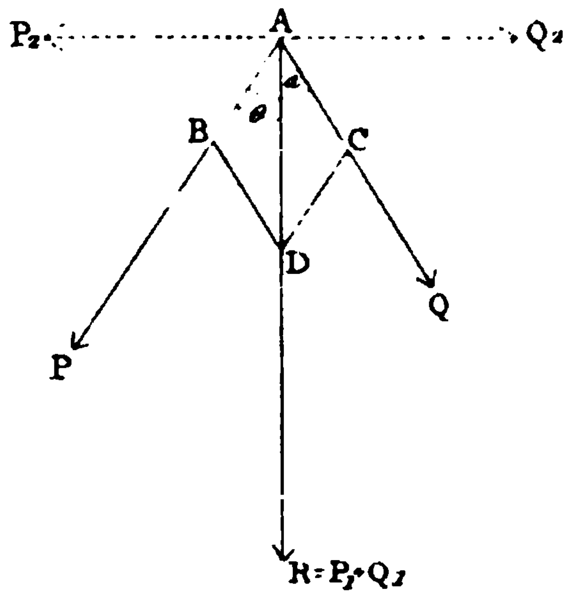
PAGE	LINE	FOR	READ
18	6	Arts. 30, 31.	Arts. 29, 30.
21	20, 22	Q_1	Q,
30	20	and the	and (as in Art. 47) the
36	2	$y'' + y'$	$y'' - y'$
58	9	$4 + n^2$	$1 + n^2$
129	14	*dele* $= \dfrac{(W+W')\sin\beta}{W'\sin\alpha\sin(\alpha+\beta)} - \dfrac{\sin(2\alpha+\beta)}{\sin\alpha\sin(\alpha+\beta)}$.	
131	8	$\sin\psi$ ...	$\sin\psi = 0$ ...
—	15	$\sin(\alpha+\theta')$	$\sin(\alpha+\theta)$
138	10	the lower cylinders will slip first	the cylinders will never slip
145	16	$\pi\rho c$	$\pi\rho$
157	3 *note*,	OX	ox
161	9	equate	equate the coefficient of
—	4 *from bottom*,	unity	r'
164	12	r^2	r^4
170	4	Q'	Q'_i
172	15	*dele* (Art. 171)	
172	22 *and last*,	Art. 191	Art. 190
205	2	Art. 216	Art. 212
215	5 *from bottom*,	$\dfrac{2\mu}{x}$	$\dfrac{2\mu}{a}$
257	2 *from bottom*,	Art. 169	Art. 167
268 269	9 10	*with respect to the Earth*,	*with respect to the line joining the Earth and Sun*,
304	10	Art. 317	Art. 337
305	6	regression	progression
384	20	$\frac{1}{2}\pi b$	$(\frac{1}{2}\pi - 1)\, b$
385	20	corresponding	conjugate
393	2 *from bottom*,	PQ'	PP'
394	1	$\int_{-\infty}$	$\int_{-\infty}^{\infty}$
405, 407	*heading*,	CATER	KATER
421	15	ωk^2	ωk
426	3	$S\Sigma$	Σ
429	11	The first edition of this work was published in 1836: and 1836 + 720 = 2556.	
432	3	n sin	N sin
443	3	$M\sin\nu$	$N\sin\nu$
—	8	M^2	M
455	10 *from bottom*,	Art. 489	Art. 484
478	5 *from bottom*,	where	when
479	5 *from bottom*,	XI.	X.

PAGE	LINE	FOR	READ
482	6	on	or
—	8 *from bottom*,	224	225
492	5	(6)	(5)
—	11 *from bottom*, *dele* β the angle......is measured:		
497	10, 11, *the quantities under* Σ *must be interchanged in these two lines.*		
498	PROB. 27	given	certain
—	——	prove that the path......circle.	find the velocity of projection and the friction when the path is a circle.
500	*last line*,	(2)	(3)
501	1	e	P
518	6, 7 *from bottom*,	$+C$	$-C$
525	5	$i=0$	$i=2$
528	7	(1)	(2)
529	2	$\int_0$	$\int_0^{\pi}$.
531	4	533	535
532	3 *twice*,	a	aϵ
534	2	a'^4	a'^5
545	15	to	to be
568	3 *from bottom*,	a	$\frac{1}{a}$
575	2	$\frac{d^2y}{dt^2}$	$-\frac{d^2y}{dt^2}$
530	*The formulæ deduced in Art.* 581 *are true for orbits of any eccentricity.*		
588	1	Article following that	COR. of the Article.

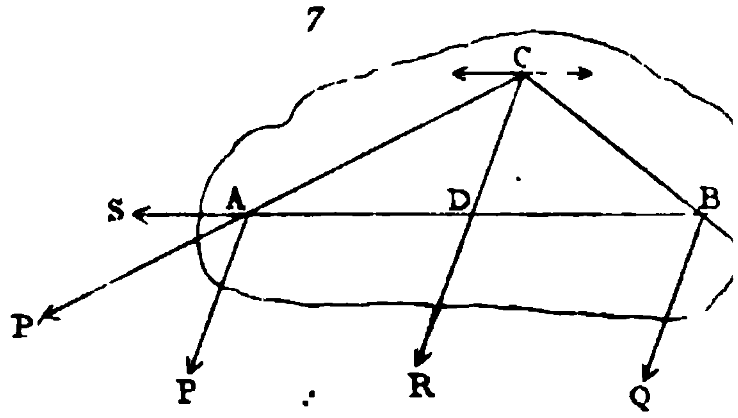
IN THE FIGURES.

Fig. 13, for P_1 read R.
Fig. 73, At the extremity of AC produced write B.
Fig. 103, A is omitted at the top.

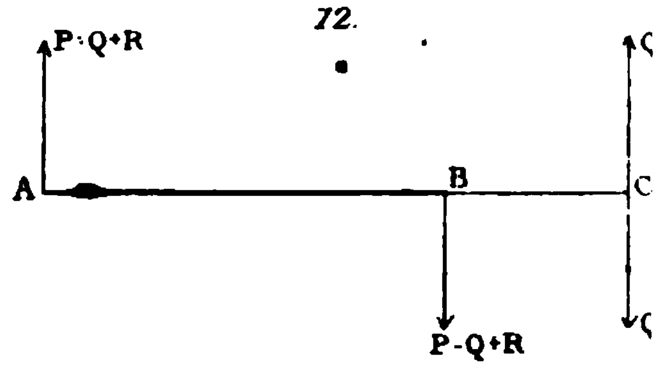
1.



7



12.



17

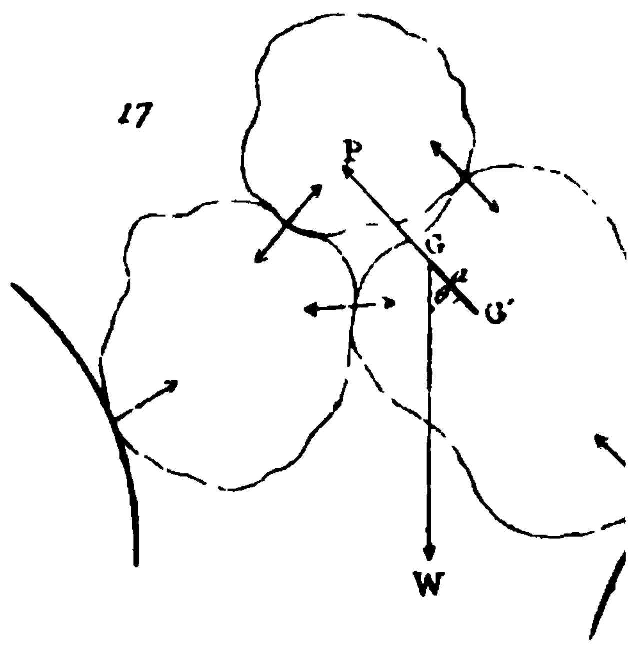


Plate II

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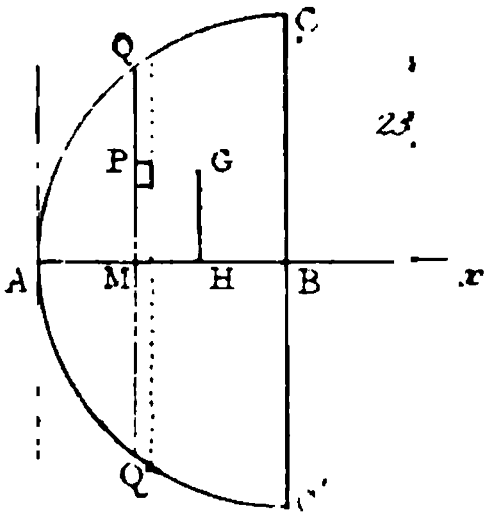
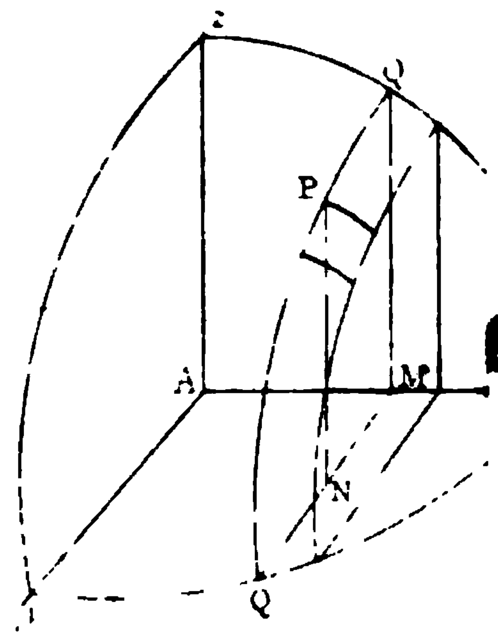
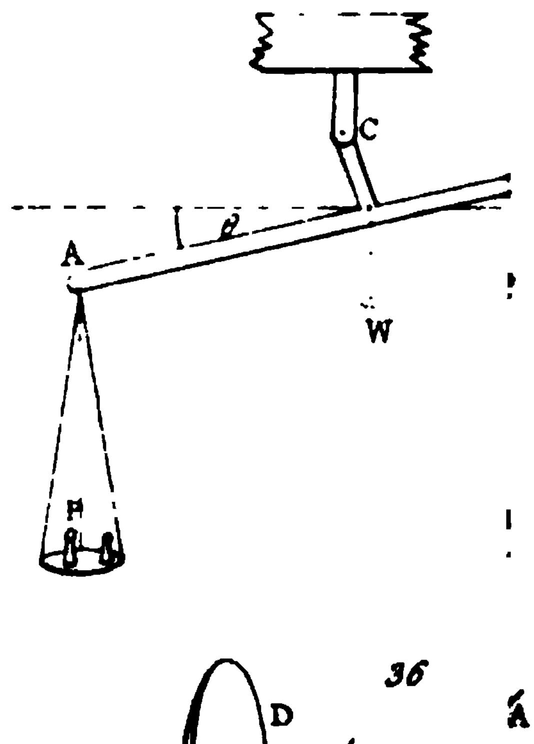
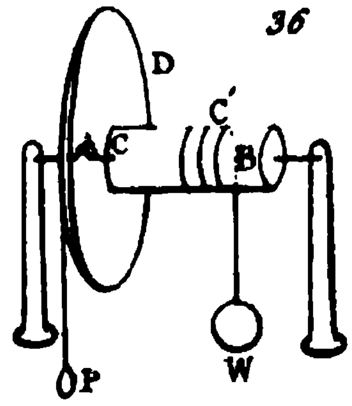
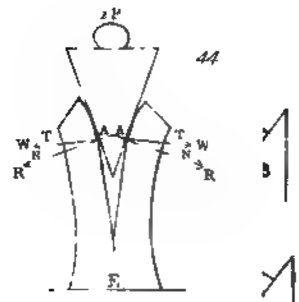




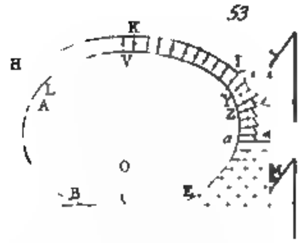


36





44



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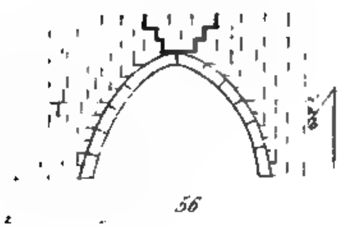

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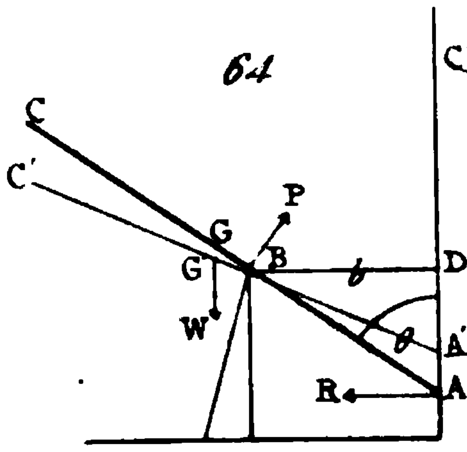

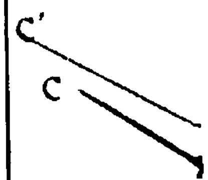

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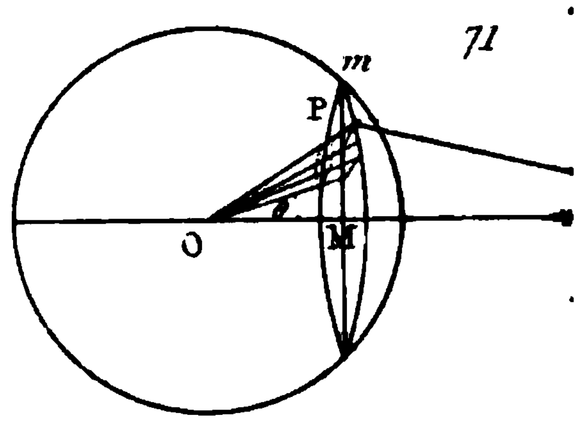

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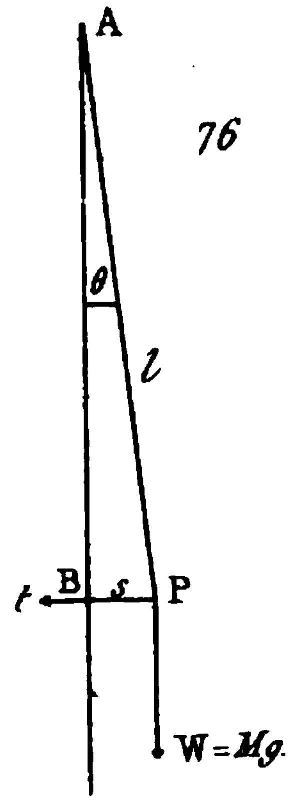


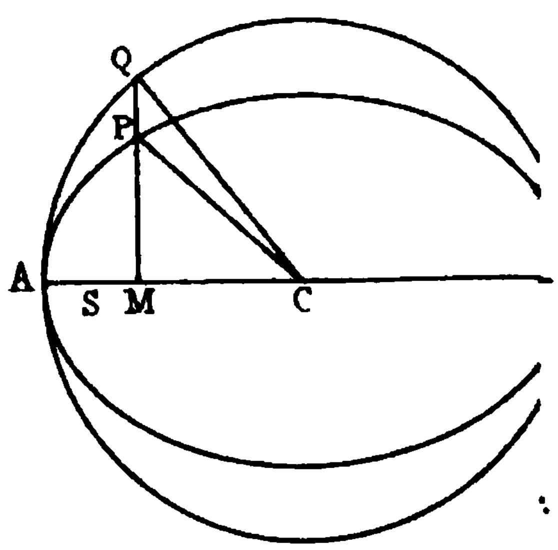
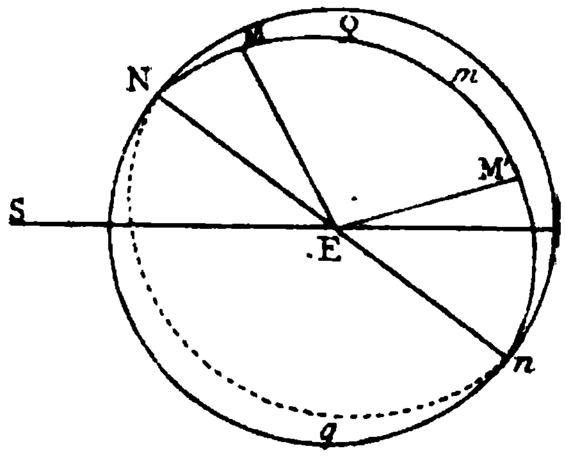
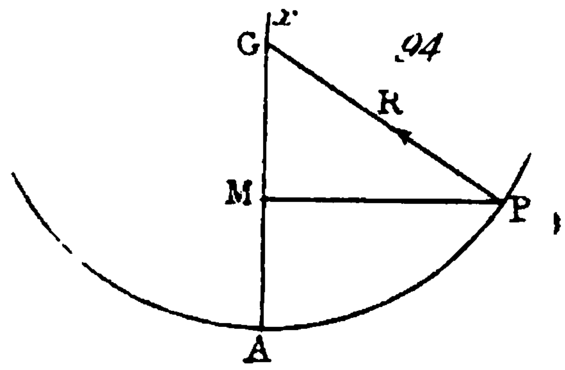


Plate V.

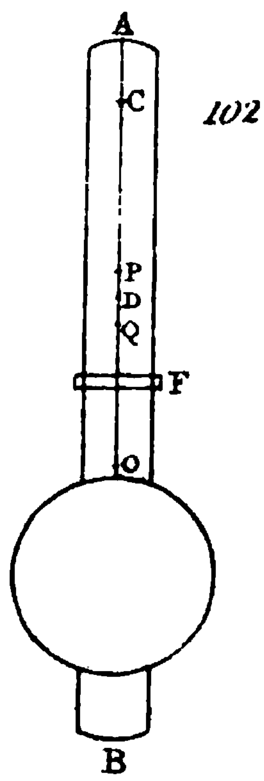
89

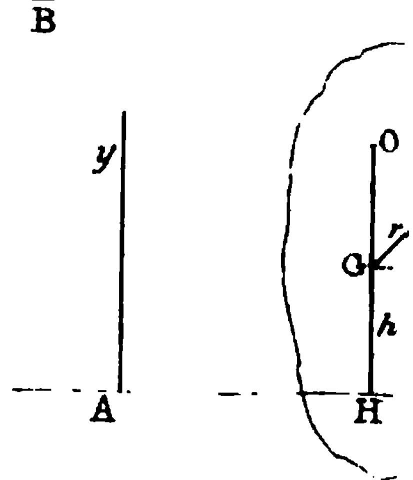


94



102





Basire sc.